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DUNKL TRANSFORM OF DINI-LIPSCHITZ FUNCTIONS

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ABSTRACT. Using a Dunkl translation operator, we obtain an analog of Younis's theorem for the Dunkl transform for functions satisfying the Dini-Lipschitz condition in the space $\mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx)$, where $1 < p \leq 2$ and $\alpha > -\frac{1}{2}$.

1. INTRODUCTION AND PRELIMINARIES

In [6], Younis proved the theorem related to Fourier transform and Dini-Lipschitz functions, Younis characterized the set of functions in $\mathrm{L}^p(\mathbb{R})$ with $1 < p \leq 2$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely

Theorem 1 Let $f(x) \in \mathrm{L}^p(\mathbb{R})$ with $1 < p \leq 2$ such that

$$\|f(x+h) - f(x)\|_{\mathrm{L}^p(\mathbb{R})} = O\left(\frac{h^\alpha}{\log(\frac{1}{h})^\gamma}\right),$$

where $0 < \alpha \leq 1$ as $h \longrightarrow 0$. Then $\mathcal{F}(f) \in \mathrm{L}^\beta(\mathbb{R})$ for

$$\frac{p}{p + \alpha p - 1} < \beta \leq p' = \frac{p}{p-1}$$

and

$$\frac{1}{\beta} < \gamma$$

where $\mathcal{F}(f)$ stands for the Fourier transform of f.

In this paper, we prove an analog of theorem 1 in the Dunkl transform. For this purpose, we use a Dunkl translation operator.

The Dunkl operator is a differential-difference operator D_α

$$\mathrm{D}_\alpha f(x) = \frac{df(x)}{dx} + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x},\ \alpha > -\frac{1}{2},$$

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where $f \in \mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx),\ (1 < p \leq 2)$.

Let $j_\alpha(x)$ is a normalized Bessel function of the first kind, i.e.,

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha},$$

where $J_\alpha(x)$ is a Bessel function of the first kind ([1]).

The function $j_\alpha(x)$ is infinitely differentiable and is defined also by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)},\ z \in \mathbb{C} \tag{1}$$

From (1), we see that

$$\lim_{z \longrightarrow 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Longrightarrow |j_\alpha(z) - 1| \geq c|z|^2. \tag{2}$$

Let kernel Dunkl [2] function is defined by the formula

$$e_\alpha(x) = j_\alpha(x) + i(2\alpha+2)^{-1} x j_{\alpha+1}(x). \tag{3}$$

The function $y = e_\alpha(x)$ satisfies the equation $\mathrm{D}_\alpha y = iy$ with the initial condition $y(0) = 1$. In the limit case with $\alpha = -\frac{1}{2}$ the kernel function coincides with the usual exponential function e^{ix}.

The formula (3) gives

$$|1 - j_\alpha(hx)| \leq |1 - e_\alpha(hx)| \tag{4}$$

The Dunkl transform is defined by

$$\widehat{f}(\lambda) = \int_{-\infty}^{+\infty} f(x) e_\alpha(\lambda x) |x|^{2\alpha+1} dx,\ \lambda \in \mathbb{R}$$

The inverse Dunkl transform is defined by the formula

$$f(x) = \frac{1}{(2^{\alpha+1}\Gamma(\alpha+1))^2} \int_{-\infty}^{+\infty} \widehat{f}(\lambda) e_\alpha(-\lambda x) |\lambda|^{2\alpha+1} d\lambda.$$

Plancherel's theorem and the Marcinkiewicz interpolation theorem (see [4]) we get for $f \in \mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx)$ with $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\widehat{f}\|_q \leq C_0 \|f\|_p, \tag{5}$$

where C_0 is a positive constant and

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} dx \right)^{1/p}.$$

K. Trimèche has introduced in [5] the Dunkl translation operators $\tau_h,\ h \in \mathbb{R}$.
Proposition 1[3]

(1) For all $x \in \mathbb{R}$ and $f \in \mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx)$

$$\|\tau_h f\|_p \leq 4\|f\|_p$$

(2) For all $f \in \mathrm{L}^1(\mathbb{R}, |x|^{2\alpha+1}dx)$, we have

$$\widehat{(\tau_h f)}(\lambda) = e_\alpha(\lambda h)\widehat{f}(\lambda)$$

2. Main Results

Definition 1 Let $f(x) \in \mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx)$ is said to be in the Dini-Lipschitz Functions class, denoted by $DLip(\alpha, p)$, if

$$\|\tau_h f(x) - f(x)\|_p = O\left(\log(\frac{1}{h})\right)^{-1}$$

as $h \longrightarrow 0$.

A still further extension is possible if we write

$$\|\tau_h f(x) - f(x)\|_p = O\left(\log(\frac{1}{h})\right)^{-\gamma},$$

for some γ

Theorem 2 Let $f(x)$ belong to $\mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx)$ with $\alpha > -\frac{1}{2}$ and $1 < p \leq 2$ such that

$$\|\tau_h f(x) - f(x)\|_p = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right)$$

as $h \longrightarrow 0$ and $0 < \delta \leq 1$. Then $\widehat{f} \in \mathrm{L}^\beta(\mathbb{R}, |x|^{2\alpha+1}dx)$ for

$$\frac{2\alpha p + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta \leq q = \frac{p}{p-1}$$

and

$$\frac{1}{\beta} < \gamma$$

Proof. From proposition 1, the Dunkl transform of $\tau_h f(x) - f(x)$ is $(e_\alpha(xh) - 1)\widehat{f}(x)$.

