A BOUNDARY VALUE PROBLEM OF FRACTIONAL DIFFERENTIAL INCLUSIONS WITH THREE-POINT NONLOCAL FRACTIONAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, the existence of solutions for a boundary value problem of differential inclusions of order $q \in (1, 2]$ with three-point nonlocal fractional boundary conditions involving convex and non-convex multivalued maps is studied by applying the nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory.

1. INTRODUCTION

In the past decades, the theory of fractional differential equations and inclusions has become an important area of investigation because of its wide applicability in many branches of physics, economics and technical sciences [32, 38, 40, 44, 45]. An important characteristic of fractional-order differential operator that distinguishes it from the integer-order differential operator is its nonlocal behavior, that is, the future state of a dynamical system or process involving fractional derivative depends on its current state as well its past states. In other words, differential equations of arbitrary order describe memory and hereditary properties of various materials and processes. This is one of the features that has contributed to the popularity of the subject and has motivated the researchers to focus on fractional order models, which are more realistic and practical than the classical integer-order models. There has been much interest in developing theoretical analysis like periodicity, asymptotic behavior and numerical methods for fractional equations. For some recent work on the topic, see [2, 3, 8, 9, 10, 11, 27, 41] and the references therein.

Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors, see [39, 42, 46] and references therein. For some recent work on differential inclusions of fractional order, we refer the reader to the references [1, 5, 6, 7, 30, 43].

In this paper, we consider the following fractional differential inclusions with three-point nonlocal fractional boundary conditions

$$\begin{cases} D^q x(t) \in F(t, x(t)), \ t \in [0,1], \ 1 < q \le 2, \\ D^{(q-1)/2} x(0) = 0, \quad a D^{(q-1)/2} x(1) + x(\eta) = 0, \ 0 < \eta < 1, a \in \mathbb{R}^+, \end{cases} \quad (1)$$

where D is the standard Riemann-Liouville fractional derivative, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

Nonlocal conditions were initiated by Bitsadze [14]. As remarked by Byszewski [16, 17, 18], the nonlocal conditions can be more useful than the standard initial condition to describe some physical phenomena. In fact, the nonlocal conditions yield better effects than the classical initial conditions in physics. For example, a nonlocal condition $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$ where $c_i, i = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \le T$. Fractional nonlocal boundary conditions generalize the integer order nonlocal conditions. For application of nonlocal fractional order boundary conditions, see [34, 35]. For recent papers on nonlocal fractional boundary value problems the interested reader is referred to [4], [12], [13], [47] and the references cited therein. Recently, in [11], the authors studied nonlinear fractional integro-differential equations with three-point nonlocal fractional boundary conditions.

The aim here is to establish existence results for the problem (1), when the right hand side is convex as well as nonconvex valued. In the first result (Theorem 3.1) we consider the case when the right hand side has convex values, and prove an existence result via Nonlinear alternative for Kakutani maps. In the second result (Theorem 3.2), we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result (Theorem 3.3), we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

The method used is standard, however their exposition in the framework of problem (1) is new.

2. PRELIMINARIES

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in [0,1]} |x(t)|$. Let $L^1([0,1], \mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)| dt$.

Now we recall some basic definitions on multi-valued maps [22, 31]

For a normed space $(X, \|.\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x \in \mathbb{B}} \{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a

closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. *A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

   (i) $t \longmapsto F(t, x)$ *is measurable for each $x \in \mathbb{R}$;*
   (ii) $x \longmapsto F(t, x)$ *is upper semicontinuous for almost all $t \in [0, 1]$;*
Further a Carathéodory function F is called L^1-Carathéodory if
   (iii) *for each $k > 0$, there exists $\varphi_k \in L^1([0, 1], \mathbb{R}^+)$ such that*

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_k(t)$$

   for all $\|x\| \leq k$ and for a. e. $t \in [0, 1]$.

