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EXISTENCE OF PERIODIC SOLUTIONS FOR A $2n$ TH-ORDER NEUTRAL NONLINEAR DIFFERENCE EQUATION

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Abstract. In this paper, applying Linking theorem and critical point theory, we consider a class of $2n$th-order neutral difference equations. By establishing the variational framework of this class equation and transferring the existence of periodic solutions of it into the existence of critical points of some functional, the existence and multiplicity of periodic solutions of such equations were obtained based on some new criteria.

1. Introduction

As is well known, the theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. The difference equations theory rapid development as the need of theoretics and actuality in the last decade. There have been many scholars contribute to the research of difference equations, most literature are about the qualitative properties of difference equations, such as stability, attraction, oscillation and so on.

Periodic phenomena widely exist in the nature and the people's social practice, so many scholars invest much efforts in the research of periodicity of differential equations in the last decade. Although the development of the study of periodic solutions of differential equations is relatively rapid, we can refer to refs [1–3] and the references therein, the paper for the difference equations is not sufficient [4–14]. Among many of the branches of difference equations, the development of periodicity and boundary value problems are relatively slowness, and the main reason is lacking in the powerful technical and methods to deal with the existence of periodic solutions for discrete system.

Many experts and scholars such as J. S. Yu, Z. M. Guo, A. Peterson, L. H. Erbe, C. D. Ahlbrandt, M. Bohner, P. W. Eloe, D. Reid, F. M. Atici, F. M. Atici, G. Sh.

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Guseinov, Y. Raffoul, et al. bend themselves to improve and perfect the theory of difference equations, and made the fixed point theory, Kaplan–Yorke method, critical point theory, coincidence degree theory, bifurcation theory be the useful tools to study the difference equations. A series of existence results for periodic solutions have been obtained by many researcher in there papers. Among these approaches, the critical point theory seems to be a very powerful tool to deal with these problems.

In [4], Guo and Yu consider the existence of periodic and subharmonic solutions for second-order superlinear difference equations

$$\Delta^2 x_{n-1} + f(n, x_n) = 0,$$

by the critical point theory for the first time, where $f \in C(R \times R, R)$. Some new results are obtained for the above problems when $f(t, z)$ has superlinear growth at zero and at infinity in z. At the same time, Zhou, Yu and Guo in [6] have been applying critical point theory to investigate the existence of periodic and subharmonic solutions for system

$$\Delta^2 X_{n-1} + f(n, X_n) = 0,$$

where $f = (f_1, f_2, \cdots, f_m) \in C(R \times R^m, R^m)$. The existence of periodic and subharmonic solutions have been discussed when the above system is suplinear, which extent the result of reference [4]. These papers show that the critical point method is an effective approach to the study of periodic solutions of second-order difference equations.

Compared to one-order or second-order difference equations, the study of higher-order equations has received considerably less attention. In 1994, Ahlbrandt and Peterson [7] studied the $2n$th-order difference equations of the form

$$\sum_{i=0}^{n} \Delta^i (r_i(t-i) \Delta^i y(t-i)) = 0$$

in the context of the discrete calculus of variations.

The paper [14] consider the $2n$ th-order difference equations

$$\Delta^n (r_{t-n} \Delta^n x_{t-n}) + f(t, x_t) = 0$$

by using critical point theory. By establishing the variational framework of the above equations and transferring the existence of periodic solutions of it into the existence of critical points of some functional, the authors obtain some sufficient conditions for the existence of periodic solutions of above equations.

But there are some errors in [14], all the results are righter when n as a even number, but when n as an odd number all the results can't obtained.

The paper is organized as following: In Section 2, we introduce some basic notations. In Section 3, we establish the variational framework of (1.1) and transfer the existence of periodic solutions of (1.1) into the existence of critical points of some functional, obtain some sufficient conditions for the existence of periodic solutions of (1.1). In the case when $a_t \equiv 0$ our results reduce to existence of periodic solutions in [14]. In Section 4, we give an examples to illustrate Theorem 1.1.

