Electronic Journal of Mathematical Analysis and Applications
Vol. 1(1) Jan. 2013, pp. 40-46.
ISSN: 2090-792X (online)
http://ejmaa.6te.net/

SOME APPLICATIONS OF THE PRINCIPLE OF DIFFERENTIAL SUBORDINATION

KAZUO KUROKI, H. M. SRIVASTAVA, SHIGEYOSHI OWA

Abstract. By applying the principle of differential subordination between analytic functions to a substantially general family of linear transformations, we derive an extension of the Briot-Bouquet differential subordination relations given in the monograph on the subject by Miller and Mocanu [8]. We also present a systematic discussion of various other linear transformations associated with differential subordinations.

1. Introduction and Definitions

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For a positive integer n and a complex number a, let $\mathcal{H}[a, n]$ be the class of functions $f(z) \in \mathcal{H}$ of the following form:

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k$$

Also let $\mathcal{A}_n$ denote the class of functions $f(z) \in \mathcal{H}$ of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

and suppose that

$$\mathcal{A}_1 =: \mathcal{A}.$$

If $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1)$$

then the function $f(z)$ is said to be starlike of order α in $\mathbb{U}$. We denote this class of starlike functions of order α in $\mathbb{U}$ by $\mathcal{S}^*(\alpha)$. Similarly, we say that $f(z)$ belongs to

---

2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.

Key words and phrases. Principle of differential subordination; Analytic functions; Differential subordination; Briot-Bouquet differential equation; Univalent functions; Starlike functions; Convex functions; Linear transformations.

Submitted October 3, 2012. Published January 1, 2013.

the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1).$$

See, for details, the earlier works [3] and [15].

For real numbers A and B with $-1 \leqq B < A \leqq 1$, Janowski [4] investigated the following linear transformation:

$$p(z) = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

which is analytic and univalent in $\mathbb{U}$. This function $p(z)$ is usually called the Janowski function. Moreover, as a generalization of the Janowski function $p(z)$, Kuroki and Owa [6] discussed the Janowski functions for some complex parameters A and B which satisfy the following constraints:

$$A \neq B,\ |B| \leqq 1 \qquad \text{and} \qquad |A-B| + |A+B| \leqq 2. \tag{1.1}$$

We note that the Janowski function $p(z)$ defined under the conditions in (1.1) is analytic and univalent in $\mathbb{U}$ and has a positive real part in $\mathbb{U}$ (see [6]).

We next introduce the familiar principle of differential subordination between analytic functions. Let $p(z)$ and $q(z)$ be members of the class $\mathcal{H}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}), \tag{1.2}$$

if there exists a Schwarz function $w(z)$ which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \qquad (z \in \mathbb{U})$$

and is such that

$$p(z) = q\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

From the definition of differential subordination between analytic functions, it is easy to show that the subordination (1.2) implies that

$$p(0) = q(0) \qquad \text{and} \qquad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.3}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.2) is equivalent to the condition (1.3). For various applications of the principle of differential subordination in several problems involving univalent and multivalent analytic or meromorphic functions, the interested reader is referred (for example) to the earlier works [1], [2], [5], [9], [10], [11], [12], [13] and [14] (and also to the references cited in each of these works).

Miller and Mocanu [8] developed a definitive result concerning the Briot-Bouquet differential subordinations as follows.

Lemma 1. *Let n be a positive integer and let β and γ be complex numbers with $\beta \neq 0$. Suppose also that the function $h(z)$ is convex and univalent in $\mathbb{U}$ with $h(0) = a$ and that*

$$\Re\big(\beta h(z) + \gamma\big) > 0 \qquad (z \in \mathbb{U}) \qquad \big(\Re(\beta a + \gamma) > 0\big). \tag{1.4}$$

If the function $p(z) \in \mathcal{H}[a,n]$ with $p(z) \not\equiv a$ satisfies the following differential subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \qquad (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \prec h(z) \qquad (z \in \mathbb{U}),$$

where $q(z)$ with $q(0) = a$ is the univalent solution of the differential equation:

$$q(z) + \frac{nzq'(z)}{\beta q(z)+\gamma} = h(z) \qquad (z \in \mathbb{U}).$$

As applications of Lemma 1, Miller and Mocanu [8] derived some subordination relations for certain linear transformations.

Lemma 2. *Let n be a positive integer. Also let β, γ and A be complex numbers with $\Re(\beta+\gamma) > 0$. Suppose that B is a real number with $-1 \leqq B \leqq 0$. If β, γ, A and B satisfy either*

$$\Re\big(\beta(1+AB) + \gamma(1+B^2)\big) \geqq |\beta A + \overline{\beta} B + 2B\Re\gamma| \qquad (-1 < B \leqq 0)$$

or

$$\beta(1+A) > 0 \quad \text{and} \quad \Re\big(\beta(1+A)+2\gamma\big) \geqq 0 \qquad (B=-1),$$

then $p(z) \in \mathcal{H}[1,n]$ with $p(z) \not\equiv 1$ satisfies the following subordination relation:

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec \frac{1+Az}{1+Bz} \quad \text{implies} \quad p(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}), \quad (1.5)$$

where $q(z)$ with $q(0) = a$ is the univalent solution of the differential equation:

$$q(z) + \frac{nzq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}). \qquad (1.6)$$

In the present paper, by applying the above-mentioned principle of differential subordination between analytic functions, we propose to determine the best conditions for the complex numbers β, γ, A and B in order to satisfy the condition (1.4) when

$$h(z) = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

in Lemma 1. We also deduce an extension of Lemma 2.

