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THE FUNDAMENTAL SOLUTIONS FOR MULTI-TERM MODIFIED POWER LAW WAVE EQUATIONS IN A FINITE DOMAIN

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Abstract. Fractional partial di?erential equations with more than one fractional derivative term in time, such as the Szabo wave equation, or the power law wave equation, describe important physical phenomena. However, studies of these multi-term time- space or time fractional wave equations are still under development.
In this paper, multi-term modifed power law wave equations in a finite domain are considered. The multi-term time fractional derivatives are defined in the Caputo sense, whose orders belong to the intervals $(1, 2]$, $[2, 3)$, $[2, 4)$ or $(0, n)$ $(n > 2)$, respectively. Fundamental solutions of the multi-term modi?ed power law wave equations are derived. These new techniques are based on Luchko's Theorem, a spectral representation of the Laplacian operator, a method of separating variables and fractional derivative techniques. Then these general methods are applied to the special cases of the Szabo wave equation and the power law wave equation. These methods and techniques can also be extended to other kinds of the multi-term time-space fractional models including fractional Laplacian.

1. Introduction

In recent years, a growing number of works by many authors from various fields of science and engineering deal with dynamical systems described by fractional partial differential equations (see [3, 14, 15, 23]). These new models improve on the previously used integer-order models. Generalized fractional partial differential equations have been used for describing important physical phenomena [20]. Baleanu et al. [1, 2] investigated an existence result for a superlinear fractional differential equation and on the global existence of solutions to a class of fractional differential equations. However, studies of the multi-term time and space fractional wave equations are still under development.
The effect of the attenuation plays a prominent role in many acoustic and ultrasound applications, for instance, the ultrasound second harmonic imaging and high

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intensity focused ultrasound beam for therapeutic surgery. The attenuation coefficient for biological tissue may be approximated by a power law [8] over a wide range of frequencies used in ultrasonic imaging and thermal therapy applications. Frequency-dependent loss and dispersion are typically modeled with a power-law attenuation coefficient, where the power-law exponent ranges from 0 to 2. To facilitate analytical solution, a fractional partial di?erential equation is derived that exactly describes power-law attenuation and the Szabo wave equation [24] is an approximation to this equation.

Measurements on the changes in phase velocity with frequency have been reported for at least 70 years, and much of this data, together with the results of more recent measurements, have been reported by Szabo [25]. He compared the experimental results with those predicted from absorption measurements based on causal dispersion relations. The Szabo wave equation was originally presented as an integro-differential equation for fractional power law media.

Stojanovic [26] found solutions for the diffusion-wave problem in 1-D with n-term time fractional derivatives whose orders belong to the intervals $(0, 1)$, $(1, 2)$ and $(0, 2)$, respectively, using the method of the approximation of the convolution by Laguerre polynomials in the space of tempered distributions. Time domain wave-equations for lossy media obeying a frequency power-law [13]. Mathematically, the power-law frequency dependence of the attenuation coefficient cannot be modeled with standard dissipative partial differential equations with integer-order derivatives.

However, the multi-term time-space fractional wave equations successfully capture this power-law frequency dependence. Kelly [13] considered a power law wave equation:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_0}{c_0\cos(\pi y/2)}\frac{\partial^{y+1} p}{\partial t^{y+1}} - \frac{\alpha_0^2}{\cos^2(\frac{\pi y}{2})}\frac{\partial^{2y} p}{\partial t^{2y}} = 0$$

The power law wave equation with the n-term time fractional derivatives whose orders belong to the intervals $[2, 3)$ (if $1 < y < 3/2$) and $[2, 4)$ (if $3/2 < y < 2$), respectively. The Riemann-Liouville fractional derivative $\partial_t^y =_0 D_t^y p$

$$_0D_t^y p(t) = \frac{1}{\Gamma(m-y)}\frac{d^m}{dt^m}\int_0^t \frac{p(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}.$$

Kelly [13] also considered the Szabo wave equation:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_0}{c_0\cos(\pi y/2)}\frac{\partial^{y+1} p}{\partial t^{y+1}} = 0.$$

The Szabo wave equation with the n-term time fractional derivatives whose orders belong to the intervals $(1, 2]$ (if $0 < y < 1$) and $[2, 3)$ (if $1 < y < 2$), respectively. The third term is defined by the Riemann-Liouville fractional derivative of order $y + 1$.

