

DISCONTINUOUS DYNAMICAL SYSTEMS GENERATED BY A SEMI-DISCRETIZATION PROCESS

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ABSTRACT. In this paper, a semi-discretization process is applied to the well-known Logistic differential equation. The new system after discretization will be discontinuous. Stability of fixed points for the resultant discontinuous system is studied. Bifurcation analysis and chaos are discussed.

1. INTRODUCTION

A new class of discontinuous dynamical systems generated by retarded functional equations has been defined in [3]-[7]. Here we are concerned with the well-known Logistic equation given by

$$x'(t) = \rho x(t)(1 - x(t)), \quad t \in (0, T],$$

with the initial condition

$$x(0) = x_0,$$

where ρ is a positive constant. This equation has been studied and widely discussed in many papers and textbooks of dynamical systems. In this work, we are interested in the discontinuous counterpart of this equation which we are going to define and discuss its bifurcation and chaos.

2. DISCONTINUOUS DYNAMICAL SYSTEMS

Consider the problem of retarded functional equation

$$x(t) = f(x(t-r)), \quad t \in (0, T], \tag{2.1}$$

with the initial condition

$$x(\tau) = \phi(\tau), \quad \tau \leq 0. \tag{2.2}$$

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If T is a positive integer, $r = 1$, $\phi(0) = x_o$ and $t = n = 1, 2, 3, \dots$, then the problem (2.1)-(2.2) will be the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1, 2, 3, \dots, T \quad (2.3)$$

$$x(0) = x_o. \quad (2.4)$$

This shows that the discrete dynamical system (2.3)-(2.4) is a special case of the problem of the retarded functional equation (2.1)-(2.2).

Let $t \in (0, r]$, then $t - r \in (-r, 0]$ and the solution of (2.1)-(2.2) is given by

$$x(t) = x_1(t) = f(\phi(0)), \quad t \in (0, r].$$

For $t \in (r, 2r]$, then $t - r \in (0, r]$ and the solution of (2.1) is given by

$$x(t) = x_2(t) = f(x_1(t)) = f(f(\phi(0))) = f^2(\phi(0)), \quad t \in (r, 2r].$$

Repeating the process we can easily deduce that the solution of (2.1) is given by

$$x(t) = x_n(t) = f^n(\phi(0)), \quad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr]$, $k = 1, 2, 3, \dots, n$, but

$$\lim_{t \rightarrow kr^+} x_{(k+1)r}(t) = f^{k+1}(\phi(0)) \neq x_{kr},$$

which implies that the solution of the problem (2.1)-(2.2) is discontinuous (sectionally continuous) on $(0, T]$ and thus we have proved the following theorem [3].

Theorem 1. *The solution of the problem of retarded functional equation (2.1)-(2.2) is discontinuous (sectionally continuous) even if the functions f and ϕ are continuous.*

Now let $f : [0, T] \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ and $r_1, r_2, \dots, r_n \in \mathbb{R}^+$. Then, the following definition can be given

Definition 1. *The discontinuous dynamical system is the problem of retarded functional equation*

$$x(t) = f(t, x(t-r_1), x(t-r_2), \dots, x(t-r_n)), \quad t \in (0, T], \quad (2.5)$$

$$x(\tau) = \phi(\tau), \quad \tau \leq 0. \quad (2.6)$$

Definition 2. *The fixed points of the discontinuous dynamical system (2.5)-(2.6) are the solution of the equation*

$$x(t) = f(t, x, x, \dots, x). \quad (2.7)$$

3. A SEMI-DISCRETIZATION PROCESS

Consider the Logistic differential equation given by

$$x'(t) = \rho x(t)(1 - x(t)), \quad t \in (0, T], \quad (3.1)$$

with the initial condition

$$x(0) = x_o.$$

Now we are going to apply a semi-discretization process to this equation in order to obtain its discontinuous counterpart.