For each $y \in C([0, 1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0, 1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, 1]\}.$$

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ is a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0, 1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0, 1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Definition 2.2. *A subset $\mathcal{A}$ of $L^1([0, 1], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0, 1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.*

Lemma 2.3. *([15]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a lower semi-continuous (l.s.c.) multivalued operator with nonempty closed and decomposable values. Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0, 1], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a; b)$ and $d(a, B) = \inf_{b \in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space (see [33]).

Definition 2.4. *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*
   (a) $\gamma-Lipschitz$ *if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

   (b) *a contraction if and only if it is $\gamma-Lipschitz$ with $\gamma < 1$.*

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps and Covitz and Nadler fixed point theorem.

Lemma 2.5. *(Nonlinear alternative for Kakutani maps)*[28]. *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*

    (i) *F has a fixed point in $\overline{U}$, or*
    (ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Lemma 2.6. *(Covitz and Nadler)* [20] *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

The following lemma and propositions are used in the sequel.

Lemma 2.7. [26] *Let X be a Banach space. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$, then the operator*

$$\Theta \circ S_F : C([0,1], X) \to \mathcal{P}_{cp,c}(C([0,1], X)), \ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

Proposition 2.8. [19, Proposition III.4] *If Γ_1 and Γ_2 are compact valued measurable multifunctions then the multifunction $t \to \Gamma_1(t) \cap \Gamma_2(t)$ is measurable. If (Γ_n) is a sequence of compact valued measurable multifunctions then $t \to \cap \Gamma_n(t)$ is measurable, and, if $\overline{\cup \Gamma_n(t)}$ is compact $t \to \overline{\cup \Gamma_n(t)}$ is measurable.*

Proposition 2.9. [19, Proposition III.6] *Let X be a separable metric space, $(T, \mathcal{C})$ a measurable space, Γ a multifunction from T to complete non empty subsets of X. If for each open set U in X, $\Gamma^-(U)(= \{t | \Gamma(t) \cap U \neq \emptyset\})$ belongs to $\mathcal{C}$, then Γ admits a measurable selection.*

Let us recall some definitions on fractional calculus [32, 44, 45].

Definition 2.10. *The Riemann-Liouville fractional integral of order q for a continuous function g is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \ q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.11. *The Riemann-Liouville fractional derivative of order q for a continuous function g is defined by*

$$^L D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \ n = [q] + 1, \ q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.12. *The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^c D^q f(t) = {}^L D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \ n - 1 < q < n.$$

Remark 2.13. *(i) If $f(t) \in C^n[0, \infty)$ then*

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(}n)(s)ds = I^{n-q} f^{(n)}(t), \quad t > 0, \ n-1 < q < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) The definition of Riemann-Liouville fractional derivative played a key role in developing the theory of fractional derivatives and integrals. For a class of functions $f(t)$ possessing n continuous derivatives for $t \geq a$, the Riemann-Liuoville definition of fractional derivative is equivalent to its discretized counterpart, the Grunwald-Letnikov derivatives, which is most suited for finite-difference numerical modeling [44]. Hanyga [29] has extended and deepened Biot's approach relating the strain-like quantity to the stress-like quantity in the frequency domain by using Riemann-Liouville fractional derivatives. That is why we prefer to study (1) with Riemann-Liouville differential operator instead of Caputo's derivative.

Lemma 2.14. *For $q > 0$, the general solution of the fractional differential equation $D_{0+}^q x(t) = 0$ is given by*

$$x(t) = c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}, \ i = 1, 2, \ldots, n \ (n = [q] + 1)$.