The generalized time-fractional diffusion equation corresponds to a continuous time random walk model where the characteristic waiting time elapsing between two successive jumps diverge, but the accumulated jump length variance remains finite and is proportional to t^α. The exponent α of the mean square displacement proportional to t^α often does not remain constant and changes. To adequately describe these phenomena with fractional models, multi-term modified power law wave equations and several approaches have been suggested in the literature (for example, [6, 9, 10, 12, 16, 18, 19, 20, 21, 22]). However, studies of the multi-term modified power law wave equations are still limited.

In this paper, a method of separating variables is used to solve the multi-term modified power law wave equations in a finite domain. It is found that these models can be recast with the Riemann-Liouville fractional derivative or Caputo fractional derivative for noninteger power. Then, the basic strategy of this study is that the Riemann-Liouville fractional derivative is replaced by the Caputo fractional derivative to derive a modified Szabo model, where the initial conditions can be easily prescribed. We discuss and derive the analytical solutions of these equations with nonhomogeneous Dirichlet boundary conditions.

The rest of this paper is organized as follows. In section 2, we give some relevant definitions and lemmas. The analytical solutions of the multi-term modified power law wave equations with nonhomogeneous Dirichlet boundary conditions are derived in sections 3. Some special cases are considered in section 4. Finally, we summarise the main research findings of our work in section 5.

2. Background Theory

For convenience, we introduce the following definitions and theorems, which are used throughout this paper. A modified time fractional wave equation can be written in the following form:

$$D_t^\alpha u(x,t) = k\nabla^2 u(x,t) + f(x,t), 0 < x < L, \qquad t > 0, \tag{1}$$

where x and t are the space and time variables, k is an arbitrary positive constant, $\nabla^2 = \triangle = \frac{\partial^2 u(x,t)}{\partial x^2}$, $(\triangle)$ is Laplacian, $f(x,t)$ is a sufficiently smooth function, $0 < \alpha \leq 4$ and D_t^α is a Caputo fractional derivative of order α defined as [23]

$$D_t^\alpha f(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{1+\alpha-m}}, \\ \frac{d^m}{dt^m} f(t), \end{cases} \tag{2}$$

where $m-1 < \alpha < m, \alpha = m \in N$.

When $1 < \alpha < 2$, Eq. (1) is the time-fractional wave equation and when $0 < \alpha < 1$, Eq. (1) is a fractional diffusion equation. When $\alpha = 2$, it represents a traditional wave equation; while if $\alpha = 1$, it represents a traditional diffusion equation.

In some practical situations the underlying processes cannot be described by Eq. (1), but by its modified multi-term time-fractional wave and diffusion equations that are given by [19], namely

$$P_{\alpha,\alpha_1,\cdots,\alpha_n}(D_t)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{3}$$

where $0 < x < L, \quad t > 0$,

$$P_{\alpha,\alpha_1,\cdots,\alpha_n}(D_t)u(x,t) = D_t^\alpha + \sum_{i=1}^{n} a_i D_t^{\alpha_i}, \tag{4}$$

$0 \leq \alpha_n < \ldots < \alpha_{h_1+1} \leq 1 < \alpha_{h_1} < \cdots < \alpha_{h_{m-1}+1} \leq r-1 < \alpha_{h_{m-1}} < \cdots < \alpha_1 < \alpha \leq m$, and $a_i, i = 1, \ldots, n$, $n \in N$. $D_t^{\alpha_i}$ is a Caputo fractional derivative of order α_i with respect to t.

In this paper, we will discuss and derive the analytical solution of (3) with nonhomogeneous Dirichlet boundary conditions using the method of separation of variables, respectively. The initial conditions are

$$u(x,0) = \phi_0(x), \frac{\partial^i u(x,0)}{\partial t^i} = \phi_i(x), i = 1, \cdots, m-1. \tag{5}$$

where $0 < x < L$.

Definition 1. (See [7]) A real or complex-valued function $f(x), x > 0$, is said to be in the space $C_\alpha, \alpha \in R$, if there exists a real number $p > \alpha$ such that

$$f(x) = x^p f_1(x), \tag{6}$$

for $f_1(x) \in C[0, \infty]$.

Definition 2. (See [17]) A function $f(x), x > 0$, is said to be in the space $C_\alpha^m, m \in N_0 = N \cup \{0\}$, if and only if $f^m \in C_\alpha$.

