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SQUARE-MEAN ASYMPTOTICALLY ALMOST AUTOMORPHIC SOLUTIONS FOR NONLOCAL NEUTRAL STOCHASTIC FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACES

ZUOMAO YAN, HONG-WU ZHANG

Abstract. In this paper, we prove the existence and uniqueness of square-mean asymptotically almost automorphic mild solution of a class of partial nonlocal neutral stochastic functional integro-differential equations with resolvent operators in a real separable Hilbert space. An example illustrating our main result is given.

1. Introduction

In this paper we study the existence and uniqueness of square-mean asymptotically almost automorphic solutions for the following nonlocal neutral stochastic functional integro-differential equations in the abstract form

$$dN(t,x(t)) = AN(t,x(t))dt + \int_0^t B(t-s)N(s,x(s))dsdt$$
$$+h(t,x(\gamma_2(t)))dt + f(t,x(\gamma_3(t)))dW(t), \quad t \geq 0, \tag{1}$$
$$x(0) + g(x) = x_0, \tag{2}$$

where $A : D(A) \subseteq L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H}), B(t) : D(B(t)) \subseteq L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H}), t \geq 0$, are linear, closed, and densely defined operators on $L^2(\mathbb{P},\mathbb{H})$, and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. Here $N(t,x(t)) = x(0) + a(t,x(\gamma_1(t))), a, h, f, g$ and $\gamma_i, i = 1,2,3$, are appropriate functions to be specified later.

The existence of almost automorphic, asymptotically almost automorphic, and pseudo-almost automorphic solutions is one of the most attracting topics in the qualitative theory of differential equations, due both to its mathematical interest and to the applications. Some recent contributions on the existence of such solutions for abstract differential equations, partial differential equations, functional-differential equations and integro-differential equations have been made; we refer the

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reader to the monographs of N'Guérékata [33],[34], the papers [3],[4],[14],[15],[16],[20],[26],[29],[30],[37] and the references therein. Ding et al.[17] established asymptotically almost automorphic solutions for integrodifferential equations with nonlocal initial conditions in Banach spaces. Diagana et al. [13] concerned with the existence and uniqueness of an asymptotically almost automorphic mild solution for partial neutral integro-differential equations with unbounded delay. Dos Santos and Cuevas [18] studied asymptotically almost automorphic solutions of abstract fractional integrodifferential neutral equations. Caicedo et al. [7] also considered asymptotically almost automorphic solution of some semilinear functional differential and integro-differential equations with infinite delay a complex Banach space. In recent years, stochastic differential equations have attracted great interest due to their applications in characterizing many problems in physics, biology, mechanics and so on; see [1],[10],[19],[22],[23],[28],[35],[36] and the references therein. Very Recently, Fu et al. [21] introduced a new concept of a square-mean almost automorphic stochastic process. The paper discussed the existence and uniqueness of square-mean almost automorphic mild solutions for stochastic differential equations in Hilbert spaces, which further generalized the almost automorphic theory from the deterministic version to the stochastic one. Chang et al. [8] proved the existence and uniqueness of square-mean almost automorphic mild solutions for a class of non-autonomous stochastic differential equations. In [9], the authors also established a new composition theorem for square-mean almost automorphic functions and applications to stochastic differential equations. Bezandry and Diagana [2] studied the existence of square-mean almost periodic solutions to the second order stochastic differential equations on a Hilbert space.

On the other hand, the nonlocal Cauchy problem was considered by Byszewski [5],[6] and the importance of nonlocal conditions in different fields has been discussed in [1],[11],[31],[38] and the references therein. In addition, the nonlinear integro-differential equations with resolvent operators serve as an abstract formulation of partial integro-differential equations that arise in many physical phenomena. One can see [12],[23],[24]and references therein.

In this paper, we will introduce the notion of square-mean asymptotically almost automorphy for stochastic processes and apply this new concept to investigate the existence and uniqueness of square-mean asymptotically almost automorphy mild solutions for the partial nonlocal neutral stochastic functional integro-differential equations in a real separable Hilbert space.

The paper is organized as follows. In Section 2, we recall briefly some basic notations and definitions, lemmas and preliminary facts which will be used throughout this paper. Section 3 includes some preliminaries results not only on the completeness of the space that consists of the square-mean asymptotically almost automorphic processes but also on the composition of such processes. Section 4 verifies the existence and uniqueness of the square-mean asymptotically almost automorphic solutions for the problem (1)-(2). Finally in Section 5, an illustrative example is provided to show the feasibility of the theoretical results developed in the paper.

2. Existence and uniqueness

In this section, we introduce some basic definitions, notations and lemmas which are used throughout this paper.

Throughout the paper, we assume that $(\mathbb{H}, \| \cdot \|, \langle \cdot, \cdot \rangle)$ and $(\mathbb{K}, \| \cdot \|_{\mathbb{K}}, \langle \cdot, \cdot \rangle_{\mathbb{K}})$ are two real separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The notation $L^2(\mathbb{P}, \mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that

$$E \parallel x \parallel^2 = \int_\Omega \parallel x \parallel^2 d\mathbb{P} < \infty.$$

For $x \in L^2(\mathbb{P}, \mathbb{H})$, let

$$\parallel x \parallel_2 = \left(\int_\Omega \parallel x \parallel^2 d\mathbb{P} \right)^{\frac{1}{2}}.$$

It is routine to check that $L^2(\mathbb{P}, \mathbb{H})$ is a Hilbert space equipped with the norm $\| \cdot \|$. We let $L(\mathbb{K}, \mathbb{H})$ be the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\| \cdot \|_{L(\mathbb{K},\mathbb{H})}$; in particular, this is simply denoted by $L(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$.

We denote by $C_0(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$ the space of all continuous functions $h : \mathbb{R}^+ \to L^2(\mathbb{P}, \mathbb{H})$ such that $\lim_{t\to\infty} E \parallel h(t) \parallel^2 = 0$, and by $C_0(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ the space of all continuous functions $h : \mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ such that $\lim_{t\to\infty} E \parallel h(t, x) \parallel^2 = 0$ uniformly for x in any compact subset of $L^2(\mathbb{P}, \mathbb{H})$.