Let $r > 0$ be given. Using The approximations

$$x'(t) \simeq \frac{x(t+r) - x(t)}{r},$$

in (3.1) we get

$$\begin{aligned} \frac{x(t+r) - x(t)}{r} &\simeq \rho x(t)(1 - x(t)), \\ x(t+r) &= x(t) + \rho x(t)(1 - x(t)), \\ x(t) &= x(t-r) + \rho x(t-r)(1 - x(t-r)). \end{aligned}$$

That is, we have the following discontinuous counterpart of the Logistic equation which is given by

$$x(t) = x(t-r) + r\rho x(t-r)(1 - x(t-r)), \quad t \in (0, T], \tag{3.2}$$

with the initial condition

$$x(t) = x_0, \quad t \leq 0.$$

By the same way as in section 2, we can show that the solution of system (3.2) for $t \in (nr, (n+1)r]$ is given by the loop

$$x_{n+1}(t) = x_n(nr) + \rho r x_n(nr)(1 - x_n(nr)), \quad n = 1, 2, 3, \dots$$

It is worth to mention here that the previous semi-discretization process can be obtained by Taylor expansion as follows:

$$\begin{aligned} x(t) &= x(t-r) + x'(t-r) \frac{t - (t-r)}{1!} + \dots \\ &\simeq x(t-r) + r\rho x(t-r)(1 - x(t-r)). \end{aligned}$$

To summarize, we showed that there exists a semi-discretization process which generate a discontinuous dynamical system. This discontinuous dynamical system generalizes the discrete one studied in [1].

Figure (1) shows the trajectory of the discontinuous system (3.2) when $r = 1, 0.5$ and 0.25 , while Figure (2) shows the solution of the continuous system (3.1).

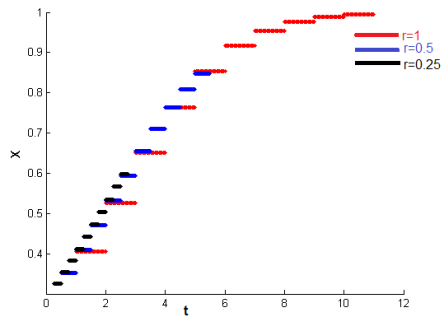


FIGURE 1. Trajectory of (3.2) with $\rho = 0.5$.

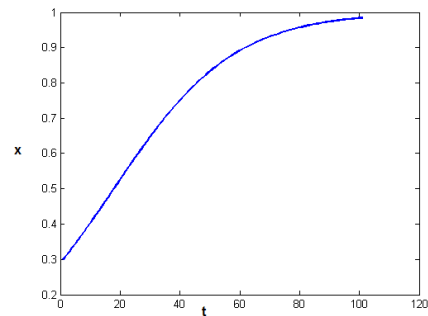


FIGURE 2. Solution of (3.1) with $\rho = 0.5$.

3.1. Approximate Solution. In this part we show that the proposed semi-discretization process best approximates the solution of the discontinuous dynamical system (3.2) to the exact solution of the continuous system (3.1) as shown in the table below. Now for $t = \frac{n+n+1}{2}r$, the following table gives the absolute error = $|exact - approximate|$ for some different values of n and r . Figures (3)-(6) illustrate the approximate solution for (3.2)

n	r=0.1	r=0.2	r=0.3	r=0.4	r=1
10	0.0656	0.1382	0.2019	0.2443	0.1338
20	0.0697	0.1123	0.0968	0.0591	1.0000e-004
30	0.0647	0.0583	0.0231	0.0064	0
40	0.0529	0.0229	0.0043	6.0000e-004	0
50	0.0389	0.0079	7.0000e-004	0	0

TABLE 1. Absolute error

when $r = 0.1, 0.2, 0.3, 0.4$ and 1 , respectively.

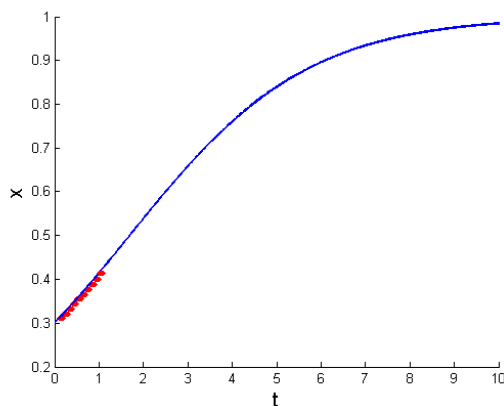


FIGURE 3. Exact(solid) of (3.1) and approximate(dashed) of (3.2) when $r = 0.1$.