Lemma 2.15. *(1) If $x \in L^1(0, 1), \nu > \sigma > 0$, then*

$$I^\nu I^\sigma x(t) = I^{\nu+\sigma} x(t), \quad D^\sigma I^\sigma x(t) = I^{\nu-\sigma} x(t), \quad D^\sigma I^\sigma x(t) = x(t). \tag{2}$$

(2) If $\nu > 0, \sigma > 0$, then

$$D^\nu t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}. \tag{3}$$

In view of Lemma 2.14, it follows that

$$I_{0+}^q D_{0+}^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n}, \tag{4}$$

for some $c_i \in \mathbb{R}, \ i = 1, 2, \ldots, n \ (n = [q] + 1)$.

In order to define the solution of (1), we recall the following lemma [11]. For convenince of the reader we include the proof.

Lemma 2.16. *For a given $\sigma \in C([0, 1], \mathbb{R}) \cap L((0, 1), \mathbb{R})$, the unique solution of the boundary value problem*

$$\begin{cases} D^q x(t) = \sigma(t), \ 0 < t < 1, \ 1 < q \leq 2, \\ D^{(q-1)/2} x(0) = 0, \quad aD^{(q-1)/2}x(1) + x(\eta) = 0, \ 0 < \eta < 1, \end{cases} \tag{5}$$

is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - \frac{\Gamma\left(\frac{q+1}{2}\right) t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma\left(\frac{q+1}{2}\right)\right]} \times \\ & \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} \sigma(s)ds \right\}, \end{aligned} \tag{6}$$

where

$$a\Gamma(q) + \eta^{(q-1)}\Gamma\left(\frac{q+1}{2}\right) \neq 0.$$

Proof. In view of Lemma 2.14, the fractional differential equation in (4) is equivalent to the integral equation

$$x(t) = -I^q \sigma(t) + b_1 t^{q-1} + b_2 t^{q-2} = -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + b_1 t^{q-1} + b_2 t^{q-2}, \quad (7)$$

where $b_1, b_2 \in \mathbb{R}$ are arbitrary constants. Applying the boundary conditions for (4), we find that $b_2 = 0$ and

$$b_1 = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma\left(\frac{q+1}{2}\right)\right]} \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} \sigma(s) ds \right\}.$$

Substituting the values of b_1 and b_2 in (7), we obtain (6). This completes the proof. $\square$

Definition 2.17. *A function $x \in AC^1([0,1], \mathbb{R})$ is a solution of the problem (1) if there exists a function $f \in L^1([0,1], \mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,1]$ and*

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds - \frac{\Gamma(\frac{q+1}{2}) t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})\right]} \times$$

$$\times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s) ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(s) ds \right\}.$$

3. MAIN RESULTS

Theorem 3.1. *Assume that*

($\mathbf{H_1}$) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact convex values;*

($\mathbf{H_2}$) *there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in L^1([0,1], \mathbb{R}_+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,1] \times \mathbb{R};$$

($\mathbf{H_3}$) *there exists a number $M > 0$ such that*

$$\frac{\Gamma(q)M}{2\psi(M)\|p\|_{L^1}} > 1. \quad (3.1)$$

Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. Define an operator

$$\Omega(x) = \left\{ h \in C([0,1], \mathbb{R}) : \begin{cases} h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds \\ h(t) = \begin{cases} -\frac{\Gamma(\frac{q+1}{2}) t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})\right]} \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s) ds \right. \\ \left. + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(s) ds \right), \qquad 0 \le t \le 1, \end{cases} \end{cases} \right\}$$

for $f \in S_{F,x}$. It will be shown that the operator Ω satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps.

(i) $\Omega(x)$ *is convex for each* $x \in C([0,1], \mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$h_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s) ds - \frac{\Gamma(\frac{q+1}{2}) t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \times$$
$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f_i(s) ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f_i(s) ds \right), \quad i = 1, 2.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in [0,1]$, we have

$$[\lambda h_1 + (1 - \lambda) h_2](t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1 - \lambda) f_2(s)] ds$$
$$- \frac{\Gamma(\frac{q+1}{2}) t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} [\lambda f_1(s) + (1 - \lambda) f_2(s)] ds \right.$$
$$+ \left. a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} [\lambda f_1(s) + (1 - \lambda) f_2(s)] ds \right).$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\lambda h_1 + (1-\lambda) h_2 \in \Omega(x)$.