In this paper, we denote by $N,\ Z,\ R$ the sets of all natural numbers, integers and real numbers, respectively. For $a, b \in Z$, define $Z(a) = \{a, a+1, \cdots\}$, when $a \leq b$.

Consider the nonlinear neutral $2n$ th-order difference equation

$$a_t[\Delta^n(r_{t-n}\Delta^n(x_{t-n+1}+a_{t-n}x_{t-n}))] \\ +\Delta^n(r_{t-n-1}\Delta^n(x_{t-n}+a_{t-n-1}x_{t-n-1}))+(-1)^n f(t,x_t)=0, \quad (1.1)$$

where $n \in Z(1)$, $t \in Z$, Δ is the forward difference operator defined by $\Delta x_t = x_{t+1}-x_t$, $\Delta^2 x_t = \Delta(\Delta x_t)$ and the real sequence r_t, a_t and the function f satisfy the following conditions:
(a) $r_{t+T}=r_t>0$, $a_{t+T}=a_t>0$, for a given positive integer T and for all $t\in Z$.
(b) $f: Z\times R\to R$ is a continuous function in the second variable and $f(t+T,z)=f(t,z)$ for all $(t,z)\in Z\times R$.

Let X be a real Hilbert space, $J\in C^1(X,R)$, which means that J is a continuously Fréchet differentiable functional defined on X. J is said to satisfy the Palais-Smale condition (P-S condition for short) if any sequence $\{u_t\}\subset X$ for which $\{J(u_t)\}$ is bounded and $J'(u_t)\to 0$ as $t\to\infty$, possesses a convergent subsequence in X.

Let B_ρ be the open ball in X with radius ρ, centered at 0 and let ∂B_ρ denote its boundary. Denote $v_1=\min\limits_{t\in Z(1,T)} r_t$, $v_2=\max\limits_{t\in Z(1,T)} r_t$, $Q_1=\min\limits_{t\in Z(1,T)} a_t$, $Q_1=\max\limits_{t\in Z(1,T)} a_t$. Clearly, $v_i>0$, $Q_i>0$, for $i=1,2$.

Lemma 1.1 **(Linking Theorem)([4]).** Let X be a real Hilbert space, $X=X_1\oplus X_2$, where X_1 is a finite-dimensional subspace of X. Assume that $J\in C^1(X,R)$ satisfies the P-S condition and
(C_1) there exist constants $\sigma>0$ and $\rho>0$ such that $J|_{\partial B_\rho\cap X_2}\geq\sigma$.
(C_2) there is $e\in\partial B_\rho\cap X_2$ and a constant $R_1>\rho$ such that $J|_{\partial Q}\leq 0$, where $Q=(\overline{B}_{R_1}\cap X_1)\oplus\{re|0<r<R_1\}$.
Then J possesses a critical value $c\geq\sigma$, where $c=\inf\limits_{h\in\Gamma}\max\limits_{u\in Q}J(h(u))$, $\Gamma=\{h\in C(\overline{Q},X)|_{\partial Q}=id\}$ and id denotes the identity operator.

Now we state the main results.

Theorem 1.1 Assume that the following conditions are satisfied:
(A_1) For all $z\in R$, one has $\int_0^z f(t,s)ds\leq 0$ and $\lim_{z\to 0}\frac{f(t,z)}{z}=0$.
(A_2) There exist $R_2>0$ and $\beta>2$ such that, for every z with $|z|\geq R_2$ one has

$$zf(t,z)\leq\beta\int_0^z f(t,s)ds<0.$$

(A_3) For all a_t, $t\in Z(1,T)$ one has $1+Q_1-2Q_2^{\frac{1}{2}}>0$.
Then Eq. (1.1) has at least two nontrivial T-periodic solutions.