2. Differential Subordinations for a Family of Linear Transformations

By using the method involving a certain generalization of the Janowski functions given by Kuroki and Owa [6], we first consider subordinations for a substantially more general family of linear transformations.

Theorem 1. *Let the parameters a, A, B, C and D be complex numbers with $A \neq aB$ and $C \neq aD$. If these parameters a, A, B, C and D also satisfy the following constraints:*

$$|B| \leqq 1, \quad |D| \leqq 1 \qquad \text{and} \qquad |A - aB| + |AD - BC| \leqq |C - aD|, \qquad (2.1)$$

then

$$\frac{a+Az}{1+Bz} \prec \frac{a+Cz}{1+Dz} \qquad (z \in \mathbb{U}). \tag{2.2}$$

Proof. Let us first define the function $w(z)$ by

$$\frac{a+Az}{1+Bz} = \frac{a+Cw(z)}{1+Dw(z)} \qquad (z \in \mathbb{U}).$$

Since $|B| \leqq 1$ and $|D| \leqq 1$, we see that $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. By noting that

$$|z| = \left| \frac{(C-aD)w(z)}{(A-aB)+(AD-BC)w(z)} \right| < 1,$$

we have

$$\big|w(z)-\rho\big| < R,$$

where

$$\rho = \frac{(A-aB)(\overline{AD}-\overline{BC})}{|C-aD|^2-|AD-BC|^2}$$

and

$$R = \frac{|A-aB||C-aD|}{|C-aD|^2-|AD-BC|^2}$$

with the condition (2.1). By observing also that

$$|\rho| + R = \frac{|A-aB|}{|C-aD|-|AD-BC|},$$

if

$$|A-aB| + |AD-BC| \leqq |C-aD|,$$

we get

$$|w(z)| < 1 \qquad (z \in \mathbb{U}).$$

Therefore, from the definition of differential subordinations, the subordination relation (2.2) holds true. This completes the proof of Theorem 1. □

In particular, by letting

$$A = b, \quad C = \overline{a}e^{i\theta} \quad \text{and} \quad D = -e^{i\theta} \quad \big(a \in \mathbb{C};\ \Re(a) > 0;\ 0 \leqq \theta < 2\pi\big)$$

in Theorem 1, we are led easily to the following assertion.

Corollary 1. *Let a be a complex number with $\Re(a) > 0$. Then, for some complex numbers a, b and B with*

$$b \neq aB, \quad |B| \leqq 1 \quad \text{and} \quad |b-aB| + |b+\overline{a}B| \leqq 2\Re(a),$$

the following subordination relation holds true:

$$\frac{a+bz}{1+Bz} \prec \frac{a+\overline{a}e^{i\theta}z}{1-e^{i\theta}z} \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi).$$

This subordination relation means the following inequality:

$$\Re\left(\frac{a+bz}{1+Bz}\right) > 0 \qquad (z \in \mathbb{U}).$$

Remark 1. Upon setting $a = 1$ and $b = A$ in Corollary 1, we find the conditions in (1.1) as the conditions for the complex numbers A and B to satisfy the folowing inequality:

$$\Re\left(\frac{1+Az}{1+Bz}\right) > 0 \qquad (z \in \mathbb{U}).$$

3. The Briot-Bouquet Differential Subordinations for Linear Transformations

By using the discussion in the preceding section, and applying Lemma 1, we deduce our proposed improvement of Lemma 2 as follows.

Theorem 2. *Let n be a positive integer and let the parameters β, γ, A and B be complex numbers satisfying the following constraints*:

$$\Re(\beta+\gamma) > 0, \quad A \neq B \qquad \text{and} \qquad |B| \leqq 1.$$

If β, γ, A and B also satisfy the following inequality:

$$|\beta(A-B)| + |\beta(A-B) + 2B\Re(\beta+\gamma)| \leqq 2\Re(\beta+\gamma),$$

then $p(z) \in \mathcal{H}[1, n]$ with $p(z) \not\equiv 1$ satisfies the subordination relation (1.5), *where $q(z)$ with $q(0) = 1$ is the solution of the differential equation* (1.6).