(ii) $\Omega(x)$ *maps bounded sets into bounded sets in* $C([0,1], \mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1], \mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1], \mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds - \frac{\Gamma(\frac{q+1}{2}) t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \times$$
$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s) ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(s) ds \right).$$

Using the relations $0 \le t - s \le 1$, $0 \le \eta - s \le \eta$ and $0 \le 1 - s \le 1$ we have

$$|h(t)| \le \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |f(s)| ds + \left| \frac{\Gamma(\frac{q+1}{2}) t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \right| \times$$
$$\times \left(\int_0^\eta \frac{|\eta-s|^{q-1}}{\Gamma(q)} |f(s)| ds + |a| \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} |f(s)| ds \right)$$
$$\le \frac{2}{\Gamma(q)} \psi(\|x\|) \int_0^1 p(s) ds.$$

Thus,

$$\|h\| \le \frac{2}{\Gamma(q)} \psi(r) \|p\|_{L^1}.$$

(iii) Ω *maps bounded sets into equicontinuous sets of* $C([0,1], \mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0,1], \mathbb{R})$. For each

$h \in \Omega(x)$, we obtain

$$
|h(t'') - h(t')|
$$
$$
= \left| \int_0^{t''} \frac{(t'' - s)^{q-1}}{\Gamma(q)} f(s)ds - \int_0^{t'} \frac{(t' - s)^{q-1}}{\Gamma(q)} f(s)ds \right.
$$
$$
- \frac{\Gamma(\frac{q+1}{2})[(t'')^{q-1} - (t')^{q-1}]}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \left(\int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} f(s)ds \right.
$$
$$
\left. \left. + a \int_0^1 \frac{(1 - s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(s)ds \right) \right|
$$
$$
\leq \left| \int_0^{t'} \frac{[(t'' - s)^{q-1} - (t' - s)^{q-1}]}{\Gamma(q)} f(s)ds \right| + \left| \int_{t'}^{t''} \frac{(t'' - s)^{q-1}}{\Gamma(q)} f(s)ds \right|
$$
$$
+ \left| \frac{\Gamma(\frac{q+1}{2})[(t'')^{q-1} - (t')^{q-1}]}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \left(\int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} f(s)ds \right. \right.
$$
$$
\left. \left. + a \int_0^1 \frac{(1 - s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(s)ds \right) \right|.
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t'' - t' \to 0$. As Ω satisfies the above three assumptions, therefore it follows by Ascoli-Arzelá theorem that $\Omega : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ is completely continuous.

(iv) Ω *has a closed graph.* Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$
h_n(t) = \int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} f_n(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \times
$$
$$
\times \left(\int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} f_n(s)ds + a \int_0^1 \frac{(1 - s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f_n(s)ds \right).
$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$
h_*(t) = \int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} f_*(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \times
$$
$$
\times \left(\int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} f_*(s)ds + a \int_0^1 \frac{(1 - s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f_*(s)ds \right).
$$

Let us consider the continuous linear operator $\Theta : L^1([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ so that

$$f \mapsto \Theta(f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \times$$

$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(s)ds \right).$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \right.$$

$$- \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \right.$$

$$\left. \left. + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} (f_n(s) - f_*(s))ds \right) \right\| \to 0 \text{ as } n \to \infty.$$

Thus, it follows by Lemma 2.7 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \times$$

$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f_*(s)ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f_*(s)ds \right),$$

for some $f_* \in S_{F,x_*}$.