Definition 3. (See [17]) A multivariate Mittag-Leffler function (n dimensional cases) is defined as

$$\begin{aligned} &E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n) \\ &\equiv \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots,l_n\geq 0}} \frac{k!}{l_1! \times \cdots \times l_n!} \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b+\sum_{i=1}^n a_i l_i)} \end{aligned} \tag{7}$$

in which $b > 0, a_i > 0, |z_i| < \infty, i = 1, \ldots, n$.

Lemma 1. Let $\mu > \mu_1 > \cdots > \mu_n \geq 0, m_i - 1 < \mu_i \leq m_i, m_i \in N_0 = N \cup \{0\}, \lambda_i \in R, i = 1, \ldots, n$. The initial value problem

$$\begin{cases} (D^\mu y)(x) - \sum_{i=1}^n \lambda_i (D^{\mu_i} y)(x) = g(x), \\ y^{(k)}(0) = c_k \in R, k = 0, \ldots, m-1, \quad m-1 < \mu \leq m, \end{cases} \tag{8}$$

where the function $g(x)$ is assumed to lie in C_{-1}, if $\mu \in N$, in C_{-1}^1, if $\mu \notin N$, and the unknown function $y(x)$ is to be determined in the space C_{-1}^m, has the solution

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), \qquad x \geq 0, \tag{9}$$

where

$$y_g(x) = \int_0^x t^{\mu-1} E_{(.),\mu}(t) g(x-t) dt, \tag{10}$$

and

$$\begin{aligned} u_k(x) &= \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i x^{k+\mu-\mu_i} E_{(.),k+1+\mu-\mu_i}(x), \\ k &= 0, \ldots, m-1, \end{aligned} \tag{11}$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl}, k, l = 0, \ldots, m-1$. The function

$$E_{(.),\beta}(x) = E_{\mu-\mu_1,\ldots,\mu-\mu_n,\beta}(\lambda_1 x^{\mu-\mu_1}, \cdots, \lambda_n x^{\mu-\mu_n}). \tag{12}$$

The natural numbers $l_k, k = 0, \ldots, m-1$, are determined from the condition

$$\begin{cases} m_{l_k} \geq & k+1, \\ m_{l_k+1} & \leq k. \end{cases} \tag{13}$$

In the case $m_i \leq k, i = 1, \ldots, n$, we set $l_k := 0$, and if $m_i \geq k+1, i = 1, \ldots, n$, then $l_k := n$.

Proof. (See [17, 12]).

Definition 4. (see [11]) Suppose the Laplacian $(-\triangle)$ has a complete set of orthonormal eigenfunction φ_n corresponding to eigenvalues λ_n^2 on a bounded region $\mathcal{D}$, i.e.,$(-\triangle)\varphi_n = \lambda_n^2\varphi_n$ on a bounded region $\mathcal{D}$; $\mathcal{B}(\varphi) = 0$ on $\partial\mathcal{D}$, where $\mathcal{B}(\varphi)$ is one of the standard three homogeneous boundary conditions. Let

$$\mathcal{F} = \{f = \sum_{n=1}^{\infty} c_n\varphi_n, c_n = \langle f, \varphi_n\rangle, \sum_{n=1}^{\infty} |c_n|^2|\lambda|_n^{\beta} < \infty, \}, \tag{14}$$

then for any $f \in \mathcal{F}, (-\triangle)$ is defined by

$$(-\triangle)f = \sum_{n=1}^{\infty} c_n\lambda_n^2\varphi_n. \tag{15}$$

Lemma 2. (See [4, 11]) Suppose the one-dimensional Laplacian $(-\triangle)$ has a complete set of orthonormal eigenfunctions φ_{n1} corresponding to eigenvalues λ_{n1}^2 on a boundary region $\Omega = [0, L]$, if $(-\triangle)\varphi_{n1} = \lambda_n^2\varphi_{n1}$ on Dirichlet boundary conditions,i.e.,

$$\begin{aligned} &-\triangle\varphi = \lambda^2\varphi, \\ &\varphi(0) = 0, \\ &\varphi(L) = 0. \end{aligned}$$

The eigenvalues are $\lambda_{n1}^2 = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, \cdots$ and the corresponding eigenfunctions are nonzero constant multiples of $\varphi_{n1} = sin(\frac{n\pi x}{L})$

