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ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR SECOND ORDER THREE-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we consider the existence of positive solutions for the second order three-point boundary value problem

$$\begin{cases} x''(t) + m^2x(t) = f(t,\ x(t)) + e(t),\ t \in (0,\ 1), \\ x'(0) = 0,\ x'(1) + x'(\eta) = 0, \end{cases}$$

where $m \in (0,\ \frac{\pi}{2})$ is a constant, $\eta \in [0,\ 1)$, $e \in C[0,\ 1]$ and nonlinearity $f(t,\ x)$ may be singular at $x = 0$. The proof of the main result are based on Krasnoselskii's theorem on cone, together with a truncation technique. Our results extend and improve some known results.

1. Introduction

In this paper, we study the existence of positive solutions for the following second order three-point boundary value problem

$$\begin{cases} x''(t) + m^2x(t) = f(t,\ x(t)) + e(t),\ t \in (0,\ 1), \\ x'(0) = 0,\ x'(1) + x'(\eta) = 0, \end{cases} \tag{1.1}$$

where $m \in (0,\ \frac{\pi}{2})$ is a constant, $\eta \in [0,\ 1)$, $e \in C[0,\ 1]$ and nonlinearity $f(t,\ x)$ may be singular at $x = 0$.

Recently, many authors have been interested in studying the existence of positive solutions for singular boundary value problems (for example, see [2-4, 9, 12] and the references therein). At the same time, investigation of positive solutions of nonlocal boundary value problem, initiated by Il' in and Moiseev [13, 14], has been given considerable attention by various authors. We refer the reader to [5-8] for some references along this line. Multi-point boundary value problems describe many phenomena in the applied mathematical sciences. For examples, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see Moshinsky [10]); many problems in the theory of elastic stability can be handled by the method

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of multi-point problems (see Timoshenko [11]). So it is interesting and important to study the existence of positive solutions for the singular second order three-point boundary value problem (1.1). However, to the best of our knowledge, the existence of positive solutions for the singular second order three-point boundary value problem (1.1) have not been discussed.

2008, Chu, Sun and Chen in [1], considered the singular boundary value problem

$$\begin{cases} x''(t) + m^2 x(t) = f(t,\ x(t)) + e(t),\ t \in (0,\ 1), \\ x'(0) = 0,\ x'(1) = 0, \end{cases} \tag{1.2}$$

where $m \in (0,\ \frac{\pi}{2})$ is a constant, $e \in C[0,\ 1]$ and nonlinearity $f(t,\ x)$ may be singular at $x = 0$. They showed that the boundary value problem (1.2) has at least one positive solution under some restrictions on the nonlinear term $f(t,\ x)$.

Motivated by the results in [1-4] and [6-9], the aim of this paper is to consider the existence of positive solutions for the more general three-point boundary value problem (1.1). Comparing with [1], we discuss the second order boundary value with nonlocal boundary conditions. Our main results (see Theorem 3.1 and Corollary 3.1 below) improve and generalize the results of [1] to some degree. The proof of the main result are based on Krasnoselskii's theorem on cone. To the best of our knowledge, the first paper taking this approach is by Wang in [15].

The rest of the paper is organized as follows: In Section 2, we state some notations and prove some preliminary results. In Section 3, we state and prove our main result.

Let us fix some notation to be used. Given $\varphi \in L^1[0,\ 1]$, we write $\varphi \succ 0$ if $\varphi \geq 0$ for *a.e.* $t \in [0, 1]$ and it is positive in a set of positive measure. Let us denote by p^* and p_* the essential supremum and infimum of a given function $p \in L^1[0, 1]$, if they exist.