(v) Finally, *we discuss a priori bounds on solutions.* Let x be a solution of (1). Then there exists $f \in L^1([0,1], \mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})]} \times$$

$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(s)ds \right).$$

In view of (H_2), for each $t \in [0,1]$, we obtain

$$|x(t)| \leq \frac{2}{\Gamma(q)}\psi(\|x\|) \int_0^1 p(s)ds.$$

Consequently, we have

$$\frac{\Gamma(q)\|x\|}{2\psi(\|x\|)\|p\|_{L^1}} \leq 1,$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1], \mathbb{R}) : \|x\| < M + 1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,1], \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder

type [28], we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This completes the proof. □

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [15] for lower semi-continuous maps with decomposable values.

Theorem 3.2. *Assume that* $(H_2) - (H_3)$ *and the following conditions hold:*

($\mathbf{H_4}$) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

    (**a**) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

    (**b**) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$;

($\mathbf{H_5}$) *for each* $\sigma > 0$, *there exists* $\varphi_\sigma \in L^1([0,1], \mathbb{R}_+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \varphi_\sigma(t) \text{ for all } \|x\| \le \sigma \text{ and for a.e.} t \in [0,1].$$

Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type ([26]). Then from Lemma 2.3, there exists a continuous function $f : C([0,1], \mathbb{R}) \to L^1([0,1], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1], \mathbb{R})$.

Consider the problem

$$\begin{cases} D^q x(t) = f(x(t)), \ 0 < t < 1, \ 1 < q \le 2, \\ D^{(q-1)/2}x(0) = 0, \quad aD^{(q-1)/2}x(1) + x(\eta) = 0, \ 0 < \eta < 1, \ a \in \mathbb{R}^+. \end{cases} \quad (8)$$

Observe that if $x \in AC^1([0,1], \mathbb{R})$ is a solution of (8), then x is a solution to the problem (1). In order to transform the problem (8) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\overline{\Omega}x(t) \;\; = \;\; \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})\right]} \times$$

$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(x(s))ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} f(x(s))ds \right).$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. □

Now we prove the existence of solutions for the problem (1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [20].

Theorem 3.3. *Assume that:*

($\mathbf{H_6}$) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

($\mathbf{H_7}$) $H_d(F(t,x), F(t,\bar{x})) \le m(t)|x - \bar{x}|$ *for almost all* $t \in [0,1]$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in L^1([0,1], \mathbb{R})$ *and* $d(0, F(t,0)) \le m(t)$ *for almost all* $t \in [0,1]$.

Then the boundary value problem (1) has at least one solution on $[0,1]$ if

$$\frac{2\|m\|_{L^1}}{\Gamma(q)} < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$ by the assumption (H_6), so F has a measurable selection (Proposition 2.8). Now we show that the operator $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined at the beginning of the proof of Theorem 3.1, satisfies the assumptions of Lemma 2.6. To show that $\Omega(x) \in \mathcal{P}_{cl}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n \geq 0} \in \Omega(x)$ be such that $u_n \to u \ (n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})\right]} \times$$

$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} v_n(s)ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} v_n(s)ds \right).$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$u_n(t) \to u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})\right]} \times$$

$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} v(s)ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} v(s)ds \right).$$

Hence $u \in \Omega(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega(x), \Omega(\bar{x})) \leq \gamma \|x - \bar{x}\| \text{ for each } x, \bar{x} \in C([0,1],\mathbb{R}).$$

Let $x, \bar{x} \in C([0,1],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,1]$,

$$h_1(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds - \frac{\Gamma(\frac{q+1}{2})t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2})\right]} \times$$

$$\times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} v_1(s)ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} v_1(s)ds \right).$$