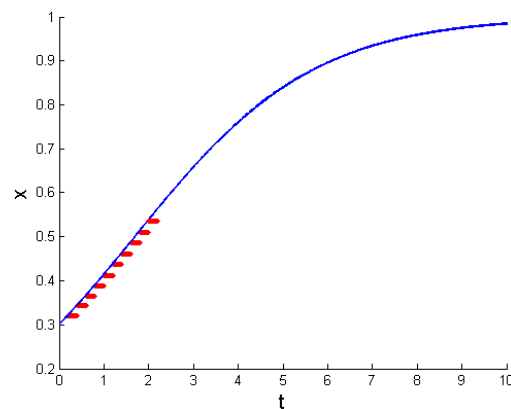


FIGURE 4. Exact(solid) of (3.1) and approximate(dashed) of (3.2) when $r = 0.2$.

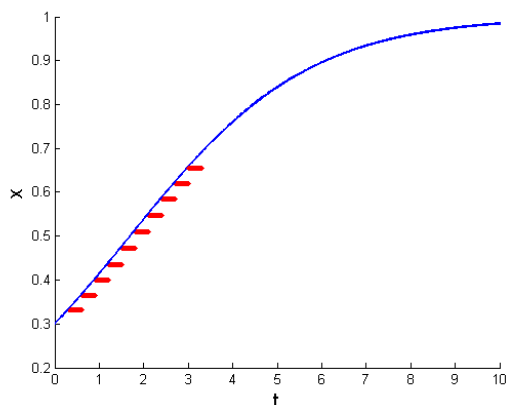


FIGURE 5. Exact(solid) of (3.1) and approximate(dashed) of (3.2) when $r = 0.3$.

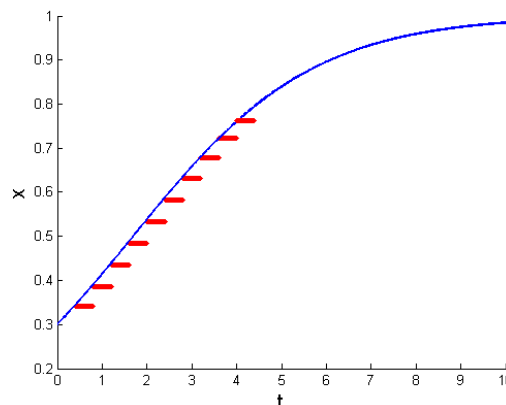


FIGURE 6. Exact(solid) of (3.1) and approximate(dashed) of (3.2) when $r = 0.4$.

4. FIXED POINTS AND STABILITY

The discontinuous dynamical system (3.2) has two fixed points namely, $x_1 = 0$ and $x_2 = 1$ which can be easily obtained by solving the equation

$$x = x + r\rho x(1 - x).$$

To study the stability of these fixed points we need the following theorem

Theorem 2. [2] *Let f be a smooth map on \mathbb{R} , and assume that x_0 is a fixed point of f .*

1. *If $|f'(x_0)| < 1$, then x_0 is stable.*
2. *If $|f'(x_0)| > 1$, then x_0 is unstable.*

In our case, $f = x + r\rho x(1 - x)$. Its derivative is $f' = 1 + \rho r - 2\rho r x$. That is, $x_1 = 0$ is stable if $|1 + r\rho| < 1$, which is impossible, this means that x_1 is unstable.

Similarly, the second fixed point $x_2 = 1$ is stable if $|1 - r\rho| < 1$.

5. BIFURCATION AND CHAOS

In this section we show by numerical experiments illustrated by bifurcation diagrams that the dynamical behavior of the discontinuous system (3.2) is completely affected by the change in both r and T .

- Take $r = 1$ and $t \in [0, 150]$ in (3.2) (Figure (7)).
- Take $r = 0.1$ and $t \in [0, 15]$ in (3.2) (Figure (8)).
- Take $r = 0.25$ and $t \in [0, 3]$ in (3.2) (Figure (9)).
- Take $r = 0.5$ and $t \in [0, 6]$ in (3.2) (Figure (10)).
- Take $r = 0.3$ and $t \in [0, 3]$ in (3.2) (Figure (11)).