2. Preliminaries and lemmas

Lemma 2.1. *Suppose* $y : [0,\ 1] \to [0,\ \infty)$ *is continuous,* $m \in (0,\ \frac{\pi}{2})$, $\eta \in [0,\ 1)$. *Then the linear problem*

$$\begin{cases} x''(t) + m^2 x(t) = y(t),\ t \in (0,\ 1), \\ x'(0) = 0,\ x'(1) + x'(\eta) = 0, \end{cases} \tag{2.1}$$

has a unique solution $x \in C^2[0,\ 1]$ *with the representation*

$$x(t) = \int_0^1 G(t,\ s) y(s)\, ds,$$

where

$$G(t,\ s) = \begin{cases} \frac{\cos m(1-s)\cos mt}{m \sin m} + \frac{\sin m(1-\eta)\cos mt \cos ms}{m \sin m(\sin m + \sin m\eta)},\ 0 \leq t \leq s \leq \eta \leq 1, \\ \frac{\cos m(1-s)\cos mt}{m \sin m} - \frac{\sin m\eta \cos mt \cos m(1-s)}{m \sin m(\sin m + \sin m\eta)},\ 0 \leq t \leq \eta \leq s \leq 1, \\ \frac{\cos m(1-t)\cos ms}{m \sin m} + \frac{\sin m(1-\eta)\cos mt \cos ms}{m \sin m(\sin m + \sin m\eta)},\ 0 \leq s \leq \eta \leq t \leq 1, \\ \frac{\cos m(1-t)\cos ms}{m \sin m} - \frac{\sin m\eta \cos mt \cos m(1-s)}{m \sin m(\sin m + \sin m\eta)},\ 0 \leq \eta \leq s \leq t \leq 1. \end{cases}$$

Proof. A general solution of $x''(t)+m^2x(t)=y(t),\ t\in(0,\ 1)$ is

$$x(t)=\int_0^t \frac{\cos m(1-t)\cos ms}{m\sin m}y(s)\mathrm{d}s+\int_t^1 \frac{\cos m(1-s)\cos mt}{m\sin m}y(s)\mathrm{d}s$$
$$+A\cos mt+B\cos m(1-t), \tag{2.2}$$

where A and B are constants. From (2.2) we have

$$x'(t)=m\sin m(1-t)\int_0^t \frac{\cos ms}{m\sin m}y(s)\mathrm{d}s-m\sin mt\int_t^1 \frac{\cos m(1-s)}{m\sin m}y(s)\mathrm{d}s$$
$$-Am\sin mt+Bm\sin m(1-t).$$

By using the boundary conditions $x'(0)=0$ and $x'(1)+x'(\eta)=0$, we obtain $B=0$,

$$A=\frac{1}{\sin m+\sin m\eta}\Big(m\sin m(1-\eta)\int_0^\eta \frac{\cos ms}{m\sin m}y(s)\mathrm{d}s$$
$$-m\sin m\eta\int_\eta^1 \frac{\cos m(1-s)}{m\sin m}y(s)\mathrm{d}s\Big).$$

Thus,

$$x(t)=\int_0^t \frac{\cos m(1-t)\cos ms}{m\sin m}y(s)\mathrm{d}s+\int_t^1 \frac{\cos m(1-s)\cos mt}{m\sin m}y(s)\mathrm{d}s$$
$$+\frac{\cos mt}{\sin m+\sin m\eta}\Big(m\sin m(1-\eta)\int_0^\eta \frac{\cos ms}{m\sin m}y(s)\mathrm{d}s$$
$$-m\sin m\eta\int_\eta^1 \frac{\cos m(1-s)}{m\sin m}y(s)\mathrm{d}s\Big).$$

Hence,

$$x(t)=\int_0^1 G(t,\ s)y(s)\mathrm{d}s. \qquad \square$$

It is easy to see that

$$\max_{0\le t,\ s\le 1} G(t,\ s)\le \frac{1}{m\sin m}+\frac{\sin m(1-\eta)}{m\sin m(\sin m+\sin m\eta)}, \tag{2.3}$$

$$\min_{0\le t,\ s\le 1} G(t,\ s)\ge \frac{\sin m\cos^2 m}{m\sin m(\sin m+\sin m\eta)}>0, \tag{2.4}$$

which imply

$$G(t,\ s)>0,\ \text{for all } t,\ s\in[0,\ 1]. \tag{2.5}$$

Let $A=\min\limits_{0\le t,\ s\le 1} G(t,\ s),\ B=\max\limits_{0\le t,\ s\le 1} G(t,\ s),\ \sigma=\frac{A}{B}$. Then $B>A>0$ and $0<\sigma<1$.