Definition 2.1. A family of bounded linear operators $\{R(t) : t \geq 0\}$ from $L^2(\mathbb{P}, \mathbb{H})$ into $L^2(\mathbb{P}, \mathbb{H})$ is a resolvent operator family for the problem

$$dx(t) = Ax(t)dt + \int_0^t B(t-s)x(s)dsdt, \ t \geq 0, \tag{3}$$

$$x_0 = x(0) \in L^2(\mathbb{P}, \mathbb{H}), \tag{4}$$

if the following conditions are verified.
(a) $R(0) = I$ (the identity operator on $L^2(\mathbb{P}, \mathbb{H})$) and the function $R(t)x$ is continuous on $[0, +\infty)$ for every $x \in L^2(\mathbb{P}, \mathbb{H})$;
(b) $R(t)D(A) \subseteq D(A)$ for all $t \geq 0$ and for $x \in D(A), AR(t)x$ is continuous on $[0, +\infty)$ and $R(t)x$ is continuously differentiable on $[0, +\infty)$;
(c) For $x \in D(A)$, the next resolvent equations hold,

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds, \ t \geq 0,$$

$$\frac{d}{dt}R(t)x = R(t)Ax + \int_0^t R(t-s)AB(s)xds, \ t \geq 0.$$

For more details on semigroup theory and resolvent operators, we refer [12],[23], [24].

Definition 2.2([21]). A stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be stochastically continuous if

$$\lim_{t\to s} E \parallel x(t) - x(s) \parallel^2 = 0.$$

Definition 2.3([21]). A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there is a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(P, H)$ such that

$$\lim_{n\to\infty} E \parallel x(t+s_n) - y(t) \parallel^2 = 0 \ \text{ and } \ \lim_{n\to\infty} E \parallel y(t-s_n) - x(t) \parallel^2 = 0$$

holds for each $t \in \mathbb{R}$.

The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is denoted by $AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$.

Lemma 2.1([21]). If x, x_1 and x_2 are all square-mean almost automorphic stochastic processes, then the following hold true:
(a) $x_1 + x_2$ is square-mean almost automorphic.
(b) λx is square-mean almost automorphic for every scalar λ.
(c) There exists a constant $M > 0$ such that $\sup_{t\in\mathbb{R}} \| x(t) \|_2 \leq M$. That is, x is bounded in $L^2(\mathbb{P}, \mathbb{H})$.

Lemma 2.2([21]). $(AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})), \| \cdot \|_\infty)$ is a Banach space when it is equipped with the norm

$$\| x \|_\infty := \sup_{t\in\mathbb{R}} \| x(t) \|_2 = \sup_{t\in\mathbb{R}} (E \| x(t) \|^2)^{\frac{1}{2}}$$

for $x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$.

Lemma 2.3([33]). Let $f \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$. Then we have
(I) $h(t) := f(-t) \in AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.
(II) $f_a(t) := f(t + a) \in AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

Definition 2.4([21]). A function $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H}), (t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P}, \mathbb{H})$ if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $\tilde{f} : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n\to\infty} E \| f(t + s_n, x) - \tilde{f}(t, x) \|^2 = 0 \text{ and } \lim_{n\to\infty} E \| \tilde{f}(t - s_n) - f(t, x) \|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in L^2(\mathbb{P}, \mathbb{H})$.

It is well known that the range $R_f = \{f(t) : t \in \mathbb{R}\}$ of an almost function f is relatively compact in $L^2(\mathbb{P}, \mathbb{H})$, thus bounded in norm (cf. [35], Theorem 1.31).

Definition 2.5. A continuous function $f : \mathbb{R}^+ \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean asymptotically almost automorphic if it can be written as $f = g + h$, where $g \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ and $h \in C_0(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$. Denote by $AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

Definition 2.6. A continuous function $f : \mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean asymptotically almost automorphic in t uniformly for x in compact subsets of $L^2(\mathbb{P}, \mathbb{H})$ if it can be written as $f = g + h$, where $g \in AA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ and $h \in C_0(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$. Denote by $AAA(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

Lemma 2.4([21]). Let $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H}), (t, x) \to f(t, x)$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P}, \mathbb{H})$, and assume that f satisfies a Lipschitz condition in the following sense:

$$E \| f(t, \phi) - f(t, \psi) \|^2 \leq \tilde{M} E \| \phi - \psi \|^2$$

for all $\phi, \psi \in L^2(\mathbb{P}, \mathbb{H})$ and for each $t \in \mathbb{R}$, where $\tilde{M} > 0$ is independent of t. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$, the stochastic process $F : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ given by $F(\cdot) = f(\cdot, x(\cdot))$ is square-mean almost automorphic.

Definition 2.7. A stochastically continuous stochastic process $x : [0, \infty) \to L^2(\mathbb{P}, \mathbb{H})$ is called a mild solution of the system (1.1)-(1.2) if $x(0) + g(x) = x_0$ and

$x(t)$ satisfies

$$x(t) = R(t)[x_0 - g(x) - a(0, x(\gamma_1(0)))] + a(t, x(\gamma_1(t)))$$
$$+ \int_0^t R(t-s)h(s, x(\gamma_2(s)))ds + \int_0^t R(t-s)f(s, x(\gamma_3(s)))dW(s), \quad t \geq 0.$$

3. Preliminary Results

Our main results on the existence of square-mean asymptotically almost automorphic solutions for the problem (1)-(2). For that, we need to introduce a few preliminary and important results.

Lemma 3.1. Assume $f \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$ admits a decomposition $f = g + h$, where $g \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})$ and $h \in C_0(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$. Then $\overline{\{g(t) : t \in \mathbb{R}\}} \subset \overline{\{f(t) : t \in \mathbb{R}^+\}}$.

One may refer to Lemma 1.7 in [17] for the proof of Lemma 3.1.

Lemma 2.2 and Lemma 3.1 together lead to the following Lemma. One can refer to Lemma 1.8 in [17] for a detailed proof.