Take $r = 0.1$ and $t \in [0, 2]$ in (3.2) (Figure (12)).

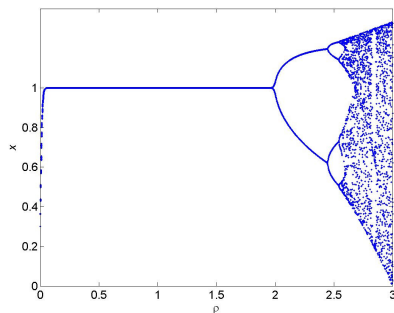


FIGURE 7. Bifurcation diagram of (3.2) when $r = 1$ and $t \in [0, 150]$.

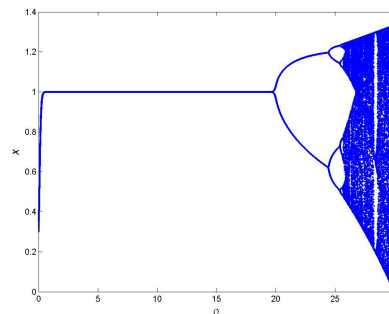


FIGURE 8. Bifurcation diagram of (3.2) when $r = 0.1$ and $t \in [0, 15]$.

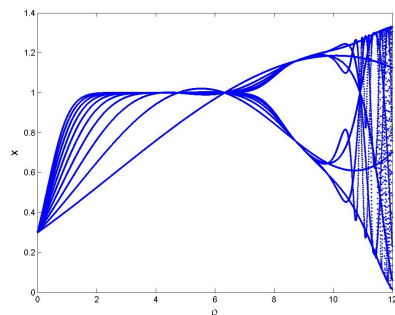


FIGURE 9. Bifurcation diagram of (3.2) when $r = 0.25$ and $t \in [0, 3]$.

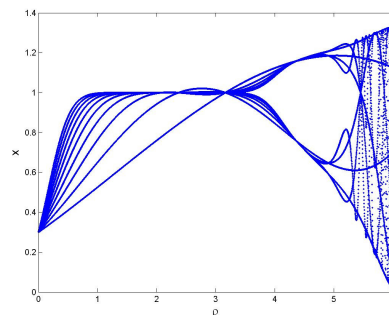


FIGURE 10. Bifurcation diagram of (3.2) when $r = 0.5$ and $t \in [0, 6]$.

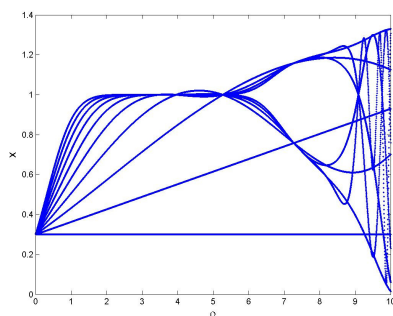


FIGURE 11. Bifurcation diagram of (3.2) when $r = 0.3$ and $t \in [0, 3]$.

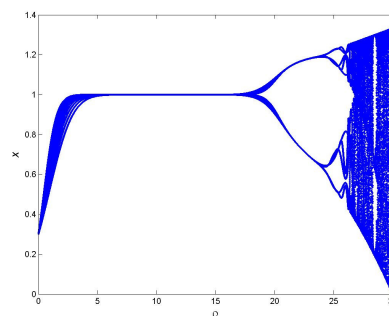


FIGURE 12. Bifurcation diagram of (3.2) when $r = 0.1$ and $t \in [0, 2]$.

6. CONCLUSION

Applying a semi-descretization process to the well-known Logistic equation resulting in a discontinuous dynamical system representing the Logistic map. Moreover, changing both the retardation parameter r together with the time $t \in [0, T]$, has the strong effect on both bifurcation and chaos behavior of the map.

Figures (11) and (12) agree with our results, while Figures (7)-(8) and (9)-(10) indicate that there is a scale which gives identical chaotic behavior.

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