In order to prove the main result of this paper, we need the following fixed-point theorem of cone expansion-compression type due to Krasnoselskii's (see[16]).

Theorem 2.1. *Let E be a Banach space and $K\subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $\theta\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let $T:\ K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be a completely continuous operator. In addition suppose either*
(i) $\|Tu\|\le\|u\|,\ \forall\ u\in K\cap\partial\Omega_1$ and $\|Tu\|\ge\|u\|,\ \forall\ u\in K\cap\partial\Omega_2$ or
(ii) $\|Tu\|\ge\|u\|,\ \forall\ u\in K\cap\partial\Omega_1$ and $\|Tu\|\le\|u\|,\ \forall\ u\in K\cap\partial\Omega_2$ holds.
Then T has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. Main results

In this section, we state and prove the main results of this paper. Let us define the function

$$\gamma(t)=\int_0^1 G(t,\ s)e(s)\mathrm{d}s,$$

which is just the unique solution of the linear problem (2.1) with $y(t)=e(t)$. For our constructions, let $E=C[0,\ 1]$, with norm, $\|x\|=\sup\limits_{0\le t\le 1}|x(t)|$. Define a cone, K, by

$$K=\big\{x\in E|\ x(t)\geq 0 \text{ on } [0,\ 1],\ \text{and}\ \min_{0\le t\le 1}x(t)\geq\sigma\|x\|\big\}.$$

Theorem 3.1. *Suppose that there exist a constant $r>0$ such that*

(H_1) *there exist continuous, nonnegative functions $g,\ h$, and k, such that*

$$0\leq f(t,\ x)\leq k(t)\big[g(x)+h(x)\big] \textit{ for all } (t,\ x)\in[0,\ 1]\times(0,\ r],$$

$g>0$ is nonincreasing and $\frac{h}{g}$ is nondecreasing in $x\in(0,\ r]$;

(H_2) $\frac{r-\gamma^*}{g(\sigma r)\left(1+\frac{h(r)}{g(r)}\right)}>K^*$, here $K(t)=\int_0^1 G(t,\ s)k(s)\mathrm{d}s$;

(H_3) there exist a continuous function $\phi_r\succ 0$ such that

$$f(t,\ x)\geq\phi_r(t) \text{ for all } (t,\ x)\in[0,\ 1]\times(0,\ r];$$

(H_4) $\phi_r(t)+e(t)\succ 0$ for all $t\in[0,\ 1]$.

Then problem (1.1) has at least one positive solution x with $0<\|x\|<r$.

Remark 3.1. Theorem 3.1 extends [1, Theorem 3.1] in the following direction: The cases $\eta\neq 0$ are considered. When $\eta=0$, then (1.1) reduce to (1.2). So Theorem 3.1 is more extensive then [1, Theorem 3.1].

Proof of Theorem 3.1.

Let $\delta=\min\limits_{0\le t\le 1}\int_0^1 G(t,\ s)\phi_r(s)\mathrm{d}s+\gamma_*$. Choose $n_0\in\{1,\ 2,\cdots\}$ such that $\frac{1}{n_0}<\sigma r_1$, where $r_1<\min\{\delta,\ r\}$ is a constant. Let $N_0=\{n_0+1,\ n_0+2,\cdots\}$. Fix $n\in N_0$. Consider the boundary value problem

$$\begin{cases} x''(t)+m^2x(t)=f_n(t,\ x(t))+e(t),\ t\in(0,\ 1),\\ x'(0)=0,\ x'(1)+x'(\eta)=0,\end{cases}\tag{3.1_n}$$

where

$$f_n(t,\ x)=\begin{cases} f(t,\ x),\ \text{if } x\geq\frac{1}{n}\\ f(t,\ \frac{1}{n}),\ \text{if } 0\leq x\leq\frac{1}{n}.\end{cases}$$