Proof. If we let

$$a = \beta+\gamma, \quad b = \beta A + \gamma B \qquad \text{and} \qquad h(z) = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

then, after some simple calculations, we find that

$$b - aB = (\beta A + \gamma B) - (\beta+\gamma)B = \beta(A-B) \neq 0$$

and

$$\begin{aligned} 2\Re(a) - (|b-aB| + |b+\overline{a}B|) &= 2\Re(a) - [|b-aB| + |b-aB+2B\Re(a)|] \\ &= 2\Re(\beta+\gamma) - \big[|\beta(A-B)| + |\beta(A-B) + 2B\Re(\beta+\gamma)|\big] \\ &\geqq 0. \end{aligned}$$

Hence, by Corollary 1, it is easy to see that

$$\Re\big(\beta h(z) + \gamma\big) = \Re\left(\frac{\beta+\gamma+(\beta A+\gamma B)z}{1+Bz}\right) = \Re\left(\frac{a+bz}{1+Bz}\right) > 0 \qquad (z \in \mathbb{U}).$$

Thus, since the conditions of Lemma 1 are satisfied, we arrive at the assertion of Theorem 2. □

Remark 2. By taking $\beta = 1$, $\gamma = 0$ and $n = 1$ in Theorem 2, and letting

$$p(z) = \frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{A}$, we obtain the following subordination implication.

Corollary 2. *If $f(z) \in \mathcal{A}$ satisfies the following subordination condition*:

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

for some complex numbers A *and* B *which satisfy the conditions in* (1.1), *then*

$$\frac{zf'(z)}{f(z)} \prec \begin{cases} \dfrac{Az}{(1+B)\left\{1-(1+Bz)^{-\frac{A}{B}}\right\}} & (A \neq 0;\ B \neq 0) \\[2ex] \dfrac{Bz}{(1+Bz)\log(1+Bz)} & (A = 0;\ B \neq 0) \\[2ex] \dfrac{Aze^{Az}}{e^{Az}-1} & (A \neq 0;\ B = 0) \end{cases}$$

for $z \in \mathbb{U}$.

Remark 3. Consider the case when

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1$$

in Corollary 2. From the definition of differential subordination between analytic functions, we thus find the implication that, if $f(z) \in \mathcal{K}(\alpha)$, then $f(z) \in \mathcal{S}^*(\beta)$, where

$$\beta = \beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & \left(\alpha \neq \dfrac{1}{2}\right) \\[2ex] \dfrac{1}{2\log 2} & \left(\alpha = \dfrac{1}{2}\right) \end{cases}$$

for each real number α with $0 \leqq \alpha < 1$. This relationship for convex and starlike functions was proven earlier by MacGregor [7].

References

[1] N. E. Cho, O. S. Kwon and H. M. Srivastava, Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu-Srivastava operator, *Integral Transforms Spec. Funct.* **21** (2010), 589–601.

[2] N. E. Cho and H. M. Srivastava, A class nonlinear integral operators preserving subordination and superordination, *Integral Transforms Spec. Funct.* **18** (2007), 95–107.

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band **259**, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[4] W. Janowski, Extremal problem for a family of functions with positive real part and for some related families, *Ann. Polon. Math.* **23** (1970), 159–177.

[5] Y. C. Kim and H. M. Srivastava, Some subordination properties for spirallike functions, *Appl. Math. Comput.* **203** (2008), 838–842.

[6] K. Kuroki and S. Owa, Notes on the Janowski functions defined by some complex parameters, *Indian J. Math.* **53** (2011), 113–123.

[7] T. H. MacGregor, A subordination for convex functions of order α, *J. London Math. Soc.* (*Ser.* 2) **9** (1975), 530–536.

[8] S. S. Miller and P. T. Mocanu, *Differential Subordinations*: *Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. **225**, Marcel Dekker Incorporated, New York and Basel, 2000.

[9] S. Owa and H. M. Srivastava, Some subordination theorems involving a certain family of integral operators, *Integral Transforms Spec. Funct.* **15** (2004), 445–454.

[10] H. Shiraishi, S. Owa, T. Hayami, K. Kuroki and H. M. Srivastava, Starlike problems for certain analytic functions concerned with subordinations, *Pan Amer. Math. J.* **21** (2011), 63–77.

[11] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, *Integral Transforms Spec. Funct.* **18** (2007), 207–216.

[12] H. M. Srivastava and A. Y. Lashin, Some applications of the Briot-Bouquet differential subordination, *J. Inequal. Pure Appl. Math.* **6** (2: Article 41) (2005), 1–7 (electronic).

[13] H. M. Srivastava and J. Patel, Applications of differential subordination to certain subclasses of meromorphically multivalent functions, *J. Inequal. Pure Appl. Math.* **6** (3: Article 88) (2005), 1–15 (electronic).

[14] H. M. Srivastava, J. Patel and G. P. Mohapatra, Some applications of differential subordination to a general class of multivalent functions, *Adv. Stud. Contemp. Math.* **3** (2001), 1–15.

[15] H. M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

Kazuo Kuroki

Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577–8502, Japan

E-mail address: freedom@sakai.zaq.ne.jp

H. M. Srivastava*

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

***Corresponding Author**

E-mail address: harimsri@math.uvic.ca

Shigeyoshi Owa

Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577–8502, Japan

E-mail address: shige21@ican.zaq.ne.jp