DUNKL TRANSFORM OF DINI-LIPSCHITZ FUNCTIONS

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Abstract. Using a Dunkl translation operator, we obtain an analog of Younis’s theorem for the Dunkl transform for functions satisfying the Dini-Lipschitz condition in the space $L^p(\mathbb{R}, |x|^{2\alpha+1}dx)$, where $1 < p \leq 2$ and $\alpha > -\frac{1}{2}$.

1. Introduction and preliminaries

In [6], Younis proved the theorem related to Fourier transform and Dini-Lipschitz functions, Younis characterized the set of functions in $L^p(\mathbb{R})$ with $1 < p \leq 2$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely

**Theorem 1** Let $f(x) \in L^p(\mathbb{R})$ with $1 < p \leq 2$ such that

$$
\|f(x+h) - f(x)\|_{L^p(\mathbb{R})} = O\left(\frac{h^\alpha}{\log(\frac{1}{h})}\right),
$$

where $0 < \alpha \leq 1$ as $h \to 0$. Then $F(f) \in L^\beta(\mathbb{R})$ for

$$
\frac{p}{p+\alpha p-1} < \beta \leq p' = \frac{p}{p-1}
$$

and

$$
\frac{1}{\beta} < \gamma
$$

where $F(f)$ stands for the Fourier transform of $f$.

In this paper, we prove an analog of theorem 1 in the Dunkl transform. For this purpose, we use a Dunkl translation operator.

The Dunkl operator is a differential-difference operator $D_\alpha$

$$
D_\alpha f(x) = \frac{df(x)}{dx} + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}, \quad \alpha > -\frac{1}{2}.
$$

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where \( f \in L^p(\mathbb{R}, |x|^{2\alpha+1} dx), \ (1 < p \leq 2). \)

Let \( j_\alpha(x) \) is a normalized Bessel function of the first kind, i.e.,
\[
j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha + 1)J_\alpha(x)}{x^\alpha},
\]
where \( J_\alpha(x) \) is a Bessel function of the first kind ([1]).

The function \( j_\alpha(x) \) is infinitely differentiable and is defined also by
\[
j_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n(z^2)^{2n}}{n!\Gamma(n + \alpha + 1)}, \quad z \in \mathbb{C}
\] (1)

From (1), we see that
\[
\lim_{z \to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0
\]
by consequence, there exist \( c > 0 \) and \( \eta > 0 \) satisfying
\[
|z| \leq \eta \implies |j_\alpha(z) - 1| \geq c|z|^2.
\] (2)

Let kernel Dunkl [2] function is defined by the formula
\[
e_\alpha(x) = j_\alpha(x) + i(2\alpha + 2)^{-1}xj_{\alpha+1}(x).
\] (3)

The function \( y = e_\alpha(x) \) satisfies the equation \( D_\alpha y = iy \) with the initial condition \( y(0) = 1. \) In the limit case with \( \alpha = -\frac{1}{2} \) the kernel function coincides with the usual exponential function \( e^{ix}. \)

The formula (3) gives
\[
|1 - j_\alpha(hx)| \leq |1 - e_\alpha(hx)|
\] (4)

The Dunkl transform is defined by
\[
\hat{f}(\lambda) = \int_{-\infty}^{+\infty} f(x)e_\alpha(\lambda x)|x|^{2\alpha+1} dx, \ \lambda \in \mathbb{R}
\]
The inverse Dunkl transform is defined by the formula
\[
f(x) = \frac{1}{(2^{\alpha+1}\Gamma(\alpha + 1))^2} \int_{-\infty}^{+\infty} \hat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1} d\lambda.
\]

Plancherel’s theorem and the Marcinkiewicz interpolation theorem (see [4]) we get for \( f \in L^p(\mathbb{R}, |x|^{2\alpha+1} dx) \) with \( 1 < p \leq 2 \) and \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1, \)
\[
\|\hat{f}\|_q \leq C_0\|f\|_p,
\] (5)

where \( C_0 \) is a positive constant and
\[
\|f\|_p = \left( \int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} dx \right)^{1/p}.
\]

K. Trimèche has introduced in [5] the Dunkl translation operators \( \tau_h, \ h \in \mathbb{R}. \)

**Proposition 1** [3]
(1) For all \( x \in \mathbb{R} \) and \( f \in L^p(\mathbb{R}, |x|^{2\alpha+1} dx) \)
\[
\|\tau_h f\|_p \leq 4\|f\|_p
\]

(2) For all \( f \in L^1(\mathbb{R}, |x|^{2\alpha+1} dx) \), we have
\[
(\tau_h f)(\lambda) = e_\alpha(\lambda h)\hat{f}(\lambda)
\]

## 2. Main Results

**Definition 1** Let \( f(x) \in L^p(\mathbb{R}, |x|^{2\alpha+1} dx) \) is said to be in the Dini-Lipschitz Functions class, denoted by \( DLip(\alpha, p) \), if
\[
\|\tau_h f(x) - f(x)\|_p = O\left(\log(\frac{1}{h})\right)^{-1}
\]
as \( h \to 0 \).