If $f(n,x_n)\equiv q_n g(x_n)+e_n$, Eq.(1.1) reduces to the following $2n$ th-order nonlinear equation

$$a_t[\Delta^n(r_{t-n}\Delta^n(x_{t-n+1}+a_{t-n}x_{t-n}))]+\Delta^n(r_{t-n-1}\Delta^n(x_{t-n}+a_{t-n-1}x_{t-n-1})), \quad (1.2)$$

where $g\in C(R,R)$, $q_{t+T}=q_t>0$, $e_{t+T}=e_t>0$ for all $t\in Z$. Then we have the following result.

Corollary 1.1 Assume that the following conditions are satisfied:
(A_4) For all a_t, $t\in Z(1,T)$ one has $1+Q_1-2Q_2^{\frac{1}{2}}>0$.
(A_5) For all $z\in R$ and $t\in Z$, one has $\int_0^z g(s)ds+zp_t\leq 0$ and $\lim_{z\to 0}\frac{g(z)+p_t}{z}=0$, where $\frac{e_t}{q_t}=p_t$.

(A_6) There exist $R_3 > 0$ and $\beta > 2$ such that, for every z with $|z| \geq R_3$, one has

$$zg(z) + p_t \leq \beta[\int_0^z g(s)ds + zp_t] < 0.$$

Then for a given positive integer T, Eq. (1.2) exists at least two nontrivial T-periodic solutions.

2. Preliminaries

The Linking Theorem is crucial for our paper, first, let us introduce some basic notations which will be used in this paper.

Let S be the set of sequences $x = (\cdots, x_{-t}, \cdots, x_{-1}, x_0, x_1, \cdots, x_t, \cdots) = \{x_t\}_{t=-\infty}^{+\infty}$, i.e., $S = \{x = \{x_t\} : x_t \in R, t \in Z\}$. For any $x, y \in S,\ a, b \in R,\ ax + by$ is defined by $ax + by := \{ax_t + by_t\}_{t=-\infty}^{+\infty}$, and then S is a vector space.

For a given positive integer T, E_T is defined as a subspace of S by $E_T = \{x = \{x_t\} \in S : x_{t+T} = x_t,\ t \in Z\}$. E_T can be endowed with the inner product $< x, y >= \sum_{i=1}^{T} x_i y_i,\ \forall x, y \in E_T$. By and the norm $\|x\| := (\sum_{i=1}^{T} x_i^2)^{\frac{1}{2}},\ \forall x \in E_T$. Clearly, E_T with the inner product is a finite-dimensional Hilbert space and linearly homeomorphic to R^T.

Defined the functional J on E^T as follows

$$J(x) = \frac{1}{2}\sum_{t=1}^{T} r_{t-1}(\triangle^n(x_t + a_{t-1}x_{t-1}))^2 - \sum_{t=1}^{T} F(t, x_t),\ \forall x \in E_T,$$

where $F(t, z) = -\int_0^z f(t, s)ds$. Clearly, $J \in C^1(E_T, R)$ and for any $x = \{x_t\}_{t \in Z} \in E_T$, by using $x_i + T = x_i$ for any $i = Z$, and $\triangle^n x_{t-1} = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} x_{t+n-k-1}$, for every $t \in Z(1, T)$, we can compute the partial derivative as

$$\frac{\partial J}{\partial x_t} = (-1)^n[\Delta^n[r_{t-n-1}\Delta^n(x_{t-n} + a_{t-n-1}x_{t-n-1})]$$
$$+ a_t\Delta^n[r_{t-n}\Delta^n(x_{t-n+1} + a_{t-n}x_{t-n})]] + f(t, x_t).$$

Then, $x = \{x_t\}_{t \in Z}$ is a critical point of J on E_T if and only if

$$a_t[\Delta^n(r_{t-n}\Delta^n(x_{t-n+1} + a_{t-n}x_{t-n}))]$$
$$+ \Delta^n(r_{t-n-1}\Delta^n(x_{t-n} + a_{t-n-1}x_{t-n-1})) + (-1)^n f(t, x_t) = 0.$$

Consider the periodicity of x_t and $f(t, z)$ in the first variable t, we can translate the existence of periodic solutions of Eq.(1.1) into the existence of critical points of J on E_T. That is, the functional J is just the variational framework of (1.1). For convenience, we identify $x \in E_T$ with $x = (x_1, x_2, \cdots, x_T)^T$.