We note that x is a solution of (3.1_n) if, and only if,

$$x(t)=\int_0^1 G(t,\ s)\big[f_n(s,\ x(s))+e(s)\big]\mathrm{d}s,\ 0\le t\le 1.\tag{3.2}$$

Define an integral operator $T_n:\ K\to E$ by

$$(T_nx)(t)=\int_0^1 G(t,\ s)\big[f_n(s,\ x(s))+e(s)\big]\mathrm{d}s,\ 0\le t\le 1,\ x\in K.$$

Then (3.2) is equivalent to the fixed point equation $x=T_nx$. We seek a fixed point of T_n in the cone K.

Set $\Omega_2 = \{x \in E|\ \|x\| < r\},\ \Omega_1 = \{x \in E|\ \|x\| < r_1\}$. If $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then

$$r \geq x(t) \geq \sigma\|x\| \geq \sigma r_1 > 0 \text{ on } [0,\ 1].$$

Notice from (2.5), (H_3) and (H_4) that, for $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, $(T_n x)(t) \geq 0$ on $[0,\ 1]$. Also, for $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, we have

$$\begin{aligned}(T_n x)(t) &= \int_0^1 G(t,\ s)\big[f_n(s,\ x(s)) + e(s)\big]\mathrm{d}s \\ &\leq \max_{0\leq t,\ s\leq 1} G(t,\ s)\int_0^1 \big[f_n(s,\ x(s)) + e(s)\big]\mathrm{d}s,\ t \in [0,\ 1],\end{aligned}$$

so that

$$\|T_n x\| \leq \max_{0\leq t,\ s\leq 1} G(t,\ s)\int_0^1 \big[f_n(s,\ x(s)) + e(s)\big]\mathrm{d}s. \tag{3.3}$$

And next, if $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, we have by (3.3),

$$\begin{aligned}\min_{0\leq t\leq 1}(T_n x)(t) &= \min_{0\leq t\leq 1}\int_0^1 G(t,\ s)\big[f_n(s,\ x(s)) + e(s)\big]\mathrm{d}s \\ &\geq \min_{0\leq t,\ s\leq 1} G(t,\ s)\int_0^1 \big[f_n(s,\ x(s)) + e(s)\big]\mathrm{d}s \\ &= \sigma \max_{0\leq t,\ s\leq 1} G(t,\ s)\int_0^1 \big[f_n(s,\ x(s)) + e(s)\big]\mathrm{d}s \\ &\geq \sigma\|T_n x\|.\end{aligned}$$

As a consequence, $T_n:\ K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. In addition, standard arguments show that T_n is completely continuous.

If $x \in K$ with $\|x\| = r$, then

$$r \geq x(t) \geq \sigma\|x\| = \sigma r > 0 \text{ on } [0,\ 1],$$

and we have by (H_1) and (H_2),

$$\begin{aligned}(T_n x)(t) &= \int_0^1 G(t,\ s)\big[f_n(s,\ x(s)) + e(s)\big]\mathrm{d}s \\ &\leq \int_0^1 G(t,\ s)k(s)\big[g(x(s)) + h(x(s))\big]\mathrm{d}s + \gamma^* \\ &\leq g(\sigma r)\big(1 + \frac{h(r)}{g(r)}\big)K^* + \gamma^* \\ &< r = \|x\|,\ t \in [0,\ 1].\end{aligned}$$