Lemma 3.2. $AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$ is a Banach space with the norm

$$\| f \|_\infty = \sup_{t \in \mathbb{R}^+} \| f(t) \|_2 = \sup_{t \in \mathbb{R}^+} (E \| f(t) \|^2)^{\frac{1}{2}}.$$

Lemma 3.3. Let $f \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$. Then the range $\mathcal{R}_f = \{f(t) : t \in \mathbb{R}^+\}$ is relatively compact in $L^2(\mathbb{P}, \mathbb{H})$.

One may refer to Lemma 1.9 in [17] for the proof of Lemma 3.3.

Let $K \subset L^2(\mathbb{P}, \mathbb{H})$ and $\mathbb{Y} \subset \mathbb{R}$. We denote by $\mathcal{C}_K(\mathbb{Y} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ the set of all the functions $f : \mathbb{Y} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ satisfying $f(t, \cdot)$ is uniformly continuous on $L^2(\mathbb{P}, \mathbb{H})$ uniformly for $t \in \mathbb{Y}$.

Lemma 3.4. Let $x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$, $K = \{x(t) : t \in \mathbb{R}\}$ and $f \in AA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) \cap \mathcal{C}_K(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$. Then $f(\cdot, x(\cdot)) \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$.

One may refer to Lemma 2.2 in [17] for the proof of Lemma 3.4.

Lemma 3.5. Let $K \subset L^2(\mathbb{P}, \mathbb{H})$ be compact and $f \in AA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) \cap \mathcal{C}_K(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$. Then $f \in \mathcal{C}_K(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$.

The proof of Lemma 3.5 can be performed along the direction of the proof of Lemma 2.4 in [17].

Lemma 3.6. Let $x \in AAA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$, $K = \overline{\{x(t) : t \in \mathbb{R}^+\}}$ and

$$f \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) \cap \mathcal{C}_K(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})).$$

Then $f(\cdot, x(\cdot)) \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$.

Lemma 3.6 can be proved by using Lemmas 3.1,3.3 and Lemmas 3.4,3.5, and one may refer to Theorem 2.2 in [17] for more details about the proof of Lemma 3.6.

Lemma 3.7. Let $(R(t))_{t \geq 0}$ be a family of continuous linear operators on $L^2(\mathbb{P}, \mathbb{H})$ satisfying $\| R(t) \| \leq Me^{-\delta t}$ for all $t \geq 0$ and let $h, f \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$, and

$$H(t) = \int_0^t R(t-s)h(s)ds, \quad t \geq 0,$$
$$F(t) = \int_0^t R(t-s)f(s)dW(s), \quad t \geq 0.$$

Then $H(\cdot), F(\cdot) \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$.

Proof. Since H is a continuous function and $h \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$, there exist

$h_1 \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})), h_2 \in C_0(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$ such that h has a decomposition $h = h_1 + h_2$. We observe that

$$\begin{aligned} H(t) &= \int_0^t R(t-s)h_1(s)ds + \int_0^t R(t-s)h_2(s)ds \\ &= \int_{-\infty}^t R(t-s)h_1(s)ds - \int_{-\infty}^0 R(t-s)h_1(s)ds + \int_0^t R(t-s)h_2(s)ds. \end{aligned}$$

Let

$$H_1(t) = \int_{-\infty}^t R(t-s)h_1(s)dw(s),\ t \in \mathbb{R},$$

and

$$H_2(t) = -\int_{-\infty}^0 R(t-s)h_1(s)ds + \int_0^t R(t-s)h_2(s)ds,\ t \geq 0.$$

We claim that $H_1 \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$. In fact, for every real sequence $(s_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} E \parallel h_1(t+s_n) - \tilde{h}_1(t) \parallel^2 = 0$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} E \parallel \tilde{h}_1(t-s_n) - h_1(t) \parallel^2 = 0$$

for each $t \in \mathbb{R}$. Also, if we let

$$\widetilde{H}_1(t) = \int_{-\infty}^t R(t-s)\tilde{h}_1(s)ds.$$

We get

$$\begin{aligned} & E \parallel H_1(t+s_n) - \widetilde{H}_1(t) \parallel^2 \\ &= E\left\| \int_{-\infty}^{t+s_n} R(t+s_n-s)h_1(s)ds - \int_{-\infty}^t R(t-s)\tilde{h}_1(s)ds \right\|^2 \\ &= E\left\| \int_{-\infty}^{t} R(t-s)h_1(s+s_n)ds - \int_{-\infty}^t R(t-s)\tilde{h}_1(s)ds \right\|^2 \\ &\leq E\bigg(\int_{-\infty}^t \parallel R(t-s) \parallel \parallel h_1(s+s_n) - \tilde{h}_1(s) \parallel ds \bigg)^2 \\ &\leq M^2 E\bigg(\int_{-\infty}^t e^{-\delta(t-s)} \parallel h_1(s+s_n) - \tilde{h}_1(s) \parallel ds \bigg)^2 \\ &\leq M^2 \bigg(\int_{-\infty}^t e^{-\delta(t-s)} ds \bigg) \bigg(\int_{-\infty}^t e^{-\delta(t-s)} E \parallel h_1(s+s_n) - \tilde{h}_1(s) \parallel^2 ds \bigg) \\ &\leq M^2 \bigg(\int_{-\infty}^t e^{-\delta(t-s)} ds \bigg)^2 \sup_{s\in\mathbb{R}} E \parallel h_1(s+s_n) - \tilde{h}_1(s) \parallel^2 \\ &\leq \frac{M^2}{\delta^2} \sup_{s\in\mathbb{R}} E \parallel h_1(s+s_n) - \tilde{h}_1(s) \parallel^2 . \end{aligned}$$

Thus, we immediately obtain that

$$\lim_{n\to\infty} E \parallel H_1(t+s_n) - \widetilde{H}_1(t) \parallel^2 = 0.$$

Analogously to the above proof, we can obtain

$$\lim_{n\to\infty} E\Big\| \int_{-\infty}^{t-s_n} R(t-s_n-s)\tilde{h}_1(s)ds - H_1(t)\Big\|^2 = 0.$$