3. A modified multi-term time power law wave equation with nonhomogeneous Dirichlet boundary conditions

Firstly, we consider the modified multi-term time power law wave equation (3) with the initial condition (5) and the Dirichlet boundary condition:

$$\begin{cases} u(0,t) = \mu_1(t), & t \geq 0, \\ u(L,t) = \mu_2(t), & t \geq 0, \end{cases} \tag{16}$$

where $\mu_1(t)$, $\mu_2(t)$ are nonzero smooth functions with order-one continuous derivatives. In order to solve the problem with nonhomogeneous boundary conditions, we firstly transform the nonhomogeneous condition into a homogeneous boundary condition. Let

$$u(x,t) = W(x,t) + V(x,t), \tag{17}$$

where

$$V(x,t) = \frac{\mu_2(t) - \mu_1(t)}{L}x + \mu_1(t) \tag{18}$$

satisfies the boundary conditions

$$\begin{cases} V(0,t) = \mu_1(t), \\ V(L,t) = \mu_2(t), \end{cases} \tag{19}$$

and the function $W_1(x,t)$ is the solution of the problem:

$$\begin{cases} P(D_t)W(x,t) + k(-\triangle)W(x,t) = f_1(x,t), \\ W_0(x,0) = \varphi_0(x), \quad \frac{\partial^i W(x,0)}{\partial t^i} = \varphi_i(x), \\ W(0,t) = 0, \\ W(L,t) = 0, \end{cases} \tag{20}$$

with

$$\begin{cases} f_1(x,t) & = -P(D_t)V(x,t) + f(x,t), \\ \varphi_0(x) & = \phi_0(x) + \frac{\mu_1(0)-\mu_2(0)}{L}x - \mu_1(0), \\ \varphi_i(x) & = \psi_i(x) + \frac{\frac{\partial^i \mu_1(0)}{\partial t^i} - \frac{\partial^i \mu_2(0)}{\partial t^i}}{L}x - \frac{\partial^i \mu_1(0)}{\partial t^i}, \end{cases} \tag{21}$$

where $0 \le x \le L$, $t \ge 0, i = 1, 2, \cdots, m-1$. Using Definition 4 , we set

$$W(x,t) = \sum_{n=1}^{\infty} c_{n1}(t) \sin \frac{n\pi x}{L}, \tag{22}$$

where φ_{n1} are eigenfunctions that given in Lemma 2 corresponding to eigenvalues λ_{n1}^2. We expand $f_1(x,t)$ in a Fourier series in the interval $[0, L]$ by the eigenfunctions $\sin \lambda_{n1} x$, namely

$$f_1(x,t) = \sum_{n=1}^{\infty} f_{n1}(t) \sin \lambda_{n1} x, \tag{23}$$

where

$$f_{n1}(t) = b_n^{-1} \int_0^L f_1(\xi, t) \sin \lambda_{n1} \xi d\xi, \tag{24}$$

$$b_n := \int_0^L \varphi_{n1}^2(x) dx = \int_0^L \sin^2 \lambda_{n1} x dx. \tag{25}$$

Using the initial condition

$$\begin{cases} \sum_{n=1}^{\infty} c_{n1}(0) \sin \lambda_{n1} x & = \phi_1(x), \\ \sum_{n=1}^{\infty} \frac{\partial^i c_{n1}(0)}{\partial t^i} \sin \lambda_{n1} x & = \varphi_i(x), i = 1, 2, \cdots, m-1. \end{cases} \tag{26}$$

We assume that $\phi_1(x), \psi_1(x), \omega_1(x)$ are continuous functions with one-order derivative. Multiplying (26) by $\sin \lambda_{n1} x$, and integrating from 0 to L respect x, we obtain

$$\begin{cases} c_{n1}(0) & = \frac{2}{L} \int_0^L \phi_1(\xi) \sin \frac{n\pi x}{L} d\xi, \\ \frac{\partial^i c_{n1}(0)}{\partial t^i} & = \frac{2}{L} \int_0^L \varphi_i(\xi) \sin \frac{n\pi x}{L} d\xi, i = 1, 2, \cdots, m-1, \end{cases} \tag{27}$$

where b_n is given in (25).
Based on Definition 4 and substituting (22) and (24) into (20) gives the following equation:

$$P(D_t) c_{n1}(t) + k\lambda_{n1}^2 c_{n1}(t) = f_{n1}(t), \tag{28}$$

where λ_{n1} are given in Lemma 2. According to Lemma 1, we have

$$\begin{aligned} c_{n1}(t) \quad & = \int_0^t G_\alpha^D(\tau) \tau^{\alpha-1} f_{n1}(t-\tau) d\tau + c_{n1}(0) u_0(t) \\ & + \sum_{i=1}^{m-1} \frac{\partial^i c_{n1}(0)}{\partial t^i} u_i(t), \end{aligned} \tag{29}$$