Denote $W = \{(x_1, x_2, \cdots, x_T)^T \in E_T : \Delta^{n-1}x_i \equiv 0, \Delta^{n-1}a_i x_i \equiv 0,\ i \in Z(1, T)\}$ then there exists $W^{\perp} = Y$ such that $E_T = Y \oplus W$. Defined the norm $\|\cdot\|_\beta$ on E_T as follows: $\|x\|_\beta = (\sum_{i=1}^{T} |x_i|^\beta)^{\frac{1}{\beta}}$, for all $x \in E_T$ and $\beta > 1$. Clearly, $\|x\|_2 = \|x\|$, Since $\|\cdot\|_\beta$ and $\|\cdot\|$ are equivalent, there exist constants $C_1,\ C_2$ such that $C_2 \geq C_1 > 0$, and $C_1\|x\| \leq \|x\|_\beta \leq C_2\|x\|,\ \ \forall x \in E_T$.

3. Proofs of the main results

In this section, combining Lemma 1.1, we shall give the proof of our main result stated in section 1. First we shall introduce two lemmas which are useful in the proof of theorem.

Lemma 3.1 Assume that $f(t,s)$ satisfies condition (A_2) of Theorem 1.1, then the functional

$$J(x)=\frac{1}{2}\sum_{t=1}^{T}r_{t-1}(\Delta^n(x_t+a_{t-1}x_{t-1}))^2-\sum_{i=1}^{T}F(t,x_t)$$

is bounded from above on E_T.

Proof. Combining the reference [6] and condition (A_2), there exist two positive constants M and N such that, for all $z\in R$, $-\int_0^z f(t,s)ds\geq M|z|^\beta-N$. For every $x\in E_T$, let $z_{t-1}=x_t+a_{t-1}x_{t-1}$. We have

$$\|z\|^2=\sum_{t=1}^{T}|z_t|^2=\sum_{t=1}^{T}|x_{t+1}+a_tx_t|^2\leq\sum_{t=1}^{T}|x_{t+1}|^2+\sum_{t=1}^{T}|a_tx_t|^2+2\sum_{t=1}^{T}|a_tx_tx_{t+1}|$$

$$\leq\|x\|^2+Q_2\|x\|^2+\sum_{t=1}^{T}|x_{t+1}|^2+\sum_{t=1}^{T}|a_tx_t|^2=2(1+Q_2)\|x\|^2.$$

Then we have

$$\begin{aligned}J(x)\leq\ &\tfrac{v_2}{2}\sum_{t=1}^{T}(\Delta^{n-1}z_t-\Delta^{n-1}z_{t-1})^2-M\sum_{t=1}^{T}|x_t|^\beta+NT\\&\leq\tfrac{v_2}{2}\sum_{t=1}^{T}2[(\Delta^{n-1}z_t)^2+(\Delta^{n-1}z_{t-1})^2]-M\sum_{t=1}^{T}|x_t|^\beta+NT\\&=2v_2\sum_{t=1}^{T}(\Delta^{n-1}z_t)^2-M\sum_{t=1}^{T}|x|^\beta+NT\leq 8v_2\sum_{t=1}^{T}(\Delta^{n-2}z_t)^2-M\|x\|_\beta^\beta+NT\\&\leq\tfrac{v_24^n}{2}\|z\|^2-M(C_1)^\beta\|x\|^\beta+NT.\end{aligned}$$

Then obtain $J(x)\leq v_24^n(1+Q_2)\|x\|^2-M(C_1)^\beta\|x\|^\beta+NT$.

Since $\beta>2$ and the above inequality, there exists a constant M_1 such that, for every $x\in E_T,\ J(x)\leq M_1$. The proof is complete.