By (H_7), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w(t) \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w(t)| \leq m(t)|x(t) - \bar{x}(t)|, \ t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w(t)| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition 2.9), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0, 1]$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0, 1]$, let us define

$$
\begin{aligned}
h_2(t) \quad = \quad & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s) ds - \frac{\Gamma(\frac{q+1}{2}) t^{q-1}}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2}) \right]} \times \\
& \times \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} v_2(s) ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} v_2(s) ds \right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|h_1(t) - h_2(t)| \quad \le \quad & \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)| ds \\
& + \left| \frac{\Gamma(\frac{q+1}{2})}{\left[a\Gamma(q) + \eta^{(q-1)}\Gamma(\frac{q+1}{2}) \right]} \right| \left(\int_0^\eta \frac{|\eta-s|^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)| ds \right. \\
& \left. + |a| \int_0^1 \frac{|1-s|^{(q-1)/2}}{\Gamma(\frac{q+1}{2})} |v_1(s) - v_2(s)| ds \right) \\
& \le \frac{2}{\Gamma(q)} \int_0^1 m(s) \|x - \bar{x}\| ds.
\end{aligned}
$$

Hence

$$
\|h_1(t) - h_2(t)\| \le \frac{2\|m\|_{L^1}}{\Gamma(q)} \|x - \bar{x}\|.
$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$
H_d(\Omega(x), \Omega(\bar{x})) \le \gamma \|x - \bar{x}\| \le \frac{2\|m\|_{L^1}}{\Gamma(q)} \|x - \bar{x}\|.
$$

Since Ω is a contraction, it follows by Lemma 2.6 that Ω has a fixed point x which is a solution of (1). This completes the proof.   □

Example 3.4. *Consider the following fractional inclusion boundary value problem*

$$
\begin{cases}
D^{7/4} x(t) \in F(t, x(t)), \ t \in [0, 1], \ 1 < q \le 2, \\
D^{3/8} x(0) = 0, \quad D^{3/8} x(1) + x(1/4) = 0,
\end{cases}
\tag{9}
$$

where $a = 1$, $q = 7/4$, $\eta = 1/4$, and $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$
F(t, x) = \left[\frac{\sqrt{t}|x|^3}{(|x|^3 + 3)}, \frac{\sqrt{t}|x|}{2(|x| + 1)} \right].
$$

For $f \in F$, we have

$$
|f| \le \max \left\{ \frac{\sqrt{t}|x|^3}{(|x|^3 + 3)}, \frac{\sqrt{t}|x|}{2(|x| + 1)} \right\} \le \sqrt{t}, \ x \in \mathbb{R}, \ t \in [0, 1].
$$

Thus,

$$
\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \le \sqrt{t} = p(t)\psi(\|x\|), \ x \in \mathbb{R},
$$

with $p(t) = \sqrt{t}$, $\psi(\|x\|) = 1$. Further, using the condition (H_3), it is found that $M > 4/3\Gamma(7/4)$. Clearly, all the conditions of Theorem 3.1 are satisfied. So there exists at least one solution of the problem (9) on $[0, 1]$.

Example 3.5. *Consider the following fractional inclusion boundary value problem*

$$\begin{cases} D^{7/2}x(t) \in F(t, x(t)), \ t \in [0, 1], \ 1 < q \leq 2, \\ D^{3/8}x(0) = 0, \qquad D^{3/8}x(1) + x(1/4) = 0, \end{cases} \tag{10}$$

where $a = 1$, $q = 7/2$, $\eta = 1/4$, and $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t, x) = \left[0, \frac{\sin x}{(4 + t)^2} + 3 \right].$$

Then we have

$$\sup\{|u| : u \in F(t, x)\} \leq 3 + \frac{1}{(4 + t)^2},$$

$$H_d(F(t, x), F(t, \bar{x})) \leq \frac{1}{(4 + t)^2}|x - \bar{x}|.$$

Let $m(t) = \dfrac{1}{(4 + t)^2}$. Then

$$H_d(F(t, x, y), F(t, \bar{x}, \bar{y})) \leq m(t)|x - \bar{x}|,$$

and

$$\frac{2\|m\|_{L^1}}{\Gamma(q)} = \frac{4}{75\sqrt{\pi}} < 1.$$

By Theorem 3.3 the problem (10) has at least one solution on $[0, 1]$.

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