in which

$$G_\eta^D(t) = E_{(\nu_1, \cdots, \nu_n, \alpha), \eta}(-a_1 t^{\nu_1}, \cdots, -a_n t^{\nu_n}, -k\lambda_{n1}^2 t^\alpha),\ \nu_i = \alpha - \alpha_i,\ i = 1, 2, \cdots, n,$$

$$u_0(t) = 1 - k\lambda_{n1}^2 t^\alpha G_{1+\alpha}^D(t), \tag{30}$$

$$\begin{aligned} u_s(t) &= \frac{t^s}{s!} - \sum_{i=h_s+1}^{n} a_i t^{s+\gamma_i} G^D_{s+1+\gamma_i} \\ &-k\lambda_{n1}^2 t^{s+\alpha} G^D_{s+1+\alpha}(t), \end{aligned} \tag{31}$$

in which $s = 1, 2, \cdots, m-1$, and $\frac{\partial^i c_{n1}(0)}{\partial t^i}$ are given in (27).
Therefore we obtain the fundamental solution of the modified multi-term time power law wave equation with Dirichlet boundary condition as

$$\begin{aligned} u(x,t) &= W(x,t) + V(x,t) \\ &= \sum_{n=1}^{\infty} c_{n1}(t) \sin \frac{n\pi x}{L} + \frac{\mu_2(t) - \mu_1(t)}{L} x + \mu_1(t), \end{aligned} \tag{32}$$

where c_{n1} is given in (29), in which $c_{n1}(0)$ and $\frac{\partial^i c_{n1}(0)}{\partial t^i}$ are given in (27).

4. Special cases

In this section, we give some special cases using the results in Section 3.
Case 1. We consider a modified power law wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0 \cos(\pi y/2)}\frac{\partial^{y+1} p}{\partial t^{y+1}} + \frac{\alpha_0^2}{\cos^2(\frac{\pi y}{2})}\frac{\partial^{2y} p}{\partial t^{2y}} = \nabla^2 p, \tag{33}$$

where the fractional derivatives are defined by the Caputo fractional derivatives and whose orders belong to the interval $[2, 3)$ (if $1 \le y < \frac{3}{2}$). The initial conditions are given by (5) with $m = 3$ and boundary conditions are given by (16).
Firstly, we consider the interval $[2, 3)$ (if $1 \le y < \frac{3}{2}$).
We rewrite (33) as the following form:

$$\frac{\partial^{2y} p}{\partial t^{2y}} + \frac{2\cos(\frac{\pi y}{2})}{c_0\alpha_0}\frac{\partial^{y+1} p}{\partial t^{y+1}} + \frac{\cos^2(\frac{\pi y}{2})}{c_0^2\alpha_0^2}\frac{\partial^2 p}{\partial t^2} = \frac{\cos^2(\frac{\pi y}{2})}{\alpha_0^2}\nabla^2 p. \tag{34}$$

Let $a_1 = \frac{2\cos(\frac{\pi y}{2})}{c_0\alpha_0}, a_2 = \frac{\cos^2(\frac{\pi y}{2})}{c_0^2\alpha_0^2}, k = \frac{\cos^2(\frac{\pi y}{2})}{\alpha_0^2}$, then Eq. (34) becomes

$$\frac{\partial^{2y} p}{\partial t^{2y}} + a_1\frac{\partial^{y+1} p}{\partial t^{y+1}} + a_2\frac{\partial^2 p}{\partial t^2} = k\nabla^2 p. \tag{35}$$

Comparing Eq. (35) with Eq. (3) (let $a_3 = \cdots = a_n = 0$, $\alpha = 2y$, $\alpha_1 = y+1$, $\alpha_2 = 2$, $f(x,t) = 0$, $u(x,t) \equiv p(x,t)$), we obtain the analytical solution of (35) with initial conditions (5) and Dirichlet boundary conditions (16) as

$$\begin{aligned} u(x,t) &= W(x,t) + V(x,t) \\ &= \sum_{n=1}^{\infty} c_{n1}^{(1)}(t) \sin \mu_n x + \frac{\mu_2(t) - \mu_2(t)}{L} x + \mu_1(t), \end{aligned} \tag{36}$$