Lemma 3.2 Assume that $f(t,z)$ satisfies condition (A_2) of Theorem 1.1, then the functional $J(x)$ satisfies the P-S condition.

Proof. Let $x^{(k)}\in E_T,\ k\in Z(1)$, be such that$\{J(x^{(k)})\}$ is bounded. Then there exists M_2 such that, for every $k\in N,\ |J(x^{(k)})|\leq M_2$. From the proof of Lemma 3.1, we have for every $k\in N$,

$$-M_2\leq J(x^{(k)})\leq v_24^n(1+Q_2)\|x\|^2-M(C_1)^\beta\|x\|^\beta+NT.$$

That is

$$M(C_1)^\beta\|x\|^\beta-v_24^n(1+Q_2)\|x\|^2\leq M_2+NT,\quad\forall k\in N.$$

Thus, $\{x^{(k)}\}$ is bounded on E_T. Since E_T is finite-dimensional, there exists a subsequence of $\{x^{(k)}\}$, which is convergent in E_T, and the P-S condition satisfied.

Proof of Theorem 1.1 From (A_1), we know $f(t,0)=0$, then $\{x_t\}=0$, i.e. $x_t\equiv 0$ $(t\in Z)$ is a trivial $T-$periodic solution of Eq.(1.1). Combining Lemma 3.1, J is bounded from above. We denote C_0 as the supremum of $\{J(x),\ x\in E_T\}$. The proof of Lemma 3.2 implies $\lim_{\|x\|\to+\infty}J(x)=-\infty,\ -J$ is coercive. By continuity of J on E_T, there exists $\overline{x} \in E_T$ such that $J(\overline{x}) = C_0$, and $\overline{x}$ is a critical point of J. We claim that $C_0 > 0$.

In fact, by condition (A_1), we known $\lim_{z\to 0} \frac{F(t,z)}{z^2} = 0$. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for every z with $|z| \le \eta,\ F(t,z) \le \varepsilon z^2$.

For every $x = (x_1, x_2, \cdots, x_T)^T \in T$ with $\|x\| \le \eta,\ |x_t| \le \eta,\ t \in Z(1,T)$. When $T > 2$, we have

$$J(x) \ge \frac{v_1}{2} \sum_{t=1}^{T} (\Delta^{n-1} z_t - \Delta^{n-1} z_{t-1})^2 - \sum_{i=1}^{T} F(t, x_t) \ge \frac{v_1}{2} y^T A y - \varepsilon \sum_{i=1}^{T} |x_t|^2,$$

where $y = (\Delta^{n-1} z_1, \Delta^{n-1} z_2, \cdots, \Delta^{n-1} z_T)^T$, and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{T\times T}.$$

Clearly, 0 and $\xi = (v, v, \cdots, v)^T \in E_T$ is an eigenvalue of A and the eigenvector corresponding to 0, where $v \ne 0$ and $v \in R$. Let $\lambda_1, \lambda_2, \cdots, \lambda_{T-1}$ be the other eigenvalues of A. By matrix theory, we have $\lambda_j > 0,\ \forall j \in Z(1, T-1)$. Without loss of generality, we assume that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{T-1}$, then $J(x) \ge \frac{v_1}{2}\lambda_1 \|y\|^2 - \varepsilon\|x\|^2$. In view of $\|y\|^2 = \sum_{t=1}^{T} (\Delta^{n-2} z_{t+1} - \Delta^{n-2} z_t)^2 \ge \lambda_1 \sum_{t=1}^{T} (\Delta^{n-2} z_t)^2 \ge \lambda_1^{n-1} \|z\|^2$, Since

$$\|z\|^2 = \sum_{t=1}^{T} |z_t|^2 = \sum_{t=1}^{T} |x_{t+1} + a_t x_t|^2 \ge \sum_{t=1}^{T} |x_t|^2 + \sum_{t=1}^{T} |a_t x_t|^2 - 2\sum_{t=1}^{T} |a_t x_t x_{t+1}|,$$