Next, let us show that $H_2 \in C_0(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$. Since $h_2 \in C_0(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$, $\varepsilon > 0$, there exists a constant $T > 0$ such that $E \parallel h_2(s) \parallel^2 \leq \varepsilon$ for all $s \geq T$. Then, by using Cauchy-Schwarz inequality, we deduce for all $t \geq 2T$,

$$\begin{aligned}
&E \parallel H_2(t) \parallel^2 \\
&= E\Big\| \int_0^{\frac{t}{2}} R(t-s)h_2(s)ds + \int_{\frac{t}{2}}^t R(t-s)h_2(s)ds - \int_{-\infty}^0 R(t-s)h_1(s)ds\Big\|^2 \\
&\leq 3M^2E\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-s)} \parallel h_2(s) \parallel ds\Big)^2 + 3M^2E\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-s)} \parallel h_2(s) \parallel ds\Big)^2 \\
&+ 3M^2E\Big(\int_{-\infty}^0 e^{-\delta(t-s)} \parallel h_1(s) \parallel ds\Big)^2 \\
&\leq 3M^2\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-s)}ds\Big)\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-s)}E \parallel h_2(s) \parallel^2 ds\Big) \\
&\quad + 3M^2\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-s)}ds\Big)\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-s)}E \parallel h_2(s) \parallel^2 ds\Big) \\
&\quad + 3M^2\Big(\int_{-\infty}^0 e^{-\delta(t-s)}ds\Big)\Big(\int_{-\infty}^0 e^{-\delta(t-s)}E \parallel h_1(s) \parallel^2 ds\Big) \\
&\leq 3M^2\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-s)}ds\Big)^2 \parallel h_2 \parallel_\infty^2 + 3M^2\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-s)}ds\Big)^2\varepsilon \\
&\quad + 3M^2\Big(\int_{-\infty}^0 e^{-\delta(t-s)}ds\Big)^2 \parallel h_1 \parallel_\infty^2 \\
&\leq \frac{3M^2}{\delta^2} \parallel h_2 \parallel_\infty^2 e^{-\delta t} + \frac{3M^2}{\delta^2}\varepsilon + \frac{3M^2}{\delta^2} \parallel h_1 \parallel_\infty^2 e^{-2\delta t}.
\end{aligned}$$

Therefore, $\lim_{t\to\infty} E \parallel H_2(t) \parallel^2 = 0$, that is, $H_2 \in C_0(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$. Recalling that $H(t) = H_1(t) + H_2(t)$ for all $t \geq 0$, we get $H \in AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$.

Similarly, F is a continuous function. Since $f \in AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$, f has a decomposition $f = f_1 + f_2$, where $f_1 \in AA(\mathbb{R}, L^2(\mathbb{P},\mathbb{H})), f_2 \in C_0(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$. We observe that

$$\begin{aligned}
F(t) &= \int_0^t R(t-s)f_1(s)dW(s) + \int_0^t R(t-s)f_2(s)dW(s) \\
&= \int_{-\infty}^t R(t-s)f_1(s)dW(s) - \int_{-\infty}^0 R(t-s)f_1(s)dW(s) + \int_0^t R(t-s)f_2(s)dW(s).
\end{aligned}$$

Let

$$F_1(t) = \int_{-\infty}^t R(t-s)f_1(s)dW(s),\ t \in \mathbb{R},$$

and

$$F_2(t) = -\int_{-\infty}^0 R(t-s)f_1(s)dW(s) + \int_0^t R(t-s)f_2(s)dW(s),\ t \geq 0.$$

We claim that $F_1 \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$. In fact, for every real sequence $(s_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} E \parallel f_1(t+s_n) - \tilde{f}_1(t) \parallel^2 = 0$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} E \parallel \tilde{f}_1(t-s_n) - f_1(t) \parallel^2 = 0$$

for each $t \in \mathbb{R}$. Let $\widetilde{W}(\sigma) := W(\sigma + s_n) - W(s_n)$ for each m$\sigma \in \mathbb{R}$. We know that $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. Moreover, if we let

$$\widetilde{F}_1(t) = \int_{-\infty}^{t} R(t-s)\tilde{f}_1(s)dW(s),$$

then by making a change of variables $\sigma = s - s_n$, We get

$$\begin{aligned}
&E \parallel F_1(t+s_n) - \widetilde{F}_1(t) \parallel^2 \\
&= E\left\| \int_{-\infty}^{t+s_n} R(t+s_n-s)f_1(s)ds - \int_{-\infty}^{t} R(t-s)\tilde{f}_1(s)ds \right\|^2 \\
&= E\left\| \int_{-\infty}^{t} R(t-\sigma)[f_1(\sigma+s_n) - \tilde{f}_1(\sigma)]d\widetilde{W}(\sigma) \right\|^2.
\end{aligned}$$

Thus, using an estimate on Ito integral established in Ichikawa [27], we obtain that

$$\begin{aligned}
&E \parallel F_1(t+s_n) - \widetilde{F}_1(t) \parallel^2 \\
&\le E\bigg(\int_{-\infty}^{t} \parallel R(t-\sigma)[f_1(\sigma+s_n) - \tilde{f}_1(\sigma)] \parallel^2 ds \bigg) \\
&\le M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} E \parallel f_1(s+s_n) - \tilde{f}_1(s) \parallel ds \\
&\le M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} ds \sup_{s\in\mathbb{R}} E \parallel f_1(s+s_n) - \tilde{f}_1(s) \parallel^2 \\
&\le \frac{M^2}{2\delta} \sup_{s\in\mathbb{R}} E \parallel f_1(s+s_n) - \tilde{f}_1(s) \parallel^2 .
\end{aligned}$$

Thus, we immediately obtain that

$$\lim_{n\to\infty} E \parallel F_1(t+s_n) - \widetilde{F}_1(t) \parallel^2 = 0.$$

Analogously to the above proof, we can obtain

$$\lim_{n\to\infty} E\left\| \int_{-\infty}^{t-s_n} R(t-s_n-s)\tilde{f}_1(s)ds - F_1(t) \right\|^2 = 0.$$