where

$$\begin{aligned} c_{n1}^{(1)}(t) &= \int_0^t G^{D1}_{2y}(\tau)\tau^{2y-1} f^{(1)}_{n1}(t-\tau)d\tau + c_{n1}(0)u_0(t) \\ &\quad + c'_{n1}(0)u_1(t) + c''_{n1}(0)u_2(t), \end{aligned} \tag{37}$$

$$G^{D1}_{\eta}(t) = E_{(y-1,2y-2,2y),\eta}(-a_1 t^{y-1}, -a_2 t^{2y-2}, -k\lambda_{n1}^2 t^{2y}), \tag{38}$$

$$f_{n1}^{(1)}(t-\tau) = b_n^{-1}\int_0^L f_1^{(1)}(\xi,t-\tau)\sin\lambda_{n1}\xi d\xi, \tag{39}$$

$$f_1^{(1)}(x,t) = -\frac{\partial^{2y}V(x,t)}{\partial t^{2y}} - a_1\frac{\partial^{y+1}V(x,t)}{\partial t^{y+1}} - a_2\frac{\partial^2 V(x,t)}{\partial t^2}, \tag{40}$$

$$u_0(t) = 1-k\lambda_{n1}^2 t^{2y}G_{1+2y}^{D1}(t), \tag{41}$$

$$u_1(t) = t-k\lambda_{n1}^2 t^{1+2y}G_{2+2y}^{D1}(t), \tag{42}$$

$$u_2(t) = \frac{t^2}{2!} - a_2 t^{2y}G_{2y+1}^{D1} - k\lambda_{n1}^2 t^{2+2y}G_{3+2y}^{D1}(t). \tag{43}$$

Here $b_n^{-1}, V(x,t)$ are given in (25) and(18), $c_{n1}(0)$, $c'_{n1}(0)=\frac{\partial c_{n1}(0)}{\partial t}$ and $c''_{n1}(0)=\frac{\partial^2 c_{n1}(0)}{\partial t^2}$ are given in (27).

Case 2. Similarly, we obtain the fundamental solution of the modified power law wave equation (33) in the interval $[2,4)$ (if $\frac{3}{2}\le y<2$). The initial conditions are given by (5) with $m=4$ and boundary conditions are given by (16).

$$\begin{aligned} u(x,t) &= W(x,t)+V(x,t) \\ &= \sum_{n=1}^{\infty} c_{n1}^{(2)}(t)\sin\mu_n x + \frac{\mu_2(t)-\mu_1(t)}{L}x + \mu_1(t), \end{aligned} \tag{44}$$

where

$$\begin{aligned} c_{n1}^{(2)}(t) &= \int_0^t G_{2y}^{D2}(\tau)\tau^{2y-1}f_{n1}^{(2)}(t-\tau)d\tau + c_{n1}(0)u_0(t) \\ &\quad + c'_{n1}(0)u_1(t) + c''_{n1}(0)u_2(t) + c'''_{n1}(0)u_3(t), \end{aligned} \tag{45}$$

$$G_\eta^{D2}(t) = E_{(y-1,2y-2,2y),\eta}(-a_1t^{y-1}, -a_2t^{2y-2}, -k\lambda_{n1}^2t^{2y}), \tag{46}$$

$$u_0(t) = 1-k\lambda_{n1}^2 t^{2y}G_{1+2y}^{D2}(t), \tag{47}$$

$$u_1(t) = t-k\lambda_{n1}^2 t^{1+2y}G_{2+2y}^{D2}(t), \tag{48}$$

$$u_2(t) = \frac{t^2}{2!} - a_2 t^{2y}G_{2y+1}^{D2} - k\lambda_{n1}^2 t^{2+2y}G_{3+2y}^{D2}(t), \tag{49}$$

$$u_3(t) = \frac{t^3}{3!} - a_2 t^{2y+1}G_{2y+1}^{D2} - k\lambda_{n1}^2 t^{3+2y}G_{4+2y}^{D2}(t). \tag{50}$$

Here $f_{n1}^{(2)}(t-\tau)$ is same as $f_{n1}^{(1)}(t-\tau)$, $c_{n1}(0)$, $c'_{n1}(0)=\frac{\partial c_{n1}(0)}{\partial t}$, $c''_{n1}(0)=\frac{\partial^2 c_{n1}(0)}{\partial t^2}$ and $c'''_{n1}(0)=\frac{\partial^3 c_{n1}(0)}{\partial t^3}$ are given in (27).