We have

$$\|\tau_h f(x) - f(x)\|_p = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right)$$

as $h \longrightarrow 0$, then

$$\|\tau_h f(x) - f(x)\|_p^p = O\left(\frac{h^{\delta p}}{(\log\frac{1}{h})^{\gamma p}}\right).$$

By proposition 1 and formula (5), we have

$$\int_{-\infty}^{\infty}|1-e_\alpha(hx)|^q|\widehat{f}(x)|^q|x|^{2\alpha+1}dx \leq C_0\left(\int_{-\infty}^{\infty}|\tau_h f(x)-f(x)|^p|x|^{2\alpha+1}dx\right)^{1/p-1}$$

From (2) and (4), we obtain

$$\int_0^{\eta/h}|hx|^{2q}|\widehat{f}(x)|^q|x|^{2\alpha+1}dx \leq C_0\frac{h^{\delta q}}{(\log\frac{1}{h})^{\gamma q}},$$

and hence

$$\int_0^{\eta/h}|x^2\widehat{f}(x)|^q|x|^{2\alpha+1}dx = O\left(\frac{h^{(\delta-2)q}}{(\log\frac{1}{h})^{\gamma q}}\right).$$

Let

$$\psi(t)=\int_1^t|x^2\widehat{f}(x)|^\beta x^{(2\alpha+1)\beta/q}dx.$$

Then, if $\beta \leq q$, and by Hölder inequality we obtain

$$\begin{aligned}\psi(t) &\leq \left(\int_1^t|x^2\widehat{f}(x)|^q x^{2\alpha+1}dx\right)^{\beta/q}\left(\int_1^t dx\right)^{1-\beta/q}\\ &= O(t^{(2-\delta)q\frac{\beta}{q}}(\log t)^{-\gamma q.\frac{\beta}{q}}t^{1-\beta/q})\\ &= O(t^{(2-\delta)\beta}(\log t)^{-\gamma\beta}t^{1-\beta/q})\\ &= O((\log t)^{-\gamma\beta}t^{1+\beta-\delta\beta+\beta/p}).\end{aligned}$$

Hence

$$\int_1^t|\widehat{f}(x)|^\beta x^{2\alpha+1}dx = \int_1^t x^{-2\beta-(2\alpha+1)\frac{\beta}{q}}\psi^{'}(x)x^{2\alpha+1}dx$$

$$\begin{aligned}&= t^{-2\beta-(2\alpha+1)\frac{\beta}{q}}t^{2\alpha+1}\psi(t)+(2\beta+(2\alpha+1)\frac{\beta}{q}-(2\alpha+1))\int_1^t x^{-2\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha}\psi(x)dx\\ &= O\left(t^{-2\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha+1}t^{1+\beta-\delta\beta+\beta/p}(\log t)^{-\gamma\beta}\right)+O\left(\int_1^t x^{-2\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha}x^{1+\beta-\delta\beta+\beta/p}(\log x)^{-\gamma\beta}dx\right)\\ &= O\left(t^{-2\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha+2+\beta-\delta\beta+\beta/p}(\log t)^{-\gamma\beta}\right).\end{aligned}$$

and for the right hand of this estimate to be bounded as $t \longrightarrow \infty$ one must have

$$-2\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha+2+\beta-\delta\beta+\beta/p<0$$

and

$$-\gamma\beta<-1$$

i.e.,

$$\beta>\frac{2\alpha p+2p}{2p+2\alpha(p-1)+\delta p-2}$$

and

$$\frac{1}{\beta} < \gamma$$

Similarly for the integral over $(-t, -1)$. This proves the theorem.

Theorem 3 Let $f(x)$ belong to $\mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx)$, $1 < p \leq 2$ such that

$$f \in DLip(\alpha, p).$$

Then $\widehat{f} \in \mathrm{L}^\beta(\mathbb{R}, |x|^{2\alpha+1}dx)$ for

$$\frac{\alpha p + 1}{p + \alpha(p-1) - 1} < \beta \leq q = \frac{p}{p-1}$$

Proof. The proof goes exactly as that of theorem 2 and yields

$$\int_1^t |\widehat{f}(x)|^\beta x^{2\alpha+1} = O\left(t^{-\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha+2+\frac{\beta}{p}}(\log t)^{-\beta}\right)$$

and for the right hand of this estimate to be bounded as $t \longrightarrow \infty$ one must have

$$-\beta - (2\alpha+1)\frac{\beta}{q} + 2\alpha + 2 + \frac{\beta}{p} < 0$$

and

$-\beta < -1$ i.e., $1 < \beta$ which is always the cas in our situation.

Then

$$\beta > \frac{\alpha p + 1}{p + \alpha(p-1) - 1}$$

and this ends the proof.

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