and $\sum_{t=1}^{T} |x_{t+1}|^2 = \sum_{t=1}^{T} |x_t|^2$, we have $\sum_{t=1}^{T} |x_t x_{t+1}| \le \sum_{t=1}^{T} |x_t|^2$, then we obtain

$$-2\sum_{t=1}^{T} |a_t x_t x_{t+1}| \ge -2Q_2^{\frac{1}{2}} \|x\|^2,$$

so we get $\|z\| \ge (1 + Q_1 - 2Q_2^{\frac{1}{2}})\|x\|^2$,

$$J(x) \ge \frac{v_1}{2} \lambda_1^{n-1} [(1 + Q_1 - 2Q_2^{\frac{1}{2}})\|x\|^2] - \varepsilon\|x\|^2 = [\frac{v_1}{2} \lambda_1^{n-1} (1 + Q_1 - 2Q_2^{\frac{1}{2}}) - \varepsilon]\|x\|^2.$$

Take $\varepsilon = \frac{v_1}{4}\lambda_1^{n-1}(1 + Q_1 - 2Q_2^{\frac{1}{2}})$, and $\delta = \frac{v_1}{4}\lambda_1^{n-1}(1 + Q_1 - 2Q_2^{\frac{1}{2}})$. By (A_3), we have $\delta > 0$, then

$$J(x) \ge \delta > 0, \quad \forall Y \cap \partial B_\eta.$$

Thus, there exists $x \in E_T$ such that $J(x) \ge \delta > 0$, and $C_0 = \sup_{x\in E_T} J(x) \ge \delta > 0$, which implies that J satisfies condition (C_1) of the Linking Theorem, and the critical point of C_0 is a nontrivial T-periodic solution of Eq.(1.1). Now, we need to verify other conditions of the Linking Theorem. By Lemma 3.2, J satisfies the P-S condition (C_2). So it suffices to verify the condition. Take $e \in \partial B_1 \cap Y$. For any $w \in W$ and $r \in R$, let $x = re + w$, we obtain,

$$\begin{aligned} J(x) &\le \tfrac{v_2}{2}\sum_{t=1}^{T}(\Delta^n z_{t-1})^2 - \sum_{t=1}^{T} F(t, re_t + w_t) \\ &\le \tfrac{v_2}{2}\sum_{t=1}^{T}(\Delta^n(re_t + w_t + re_{t-1}a_{t-1} + w_{t-1}a_{t-1}))^2 - M\sum_{t=1}^{T}|re_t + w_t|^\beta + NT \\ &\le \tfrac{v_2}{2}\sum_{t=1}^{T}(r\Delta^n(e_t + e_{t-1}a_{t-1}) + \Delta^n(w_t + w_{t-1}a_{t-1}))^2 - M\sum_{t=1}^{T}|re_t + w_t|^\beta + NT \\ &= \tfrac{v_2}{2}\sum_{t=1}^{T} r^2(\Delta^n(e_t + e_{t-1}a_{t-1}))^2 - M\sum_{t=1}^{T}|re_t + w_t|^\beta + NT. \end{aligned}$$

Let $f_{t-1} = e_t + e_{t-1}a_{t-1}$, so we have

$$\begin{aligned} \sum_{t=1}^{T}(\Delta^n f_{t-1})^2 &= \sum_{t=1}^{T}(\Delta^{n-1} f_t - \Delta^{n-1} f_{t-1})^2 \le 4\sum_{t=1}^{T}(\Delta^{n-1} f_t)^2 \le 4^n \sum_{t=1}^{T} f_t^2 \\ &= 4^n \sum_{t=1}^{T}|e_t + e_{t-1}a_{t-1}|^2 \\ &\le 2\cdot 4^n\Big(\sum_{t=1}^{T}|e_t|^2 + \sum_{t=1}^{T}|e_{t-1}a_{t-1}|^2\Big) = 2\cdot 4^n(1 + Q_2). \end{aligned}$$

So $J(x) \le 4^n v_2 r^2(1 + Q_2) - M(C_1)^\beta r^\beta - M(C_1)^\beta \|w\|^2 + NT$.