Next, let us show that $F_2 \in C_0(\mathbb{R}^+, , L^2(\mathbb{P}, \mathbb{H}))$. Since $f_2 \in C_0(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$, $\varepsilon > 0$, there exists a constant $T > 0$ such that $E \parallel f_2(s) \parallel^2 \le \varepsilon$ for all $s \ge T$. Then, by the Ito integral, we deduce for all $t \geq 2T$,

$$
\begin{aligned}
&E \parallel F_2(t) \parallel^2 \\
&= E\left\| \int_0^{\frac{t}{2}} R(t-s)f_2(s)dW(s) + \int_{\frac{t}{2}}^t R(t-s)f_2(s)dW(s) \right. \\
&\left. - \int_{-\infty}^0 R(t-s)f_1(s)dW(s) \right\|^2 \\
&\leq 3E\left(\int_0^{\frac{t}{2}} \parallel R(t-s)f_2(s) \parallel^2 ds \right) + 3E\left(\int_{\frac{t}{2}}^t \parallel R(t-s)f_2(s) \parallel^2 ds \right) \\
&\quad + 3E\left(\int_{-\infty}^0 \parallel R(t-s)f_1(s) \parallel^2 ds \right) \\
&\leq 3M^2 \int_0^{\frac{t}{2}} e^{-2\delta(t-s)} E \parallel f_2(s) \parallel^2 ds + 3M^2 \int_{\frac{t}{2}}^t e^{-2\delta(t-s)} E \parallel f_2(s) \parallel^2 ds \\
&\quad + 3M^2 \int_{-\infty}^0 e^{-2\delta(t-s)} E \parallel f_1(s) \parallel^2 ds \\
&\leq 3M^2 \left(\int_0^{\frac{t}{2}} e^{-2\delta(t-s)} ds \right) \parallel f_2 \parallel_\infty^2 + 3M^2 \left(\int_{\frac{t}{2}}^t e^{-2\delta(t-s)} ds \right) \varepsilon \\
&\quad + 3M^2 \left(\int_{-\infty}^0 e^{-2\delta(t-s)} ds \right) \parallel f_1 \parallel_\infty^2 \\
\leq\ & \frac{3M^2}{2\delta} \parallel f_2 \parallel_\infty^2 e^{-2\delta t} + \frac{3M^2}{2\delta}\varepsilon + \frac{3M^2}{2\delta} \parallel f_1 \parallel_\infty^2 e^{-2\delta t}.
\end{aligned}
$$

Therefore, $\lim_{t\to\infty} E \parallel F_2(t) \parallel^2 = 0$, that is, $F_2 \in C_0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$. Recalling that $F(t) = F_1(t) + F_2(t)$ for all $t \geq 0$, we get $F \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$.

4. Existence Results

In this section, we prove that there is a unique mild solution for the problem (1)-(2). For that, we make the following hypotheses:

(H1) There exists a resolvent operator $R(\cdot)$ of Eq. (1.1) and $R(\cdot)$ is exponentially stable, i.e., $\parallel R(t) \parallel \leq Me^{-\delta t}$ for all $t \geq 0$ and some constant $M, \delta > 0$.
(H2) The functions $a, h, f \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ and there exist continuous and nondecreasing functions $L_a, L_h, L_f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $l \geq 0$ and $E \parallel x \parallel^2, E \parallel y \parallel^2 \leq l$,

$$E \parallel a(t,x) - a(t,y) \parallel^2 \leq L_g(l) E \parallel x - y \parallel^2,$$

$$E \parallel h(t,x) - h(t,y) \parallel^2 \leq L_h(l) E \parallel x - y \parallel^2,$$

and

$$E \parallel f(t,x) - f(t,y) \parallel^2 \leq L_f(l) E \parallel x - y \parallel^2$$

for all $t \in \mathbb{R}^+$ and each $x, y \in L^2(\mathbb{P}, \mathbb{H})$.
(H3) The functions $\gamma_i \in C(\mathbb{R}^+, \mathbb{R}^+), i = 1, 2, 3$, and $g : C(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H})) \to L^2(\mathbb{P}, \mathbb{H})$ satisfies that there exists a continuous and nondecreasing function $L_g : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $l \geq 0$ and $\parallel x \parallel_\infty^2, \parallel y \parallel_\infty^2 \leq l$,

$$E \parallel g(x) - g(y) \parallel^2 \leq L_g(l) \parallel x - y \parallel_\infty^2, \quad \text{for } \; x, y \in C(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H})).$$

(H4) There exists a positive number l such that

$$l - 24M^2 L_g(l)l - 8(3M^2+1)L_a(l)l - \frac{8M^2}{\delta^2} L_h(l)l - \frac{4M^2}{\delta} L_f(l)l > \Lambda, \qquad (5)$$

where $\Lambda = 12M^2[E \parallel x_0 \parallel^2 + 2L_g^* + 2L_a^*] + 8L_a^* + \frac{8M^2}{\delta^2}L_h^* + \frac{4M^2}{\delta}L_f^*$, $L_g^* = E \parallel g(0) \parallel^2, L_a^* = \sup_{t\in\mathbb{R}^+} E \parallel a(t,0) \parallel^2, L_h^* = \sup_{t\in\mathbb{R}^+} E \parallel h(t,0) \parallel^2, L_f^* = \sup_{t\in\mathbb{R}^+} E \parallel f(t,0) \parallel^2$.

Theorem 4.1. If the assumptions (H1)-(H4) are satisfied, then the system (1)-(2) has a square-mean asymptotically almost automorphic mild solution.

Proof. Let $\mathcal{D} = \{x \in AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H})) : E \parallel x \parallel^2 \leq l\}$. Then $\mathcal{D}$ is a closed subspace of $AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$.