A still further extension is possible if we write
\[
\|\tau_h f(x) - f(x)\|_p = O\left(\log(\frac{1}{h})\right)^{-\gamma}
\]
for some \( \gamma \)

**Theorem 2** Let \( f(x) \) belong to \( L^p(\mathbb{R}, |x|^{2\alpha+1} dx) \) with \( \alpha > -\frac{1}{2} \) and \( 1 < p \leq 2 \) such that
\[
\|\tau_h f(x) - f(x)\|_p = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right)
\]
as \( h \to 0 \) and \( 0 < \delta \leq 1 \). Then \( \hat{f} \in L^\beta(\mathbb{R}, |x|^{2\alpha+1} dx) \) for
\[
\frac{2\alpha p + 2p}{2p + 2\alpha(p - 1) + \delta p - 2} < \beta \leq q = \frac{p}{p - 1}
\]
and
\[
\frac{1}{\beta} < \gamma
\]

**Proof.** From proposition 1, the Dunkl transform of \( \tau_h f(x) - f(x) \) is \((e_\alpha(xh) - 1)\hat{f}(x)\).

We have
\[
\|\tau_h f(x) - f(x)\|_p = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right)
\]
as \( h \to 0 \), then
\[
\|\tau_h f(x) - f(x)\|^p_p = O\left(\frac{h^{\delta p}}{(\log \frac{1}{h})^{\gamma p}}\right).
\]
By proposition 1 and formula (5), we have
\[ \int_{-\infty}^{\infty} |1 - e_\alpha(hx)|^q |\hat{f}(x)|^q |x|^{2\alpha+1} dx \leq C_0 \left( \int_{-\infty}^{\infty} |\tau_h f(x) - f(x)|^p |x|^{2\alpha+1} dx \right)^{1/p-1} \]

From (2) and (4), we obtain
\[ \int_{0}^{\eta/h} |hx|^{2\eta} |\hat{f}(x)|^q |x|^{2\alpha+1} dx \leq C_0 \frac{h^{\delta q}}{(\log \frac{1}{h})^{\gamma q}}. \]

and hence
\[ \int_{0}^{\eta/h} |x^2 \hat{f}(x)|^q |x|^{2\alpha+1} dx = O \left( \frac{h^{(\delta-2)q}}{(\log \frac{1}{h})^{\gamma q}} \right). \]

Let
\[ \psi(t) = \int_{t}^{t} |x^2 \hat{f}(x)|^q |x|^{2\alpha+1} dx. \]

Then, if \( \beta \leq q \), and by Hölder inequality we obtain
\[
\psi(t) \leq \left( \int_{t}^{t} |x^2 \hat{f}(x)|^q |x|^{2\alpha+1} dx \right)^{\beta/q} \left( \int_{t}^{t} dx \right)^{1-\beta/q} \\
= O(t^{(2-\delta)\beta q/(\gamma q)} (\log t)^{-\gamma q(\beta(q+1)/q - 1)}). \\
= O(t^{(2-\delta)\beta(q/q-1)} (\log t)^{-\gamma q(1-\beta/q)}) \\
= O((\log t)^{-\gamma q(1+\beta-\delta)/q}).
\]

Hence
\[
\int_{t}^{t} |\hat{f}(x)|^q |x|^{2\alpha+1} dx = \int_{t}^{t} x^{-2\beta-(2\alpha+1)\beta q} \psi' (x) x^{2\alpha+1} dx \\
= t^{-2\beta-(2\alpha+1)\beta q} x^{2\alpha+1} \psi(t) + (2\beta + (2\alpha + 1)\beta q - (2\alpha + 1)) \int_{t}^{t} x^{-2\beta-(2\alpha+1)\beta q} x^{2\alpha+1} \psi(x) dx \\
= O \left( t^{-2\beta-(2\alpha+1)\beta q} x^{2\alpha+1} \psi(t) \log t^\gamma \right) + O \left( \int_{t}^{t} x^{-2\beta-(2\alpha+1)\beta q} x^{2\alpha+1} \psi(x) \log t^\gamma dx \right) \\
= O \left( t^{-2\beta-(2\alpha+1)\beta q} x^{2\alpha+1} \psi'(x) x^{2\alpha+1} \log t^\gamma \right),
\]

and for the right hand of this estimate to be bounded as \( t \to \infty \) one must have

\[-2\beta - (2\alpha + 1)\beta q + 2\alpha + 2 + \beta - \delta \beta + \beta/p < 0\]

and

\[-\gamma \beta < -1\]

i.e.,
\[ \beta > \frac{2\alpha p + 2p}{2p + 2\alpha(p-1) + \delta p - 2} \]

and
Similarly for the integral over \((-t, -1)\). This proves the theorem.

**Theorem 3** Let \( f(x) \) belong to \( L^p(\mathbb{R}, |x|^{2\alpha+1}dx) \), \( 1 < p \leq 2 \) such that

\[
\frac{1}{\beta} < \gamma
\]

Then \( \hat{f} \in L^\beta(\mathbb{R}, |x|^{2\alpha+1}dx) \) for

\[
\frac{\alpha p + 1}{p + \alpha(p - 1) - 1} < \beta \leq \frac{p}{p - 1}
\]

**Proof.** The proof goes exactly as that of theorem 2 and yields

\[
\int_1^t |\hat{f}(x)|^\beta x^{2\alpha+1} = O \left( t^{-\beta -(2\alpha+1)\frac{\beta}{q} + 2\alpha + 2 + \beta \frac{\beta}{p} \left( \log t \right)^{-\beta}} \right)
\]

and for the right hand of this estimate to be bounded as \( t \to \infty \) one must have

\[-\beta -(2\alpha+1)\frac{\beta}{q} + 2\alpha + 2 + \beta \frac{\beta}{p} < 0
\]

and

\[-\beta < -1 \text{ i.e., } 1 < \beta \text{ which is always the case in our situation.}
\]

Then

\[
\beta > \frac{\alpha p + 1}{p + \alpha(p - 1) - 1}
\]

and this ends the proof.

**References**


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