Case 3. We consider the following modified Szabo wave equation:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_0}{c_0\cos(\pi y/2)}\frac{\partial^{y+1}p}{\partial t^{y+1}} = 0, \tag{51}$$

whose orders belong to the interval $(2,3]$ (if $1<y\le 2$). The initial conditions are given by (5) with $m=3$ and boundary conditions are given by (16).

Similarly, we rewrite this equation as

$$\frac{\partial^{y+1}p}{\partial t^{y+1}} + a\frac{\partial^2 p}{\partial t^2} = k\nabla^2 p, \tag{52}$$

where $a=\frac{\cos(\frac{\pi y}{2})}{2\alpha_0}$, $k=\frac{c_0\cos(\frac{\pi y}{2})}{2\alpha_0}$.

Comparing (51) with Eq. (3) (let $a_2=\cdots=a_n=0$, $\alpha=y+1$, $\alpha_1=2$, $f(x,t)=0$,

$u(x,t) \equiv p(x,t)$), we obtain the fundamental solution of (51) with initial conditions (5) with $m = 3$ and Dirichlet boundary conditions (16) as

$$\begin{aligned} u(x,t) &= W_1(x,t) + V_1(x,t) \\ &= \sum_{n=1}^{\infty} c_{n1}^{(3)}(t) \sin \frac{n\pi x}{L} + \frac{\mu_2(t) - \mu_1(t)}{L} x + \mu_1(t), \end{aligned} \tag{53}$$

where

$$\begin{aligned} c_{n1}^{(3)}(t) &= \int_0^t G_{y+1}^{D3}(\tau)\tau^y f_{n1}^{(3)}(t-\tau)d\tau + c_{n1}(0)u_0(t) \\ &\quad + c'_{n1}(0)u_1(t) + c''_{n1}(0)u_2(t), \quad (54) \\ G_\eta^{D3}(t) &= E_{(y-1,y+1),\eta}(-at^{y-1}, -k\lambda_{n1}^2 t^{y+1}), \quad (55) \\ f_{n1}^{(3)}(t-\tau) &= b_n^{-1} \int_0^L f_1^{(3)}(\xi, t-\tau) \sin \lambda_{n1}\xi d\xi, \quad (56) \\ f_1^{(3)}(x,t) &= -\frac{\partial^{y+1} V(x,t)}{\partial t^{y+1}} - a\frac{\partial^2 V(x,t)}{\partial t^2}, \quad (57) \\ u_0(t) &= 1 - k\lambda_{n1}^2 t^{y+1} G_{2+y}^{D3}(t), \quad (58) \\ u_1(t) &= t - k\lambda_{n1}^2 t^{2+y} G_{3+y}^{D3}(t), \quad (59) \\ u_2(t) &= \frac{t^2}{2!} - k\lambda_{n1}^2 t^{3+y} G_{4+y}^{D3}(t). \quad (60) \end{aligned}$$

Here $b_n^{-1}, V(x,t)$ are given in (25) and(18), $c_{n1}(0)$, $c'_{n1}(0) = \frac{\partial c_{n1}(0)}{\partial t}$ and $c''_{n1}(0) = \frac{\partial^2 c_{n1}(0)}{\partial t^2}$ are given in (27).

Case 4. The time-fractional telegraph equation:

$$\begin{aligned} & D_t^{2\alpha} u(x,t) + aD_t^\alpha u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (61) \\ & 0 < x < L, t > 0, \frac{1}{2} < \alpha \le 1, \\ & u(x,0) = \phi_0(x), \ \frac{\partial u(x,0)}{\partial t} = \phi_1(x), \quad (62) \\ & u(0,t) = \mu_1(t), \ u(L,t) = \mu_2(t), \quad (63) \end{aligned}$$

where $1/2 < \alpha < 1$. From Eq. (32), the analytical solution of time-fractional telegraph equation (61), (62) and (63) can be written as the following form:

$$u(x,t) = \sum_{n=1}^{\infty} c_{n1}^{(4)}(t) \sin \frac{n\pi x}{L} + \frac{\mu_2(t) - \mu_1(t)}{L} x + \mu_1(t), \tag{64}$$

where

$$c_{n1}^{(4)}(t) = \int_0^t G_{2\alpha}^{D4}(\tau)\tau^{2\alpha-1} f_{n1}^{(4)}(t-\tau)d\tau + c_{n1}(0)u_0(t) + c'_{n1}(0)u_1(t), \quad (65)$$