Define an operator Φ on $\mathcal{D}$ by

$$\begin{aligned}(\Phi x)(t) &= R(t)[x_0 - g(x) - a(0, x(\gamma_1(0)))] + a(t, x(\gamma_1(t))) \\ &\quad + \int_0^t R(t-s)h(s, x(\gamma_2(s)))ds \\ &\quad + \int_0^t R(t-s)f(s, x(\gamma_3(s)))dW(s), \quad t \geq 0.\end{aligned}$$

It follows that $s \to R(t-s)h(s, x(\gamma_2(s)))$ and $s \to R(t-s)f(s, x(\gamma_3(s)))$ are integrable on $[0,t)$ for every $t > 0$, therefore, Φx is well defined. First, let us check that $\Phi(AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))) \subset AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$. Take $x \in AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$. It is easy to prove that Φx is continuous. We define $K = \{x(t) : t \in \mathbb{R}^+\}$. It follows from (H2)-(H3) that $a, h, f \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H})) \cap \mathcal{C}_K(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H}))$. Then Lemma 3.6 gives that

$$h(\cdot, x(\cdot)), f(\cdot, x(\cdot)) \in AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H})).$$

Now, by Lemma 3.7, we have

$$H(t) = \int_0^t R(t-s)h(s, x(\gamma_2(s)))ds \in AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H})$$

and

$$F(t) = \int_0^t R(t-s)f(s, x(\gamma_3(s)))dw(s) \in AAA(\mathbb{R}^+, L^2(P, H)).$$

On the other hand, since $R(\cdot)$ is exponentially stable,

$$\lim_{t\to\infty} E \parallel R(t)[x_0 - g(x) - a(0, x(\gamma_1(0)))] \parallel^2 = 0.$$

Thus, $\Phi x \in AAA(\mathbb{R}^+, L^2(\mathbb{P},\mathbb{H}))$.

Next, we prove that $\Phi(\cdot)$ is a contraction from $\mathcal{D}$ into $\mathcal{D}$. Note that $(\Phi x)(0) = x(0)$. Moreover, if $x \in \mathcal{D}$ and $t \geq 0$, we then get

$$\begin{aligned}E \parallel (\Phi x)(t) \parallel^2 &\leq 4E \parallel R(t)[x_0 - g(x) - a(0, \gamma_1(0))] \parallel^2 + 4E \parallel a(t, \gamma_1(t)) \parallel^2 \\ &\quad + 4E \left\| \int_0^t R(t-s)h(s, \gamma_2(s))ds \right\|^2 \\ &\quad + 4E \left\| \int_0^t R(t-s)f(s, \gamma_3(s))dW(s) \right\|^2.\end{aligned}$$

By using the Cauchy-Schwarz inequality, we first evaluate the first three term of the right-hand side

$$
\begin{aligned}
&4E \parallel R(t)[x_0 - g(x) - a(0, x(\gamma_1(0)))] \parallel^2 + 4E \parallel a(t, x(\gamma_1(t))) \parallel^2 \\
&\quad + 4E \left\| \int_0^t R(t-s)h(s, x(\gamma_2(s)))ds \right\|^2 \\
&\quad \le 12M^2[E \parallel x_0 \parallel^2 + 2(E \parallel g(x) - g(0) \parallel^2 + E \parallel g(0) \parallel^2) \\
&\qquad + E \parallel a(0, x(\gamma_1(0))) \parallel^2] + 8[E \parallel a(t, x(\gamma_1(t))) - a(t,0) \parallel^2 + E \parallel a(t,0) \parallel^2] \\
&\qquad + 4M^2 E\left(\int_0^t e^{-\delta(t-s)} \parallel h(s, x(\gamma_2(t))) \parallel ds \right)^2 \\
&\quad \le 12M^2[E \parallel x_0 \parallel^2 + 2(L_g(l) \parallel x \parallel_\infty + L_g^*) + 2(L_a(l) E \parallel x(\gamma_1(0)) \parallel^2 + L_a^*)] \\
&\qquad + 8[L_a(l) E \parallel x(\gamma_1(t)) \parallel^2 + L_g^*] + 8M^2 \left(\int_0^t e^{-\delta(t-s)} ds \right) \\
&\qquad \times \left(\int_0^t e^{-\delta(t-s)} [E \parallel h(s, x(\gamma_2(s))) - h(s,0) \parallel^2 + E \parallel h(s,0) \parallel^2] ds \right) \\
&\quad \le 12M^2[E \parallel x_0 \parallel^2 + 2(L_g(l) l + L_g^*) + 2(L_a(l) l + L_a^*)] \\
&\qquad + 8[L_a(l) \sup_{t \in \mathbb{R}^+} E \parallel x(t) \parallel^2 + L_g^*] \\
&\qquad + 8M^2 \left(\int_0^t e^{-\delta(t-s)} ds \right) \left(\int_0^t e^{-\delta(t-s)} [L_h(l) E \parallel x(\gamma_2(s)) \parallel + L_h^*] ds \right) \\
&\quad \le 12M^2[E \parallel x_0 \parallel^2 + 2(L_g(l) l + L_g^*) + 2(L_a(l) l + L_a^*)] + 8[L_g(l) l + L_g^*] \\
&\qquad + 8M^2 \left(\int_0^t e^{-\delta(t-s)} ds \right) \left(\int_0^t e^{-\delta(t-s)} [L_h(l) \sup_{s \in \mathbb{R}^+} E \parallel x(s) \parallel^2 + L_h^*] ds \right) \\
&\quad \le 12M^2[E \parallel x_0 \parallel^2 + 2(L_g(l) l + L_g^*) + 2(L_a(l) l + L_a^*)] + 8[L_g(l) l + L_g^*] \\
&\qquad + 8M^2 \left(\int_0^t e^{-\delta(t-s)} ds \right)^2 [L_h(l) l + L_h^*] \\
&\quad \le 12M^2[E \parallel x_0 \parallel^2 + 2(L_g(l) l + L_g^*) + 2(L_a(l) l + L_a^*)] + 8[L_g(l) l + L_g^*] \\
&\qquad + \frac{8M^2}{\delta^2} [L_h(l) l + L_h^*]
\end{aligned}
$$

for all $t \ge 0$.