Thus, $\|T_n x\| \leq \|x\|$. Hence,

$$\|T_n x\| \leq \|x\|,\ \text{for } x \in K \cap \partial\Omega_2. \tag{3.4}$$

If $x \in K$ with $\|x\| = r_1$, then

$$r > r_1 \geq x(t) \geq \sigma\|x\| = \sigma r_1 > 0 \text{ on } [0,\ 1],$$

and we have by (H_3) and (H_4),

$$\begin{aligned}(T_nx)(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2},\ s)\big[f_n(s,\ x(s))+e(s)\big]\mathrm{d}s\\ &\geq \int_0^1 G(\frac{1}{2},\ s)[\phi_r(s)+e(s)]\mathrm{d}s\\ &\geq \min_{0\leq t\leq 1}\int_0^1 G(t,\ s)[\phi_r(s)+e(s)]\mathrm{d}s\\ &\geq \min_{0\leq t\leq 1}\int_0^1 G(t,\ s)\phi_r(s)\mathrm{d}s+\gamma_*\\ &= \delta > r_1 = \|x\|.\end{aligned}$$

Thus, $\|T_nx\|\geq\|x\|$. Hence,

$$\|T_nx\|\geq\|x\|,\ \text{for } x\in K\cap\partial\Omega_1. \tag{3.5}$$

Applying (ii) of Theorem 2.1 to (3.4) and (3.5) yields that T_n has a fixed point $x_n\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$, and $r_1\leq\|x_n\|\leq r$. As such, x_n is a solution of (3.1_n), and

$$r\geq x_n(t)\geq\sigma\|x_n\|\geq\sigma r_1>\frac{1}{n_0}>\frac{1}{n},\ t\in[0,\ 1]. \tag{3.6}$$

Next we prove the fact

$$\|x_n'\|\leq H \tag{3.7}$$

for some constant $H>0$ and for all $n\geq n_0$. To this end, integrating the first equation of (3.1_n) from 0 to 1, we obtain

$$m^2\int_0^1 x_n(t)\mathrm{d}t=\int_0^1\big[f_n(t,\ x_n(t))+e(t)\big]\mathrm{d}t.$$

Then

$$\begin{aligned}\|x_n'\| &= \max_{0\leq t\leq 1}|x_n'(t)|\\ &= \max_{0\leq t\leq 1}|\int_0^t x_n''(s)\mathrm{d}s|\\ &= \max_{0\leq t\leq 1}|\int_0^t\big[f_n(s,\ x_n(s))+e(s)-m^2x_n(s)\big]\mathrm{d}s|\\ &\leq \int_0^1\big[f_n(s,\ x_n(s))+e(s)\big]\mathrm{d}s+m^2\int_0^1 x_n(s)\mathrm{d}s\\ &= 2m^2\int_0^1 x_n(s)\mathrm{d}s\\ &\leq 2m^2r=:H.\end{aligned}$$

The fact $\|x_n\|\leq r$ and (3.7) show that $\{x_n\}_{n\in N_0}$ is a bounded and equicontinuous family on $[0,\ 1]$. Now the Arzela-Ascoli Theorem guarantees that $\{x_n\}_{n\in N_0}$ has a subsequence, $\{x_{n_k}\}_{k\in\mathbb{N}}$, converging uniformly on $[0,\ 1]$ to a function $x\in C[0,\ 1]$. From the fact $\|x_n\|\leq r$ and (3.6), x satisfies $\sigma r_1\leq x(t)\leq r$ for all $t\in[0,\ 1]$. Moreover, x_{n_k} satisfies the integral equation

$$x_{n_k}(t)=\int_0^1 G(t,\ s)\big[f_n(s,\ x_{n_k}(s))+e(s)\big]\mathrm{d}s.$$

Let $k \to \infty$ and we arrive at

$$x(t) = \int_0^1 G(t,\ s)\big[f(s,\ x(s)) + e(s)\big]\mathrm{d}s,$$

where the uniform continuity of $f(t,\ x)$ on $[0,\ 1] \times [\sigma r_1,\ r]$ is used. Therefore, x is a positive solution of boundary value problem (1.1). Finally it is not difficult to show that $\|x\| < r$. □

By Theorem 3.1, we have the following Corollary.