Let $g_1(z) = 4^n v_2 z^2(1 + Q_2) - M(C_1)^\beta z^\beta, g_2(z) = -M(C_1)^\beta\|z\|^2 + NT$. We get $\lim_{z\to+\infty} g_1(z) = -\infty$, $\lim_{z\to+\infty} g_2(z) = -\infty$, and $g_1(z)$, $g_2(z)$ are bounded from above. Thus, there exists a constant $R_4 > \eta$ such that $J(x) \le 0,\ \forall x \in \partial Q$, where $Q = (\overline{B}_{R_4} \cap W) \oplus \{re|0 < r < R_4\}$.

By the Linking theorem, J possesses critical value $c \ge \delta > 0$, where $c = \inf\limits_{h\in\Gamma}\max\limits_{u\in Q} J(h(u)), \Gamma = \{h \in C(\overline{Q}, E_T) : h|_{\partial Q} = id\}$.

Let $\widehat{x} \in E_T$ be a critical point associated to the critical value c of J, i.e. $J(\overline{x}) = c$, If $\widehat{x} \ne \overline{x}$, then the proof is complete. If $\widehat{x} = \overline{x}$, then $C_0 = J(\overline{x}) = J(\widehat{x}) = c$, that is,

$$\sup_{x\in E_T} J(x) = \inf_{h\in\Gamma}\sup_{u\in Q} J(h(u)).$$

We choose $h = id$, then $\sup_{x\in Q} J(x) = C_0$. Since the choice of $e \in \partial B_1 \cap Y$ in Q is arbitrary, we can take $-e \in \partial B_1 \cap Y$. Similarly, for any $x \in \partial Q_1$, there exists $R_5 > \eta$ such that $J(x) \le 0$, where $Q = (\overline{B}_{R_5} \cap W) \oplus \{-re|0 < r < R_5\}$.

By Lemma 1.1, J possesses critical value $c' \ge \delta > 0$, where $c' = \inf\limits_{h\in\Gamma}\max\limits_{u\in Q} J(h(u))$, $\Gamma_1 = \{h \in C(\overline{Q}_1, E_T) : h|_{\partial Q} = id\}$.

If $c' \ne C_0$, the proof is complete; otherwise $c' = C_0$, then $C_0 = \sup_{x\in Q_1} J(x)$. Due to the fact that $J|_{\partial Q} \le 0$ and $J|_{\partial Q_1} \le 0$, J attains its maximum at some points in the interior of the sets Q and Q_1. On the other hand, $Q \cap Q_1 \subset W$ and for any $x \in W,\ J(x) \le 0$. This shows that there must be a point $x' \in E_T$ such that $\widehat{x} \ne x'$ and $J(x') = c' = C_0$.

The above argument implies that, if $c < C_0$, Eq. (1.1) possesses infinitely many nontrivial $T-$periodic solutions. Otherwise $c = C_0$, Eq.(1.1) possesses infinitely many nontrivial $T-$periodic solutions. The proof of Theorem 1.1 is completed.

4. Example

We have the following example to illustrate Theorem 1.1.

Example Assume that

$$f(t,z) = -(az|z| + bz|z|)(\varphi(t) + M),$$

where $a \geq 0,\ b > 0,\ M > 0,\ \varphi(t)$ is a continuous T-periodic function and $|\varphi(t)| \leq M$.

Consider the $2n$ th-order difference equations

$$3[\Delta^{2n}(x_{t-n+1} + 3x_{t-n})] + \Delta^{2n}(x_{t-n} + 3x_{t-n-1}) + (-1)^n f(t, x_t) = 0,\ t \in Z,$$

it is easy to verify that the conditions of Theorem 1.1 are satisfied, thus this equations possesses at least two nontrivial T-periodic solutions.

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