As to the four term, by the Ito integral, we get

$$
\begin{aligned}
&4E \left\| \int_0^t R(t-s) f(s, x(\gamma_3(s))) dW(s) \right\|^2 \\
&\quad \le 4E \left(\int_0^t \parallel R(t-s) f(s, x(\gamma_3(s))) \parallel^2 ds \right) \\
&\quad \le 8M^2 \int_0^t e^{-2\delta(t-s)} [E \parallel f(s, x(\gamma_3(s))) - f(s,0) \parallel^2 + E \parallel f(s,0) \parallel^2] ds \\
&\quad \le 8M^2 \int_0^t e^{-2\delta(t-s)} [L_f(l) E \parallel x(\gamma_3(s)) \parallel + L_f^*] ds
\end{aligned}
$$

$$\le 8M^2 \int_0^t e^{-2\delta(t-s)}[L_f(l) \sup_{s\in\mathbb{R}^+} E \parallel x(s) \parallel^2 + L_f^*]ds$$
$$\le 8M^2 \int_0^t e^{-2\delta(t-s)}ds[L_f(l)l + L_f^*] \le \frac{4M^2}{\delta}[L_f(l)l + L_f^*]$$

for all $t \ge 0$. Thus, by combining the above inequality together, we obtain that,

$$E \parallel (\Phi x)(t) \parallel^2 \le 12M^2[E \parallel x_0 \parallel^2 + 2(L_g(l)l + L_g^*) + 2(L_a(l)l + L_a^*)] + 8[L_a(l)l + L_a^*] + \frac{8M^2}{\delta^2}[L_h(l)l + L_h^*] + \frac{4M^2}{\delta}[L_f(l)l + L_f^*] < l$$

for each $t \ge 0$, where (4.1) was used in the last inequality. Thus, $E \parallel \Phi x(t) \parallel^2 \le l$.

For $x, y \in \mathcal{D}$ and $t \ge 0$, we have

$$\begin{aligned}
&E \parallel (\Phi x)(t) - (\Phi y)(t) \parallel^2 \\
&\le 4E \parallel R(t)[g(x) - g(y) - a(0, x(\gamma_1(0))) + a(0, y(\gamma_1(0)))] \parallel^2 \\
&+ 4E \parallel a(t, x(\gamma_1(t))) - a(t, y(\gamma_1(t))) \parallel^2 \\
&+ 4E \left\| \int_0^t R(t-s)[h(s, x(\gamma_2(s))) - h(s, y(\gamma_2(s)))]ds \right\|^2 \\
&+ 4E \left\| \int_0^t R(t-s)[f(s, x(\gamma_3(s))) - f(s, y(\gamma_3(s)))]dW(s) \right\|^2.
\end{aligned}$$

By using the Cauchy-Schwarz inequality, we first evaluate the first three term of the right-hand side

$$\begin{aligned}
&4E \parallel R(t)[g(x) - g(y) - a(0, x(\gamma_1(0))) + a(0, y(\gamma_1(0)))] \parallel^2 \\
&\quad + 4E \parallel a(t, x(\gamma_1(t))) - a(t, y(\gamma_1(t))) \parallel^2 \\
&\quad + 4E \left\| \int_0^t R(t-s)[h(s, x(\gamma_2(s))) - h(s, y(\gamma_2(s)))]ds \right\|^2 \\
&\quad \le 8M^2[E \parallel g(x) - g(y) \parallel^2 + E \parallel a(0, x(\gamma_1(0))) - a(0, y(\gamma_1(0))) \parallel^2] \\
&\qquad + 4L_a(l)E \parallel x(\gamma_1(t)) - y(\gamma_1(t)) \parallel^2 \\
&\qquad + 4M^2 E\left(\int_0^t e^{-\delta(t-s)} \parallel h(s, x(\gamma_2(s))) - h(s, y(\gamma_2(s))) \parallel ds \right)^2 \\
&\quad \le 8M^2[L_g(l) \parallel x - y \parallel_\infty^2 + L_a(l) \sup_{s\in\mathbb{R}^+} E \parallel x(s) - y(s) \parallel^2] \\
&\qquad + 4L_a(l) \sup_{s\in\mathbb{R}^+} E \parallel x(s) - y(s) \parallel^2 + 4M^2 \left(\int_0^t e^{-\delta(t-s)}ds \right) \\
&\qquad \times \left(\int_0^t e^{-\delta(t-s)} E \parallel h(s, x(\gamma_2(s))) - h(s, y(\gamma_2(s))) \parallel^2 ds \right) \\
&\quad \le 8M^2[L_g(l)E \parallel x - y \parallel_\infty^2 + L_a(l)E \parallel x - y \parallel_\infty^2] + 4L_a(l) \parallel x - y \parallel_\infty^2 \\
&\qquad + 4M^2 L_h(l) \left(\int_0^t e^{-\delta(t-s)}ds \right) \left(\int_0^t e^{-\delta(t-s)} E \parallel x(\gamma_2(s)) - y(\gamma_2(s)) \parallel^2 ds \right) \\
&\quad \le [8M^2 L_g(l) + 4(2M^2 + 1)L_a(l) \parallel x - y \parallel_\infty^2
\end{aligned}$$

$$+ 4M^2L_h(l)\bigg(\int_0^t e^{-\delta(t-s)}ds\bigg)^2 \sup_{s\in\mathbb{R}^+} E \parallel x(s) - y(s) \parallel^2$$
$$\leq \bigg[8M^2L_g(l) + 4(2M^2+1)L_a(l) + \frac{4M^2}{\delta^2}L_h(l)\bigg] \parallel x - y \parallel_\infty^2 .$$

As to the four term, by the Ito integral, we get

$$\begin{aligned}
&4E\bigg\| \int_0^t R(t-s)[f(s,x(\gamma_3(s))) - f(s,y(\gamma_3(s)))]dW(s)\bigg\|^2 \\
&\leq 4M^2\int_0^t e^{-2\delta(t-s)}E \parallel f(s,x(\gamma_3(s))) - f(s,y(\gamma_3(s))) \parallel^2 ds \\
&\leq 4M^2L_f(l)\int_0^t e^{-2\delta(t-s)}E \parallel x(\gamma_3(s)) - y(\gamma_3(s)) \parallel^2 ds \\
&\leq 4M^2L_f(l)\int_0^t e^{-2\delta(t-s)}ds \sup_{s\in\mathbb{R}^+} E \parallel x(s) - y(s) \parallel^2 \\
&\leq \frac{2M^2}{\delta}L_f(l) \parallel x - y \parallel_\infty^2 .
\end{aligned}$$