Corollary 3.1. *Assume that there exist continuous functions* $\bar{b},\ b \succ 0$ *and* $\lambda > 0$ *such that*

(F) $0 \le \frac{\bar{b}(t)}{x^{\lambda}} \le f(t,\ x) \le \frac{b(t)}{x^{\lambda}}$, *for all* $x > 0$ *and* $t \in [0,\ 1]$.

Then problem (1.1) has at least one positive solution if one of the following two conditions holds:

(i) $e_* \ge 0$;

(ii) $e^* < 0,\ \bar{b}_* + \left(\frac{B^*}{\sigma^{\lambda}}\right)^{\frac{\lambda}{\lambda+1}} e_* > 0$, *where* $B(t) = \int_0^1 G(t,\ s)b(s)ds$.

Remark 3.2. Corollary 3.1 extends [1, Corollary 3.1] in the following direction: The cases $\eta \ne 0$ are considered. When $\eta = 0$, then (1.1) reduce to (1.2). So Corollary 3.1 is more extensive then [1, Corollary 3.1].

4. Example

Consider second order Neumann boundary value problem

$$\begin{cases} x''(t) + \frac{\pi^2}{9}x(t) = \sqrt{2}t^{24}\big[x^{-1}(t) + 1\big],\ t \in (0,\ 1), \\ x'(0) = 0,\ x'(1) = x'(\frac{1}{2}). \end{cases} \tag{4.1}$$

Here, $f(t,\ x) = \sqrt{2}t^{24}\big[x^{-1} + 1\big],\ (t,\ x) \in [0,\ 1] \times (0,\ +\infty),\ e(t) \equiv 0,\ m = \frac{\pi}{3}$. Let $k(t) = \sqrt{2}t^{24},\ g(x) = \frac{1}{x},\ h(x) \equiv 1,\ \phi_r(t) = \frac{\sqrt{2}t^{24}}{2},\ r = 2$, then we can check that $(H_1),\ (H_3)$, and (H_4) are satisfied. In addition, for $r = 2$, from (2.3)(2.4) we have

$$\frac{r - \gamma^*}{g(\sigma r)\Big(1 + \frac{h(r)}{g(r)}\Big)} = \frac{4\sigma}{3} \ge \frac{4}{3}\frac{\frac{\sin m \cos^2 m}{m \sin m(\sin m + \sin m\eta)}}{\frac{1}{m \sin m} + \frac{\sin m(1-\eta)}{m \sin m(\sin m + \sin m\eta)}} = \frac{2\sqrt{3}}{12 + 6\sqrt{3}} > \frac{2\sqrt{3}}{24}.$$

On the other hand,

$$\begin{aligned} K^* = &= \max_{0 \le t \le 1} \int_0^1 G(t,\ s)(\sqrt{2}s^{24})\mathrm{d}s \\ &\le \frac{\sqrt{2}\Big(\frac{1}{m \sin m} + \frac{\sin m(1-\eta)}{m \sin m(\sin m + \sin m\eta)}\Big)}{25} \\ &= \frac{\sqrt{2}}{25} \cdot \frac{12(2 + \sqrt{3})}{2\pi(3 + \sqrt{3})} \\ &< \frac{2\sqrt{3}}{25}. \end{aligned}$$

Hence, $\frac{r-\gamma^*}{g(\sigma r)\big(1 + \frac{h(r)}{g(r)}\big)} > K^*$. So that (H_2) is satisfied. According to Theorem 3.1, the boundary value problem (4.1) has at least one positive solution x with $0 < \|x\| < 2$.

For boundary value problem (4.1), however, we cannot obtain the above conclusion by Theorem 3.1 of paper [1]. These imply that Theorem 3.1 in this paper complement and improve those obtained in [1].

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