$$G_{\eta}^{D4}(t) = E_{(\alpha,\alpha),\eta}(-at^{\alpha}, -k\lambda_{n1}^2 t^{2\alpha}), \quad (66)$$

$$f_{n1}^{(4)}(t-\tau) = b_n^{-1}\int_0^L f_1^{(4)}(\xi, t-\tau)\sin\lambda_{n1}\xi d\xi, \quad (67)$$

$$f_1^{(4)}(x,t) = -(D_t^{2\alpha} + aD_t^{\alpha})V(x,t) + f(x,t), \quad (68)$$

$$u_0(t) = 1 - k\lambda_{n1}^2 t^{2\alpha} G_{1+2\alpha}^{D4}(t), \quad (69)$$

$$u_1(t) = t - at^{2+\alpha} G_{2+\alpha}^{D4}(t) - k\lambda_{n1}^2 t^{1+2\alpha} G_{2+2\alpha}^{D4}(t). \quad (70)$$

Here $b_n^{-1}, V(x,t)$ are given in (25) and(18), $c_{n1}(0)$, $c'_{n1}(0) = \frac{\partial c_{n1}(0)}{\partial t}$ and $c''_{n1}(0) = \frac{\partial^2 c_{n1}(0)}{\partial t^2}$ are given in (27).

5. Numerical example

In this part, we give an numerical example to illustrate validity of our solution techniques. We consider Case 4 in the above section and take the following conditions and parameters. The initial conditions in (62) are taken as

$$\phi_0(x) = 0, \phi_1(x) = 0. \quad (71)$$

The boundary conditions in (63) are taken as

$$\mu_1(t) = t_5, \mu_2(t) = et_5. \quad (72)$$

We take the other parameters as $a = 1$, $k = 1$, $\alpha = 0.8$ in Eq. (57). And it is easy to testify the exact solution is $u(x,t) = e^x t_5$. The exact solutions and analytic solutions are plotted in Fig. 1, which shows the present method is in good agreement with the exact solution.

6. Conclusions

In this paper, we have proposed some new solution techniques to derive fundamental solutions to multi-term modified power law wave equations with nonhomogeneous Dirichlet boundary conditions in a finite domain. The multi- term time fractional derivatives are defined in the Caputo sense, whose orders belong to the intervals $(1,2]$, $[2,3)$, $[2,4)$ or $(0,n)$ $(n>2)$, respectively. These techniques are based on Luchko's Theorem, a spectral representation of the Laplacian operator, a method of separating variables and fractional derivative techniques. By using these techniques, fundamental solutions of the modified power law wave equations and a modified Szabo wave equation are also derived. These methods and techniques can also be extended to other kinds of the multi-term time-space fractional models including fractional Laplacian.

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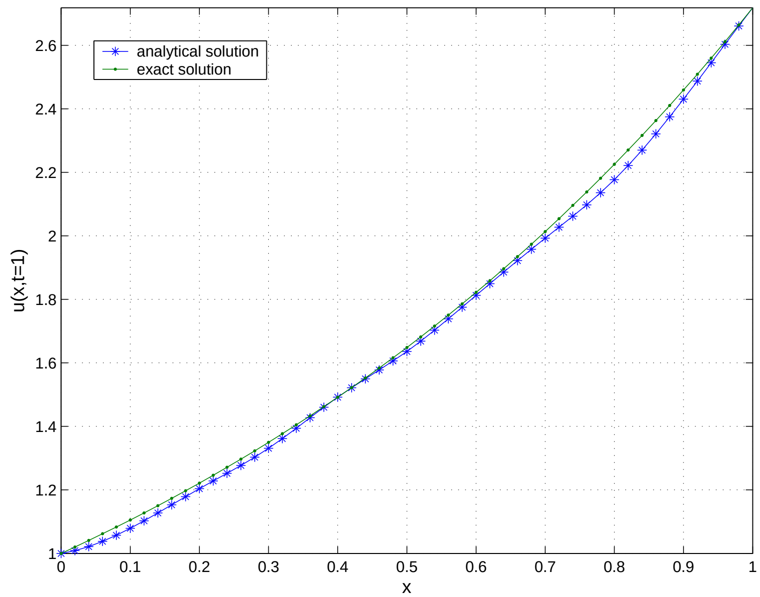

Figure 1. The analytic solution and exact solution.

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