Thus, by combining the above inequality together, we obtain that, for each $t \geq 0$,

$$\begin{aligned}
E \parallel (\Phi x)(t) - (\Phi y)(t) \parallel^2 \leq \bigg[&8M^2L_g(l) + 4(2M^2+1)L_a(l) \\
&+ \frac{4M^2}{\delta^2}L_h(l) + \frac{2M^2}{\delta}L_f(l)\bigg] \parallel x - y \parallel_\infty^2 .
\end{aligned}$$

Taking supremum over t,

$$\parallel \Phi x - \Phi y \parallel_\infty^2 \leq L_0 \parallel x - y \parallel_\infty^2,$$

where $L_0 = [8M^2L_g(l) + 4(2M^2+1)L_a(l) + \frac{4M^2}{\delta^2}L_h(l) + \frac{2M^2}{\delta}L_f(l)]$. From (4.1), we know that

$$l - 24M^2L_g(l)l - 8(3M^2+1)L_a(l)l - \frac{8M^2}{\delta^2}L_h(l)l - \frac{4M^2}{\delta}L_f(l)l > 0,$$

Therefore,

$$24M^2L_g(l) + 8(3M^2+1)L_a(l) + \frac{8M^2}{\delta^2}L_h(l) + \frac{4M^2}{\delta}L_f(l) < 1,$$

which implies that $L_0 < 1$. Thus Φ is a contraction from $\mathcal{D}$ to $\mathcal{D}$. So Φ has a unique fixed point in $\mathcal{D}$, which means there exists a square-mean asymptotically almost automorphic mild solution to the problem (1)-(2).

5. Application

Consider the following neutral stochastic partial functional integrodifferential equations of the form

$$\begin{aligned}
dN(t,u(t))(\xi) = &\frac{\partial^2 N(t,u(t))(\xi)}{\partial x^2}dt \\
&+ \int_0^t (t-s)^\eta e^{-\gamma(t-s)}\frac{\partial^2 N(t,u(t))(\xi)}{\partial x^2}dsdt \\
&+\vartheta(t,u(\lambda t+\lambda_0,\xi))dt + \theta(t,u(\lambda t+\lambda_0,\xi))dW(t), \qquad (6)
\end{aligned}$$

$$u(t,0) = u(t,\pi) = 0, \tag{7}$$

$$u(0,\xi) + \int_0^\pi k(\xi,\tau)u(0,\tau)d\tau = u_0(\xi), \tag{8}$$

for $(t,\xi) \in [0,\infty) \times [0,\pi]$, where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. In this system, the constants $\lambda > 0, \lambda_0 \geq 0$, and $k : [0,\pi] \times [0,\pi] \to \mathbb{R}$ is continuous functions. The η and γ are positive numbers and

$$N(t,u(t))(\xi) = u(t,\xi) + \mu(t, u(\lambda t + \lambda_0, \xi)).$$

Let $\mathbb{H} = L^2([0,\pi])$ with the norm $\| \cdot \|$ and define the operators $A : \mathbb{H} \to \mathbb{H}$ by $A\omega = \omega''$ with the domain

$$D(A) := \{\omega \in \mathbb{H} : \omega, \omega'' \text{ are absolutely continuous}, \omega'' \in \mathbb{H}, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 \langle \omega, \omega_n \rangle \omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of A. It is well known that A generates a strongly continuous semigroup that is analytic, and resolvent operator $R(t)$ can be extracted this analytic semigroup and given by

$$R(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \omega \in \mathbb{H}$$

with $\| R(t) \| \leq Me^{-\delta t}$ for all $t \geq 0$ and constants $M, \delta > 0$. We also consider the operator $B(t) : D(A) \subseteq \mathbb{H} \to \mathbb{H}, t \geq 0, B(t)\omega = t^\eta e^{-\gamma t}\omega''$ for $\omega \in D(A)$.

For $t \in \mathbb{R}^+, \varphi \in L^2(\mathbb{P}, \mathbb{H})$, we define respectively

$$a(t,\varphi)(\xi) = \mu(t, \varphi(\xi)), \quad N(t,\varphi)(\xi) = u(0,\xi) + \mu(t,\varphi(\xi)),$$
$$h(t,\varphi)(\xi) = \vartheta(t,\varphi(\xi)), \quad f(t,\varphi)(\xi) = \theta(t,\varphi(\xi)),$$

and

$$g(x)(\xi) = \int_0^\pi k(\xi,\tau)x(\tau)d\tau, \quad x \in C(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H})).$$

Let $\gamma_1(t) = \gamma_2(t) = \gamma_3(t) = \lambda t + \lambda_0$, and the above equation can be written in the abstract form as the system (1)-(2). Moreover, $a, h, f \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 4.1, we can conclude that the problem (6)-(8) has a unique asymptotically almost automorphic mild solution $u \in AAA(\mathbb{R}^+, L^2(\mathbb{P}, \mathbb{H}))$.

In particular, we can take

$$\mu(t, u(t,\xi)) = u(t,\xi) \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t},$$
$$\vartheta(t, u(t,\xi)) = u(t,\xi) \sin \frac{1}{2 + \cos t + \cos \sqrt{3}t},$$
$$\theta(t, u(t,\xi)) = u(t,\xi) \sin \frac{1}{2 + \cos t + \cos \sqrt{5}t}.$$

It follows that

$$a(t,\varphi) = \varphi \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})),$$

$$h(t,\varphi) = \varphi \sin \frac{1}{2+\cos t + \cos \sqrt{3}t} \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H})),$$

$$f(t,\varphi) = \varphi \sin \frac{1}{2+\cos t + \cos \sqrt{5}t} \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H})).$$

Clearly, the a, h and f set above satisfy the Lipschitz conditions, respectively.

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Zuomao Yan
Department of Mathematics, Hexi University, Zhangye, Gansu 734000, P.R. China
E-mail address: yanzuomao@163.com

Hong-Wu Zhang
Department of Mathematics, Hexi University, Zhangye, Gansu 734000, P.R. China
E-mail address: zhanghw2007@lzu.cn