COUPLED SYSTEMS OF CHANDRASEKHAR'S QUADRATIC INTEGRAL EQUATIONS

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ABSTRACT. We present existence theorems for coupled systems of the famous Chandrasekhar's quadratic integral equation which has numerous application (cf. [1], [2], [3] and [12]). It arose originally in connection with scattering through a homogeneous semi- infinite plane atmosphere [12].

1. INTRODUCTION

Systems occur in various problems of applied nature, for instance, see ([9]-[11], [26], [27] and [24]). Recently, Su [31] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [32] analyzed the solutions of coupled nonlinear fractional reaction-diffusion equations. The solvability of the coupled systems of integral equations in reflexive Banach space proved in [20]-[22]. Also, a comparison between the classical method of successive approximations (Picard) method and Adomian decomposition method of coupled system of quadratic integral equations proved in [23].

In this paper, the existence of at least one continuous solution for the coupled system of quadratic integral equation of Chandrasekhar's type

$$x(t) = a_1(t) + y(t) \int_0^1 \frac{t \, \lambda_1 \, \phi_1(s)}{t + s} \, y(s) \, ds, \ t \, \in \, I,$$

$$\tag{1}$$

$$y(t) = a_2(t) + x(t) \int_0^1 \frac{t \, \lambda_2 \, \phi_2(s)}{t + s} \, x(s) \, ds, \ t \, \in \, I,$$

will be proved, where λ_i, $i = 1, 2$ are positive constants, $I = [0, 1]$ and ϕ_i, $i = 1, 2$ are essentially bounded functions need not be continuous.
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Quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory. Especially, the so-called quadratic integral equation

of Chandrasekhar's type can be very often encountered in many applications [1]. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see e.g. [1]-[8] and [14]-[15]. However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness we use Tychonoff fixed point theorem. The existence of continuous solutions for some quadratic integral equations was proved by using Schauder-Tychonoff fixed point theorem [30].

Also, the existence of some coupled systems of Chandrasekhar's integral equations will be considered as applications.

The existence of the well-known Chandrasekar's integral equation

$$x(t) = 1 + x(t) \int_0^1 \frac{t \ \lambda \ \phi(s)}{t + s} \ x(s) \ ds, \ t \ \in \ I$$

was proved under certain assumption that the so- called *characteristic function* ϕ is an even polynomial in s [12]. For such *characteristic function*, it is known that the result solutions can be expressed in terms of Chandrasekar's H - function [12]. This function is immediately related to the angular pattern or single scattering . In astrophysical applications of the Chandraskar's equation the only restriction, that $\int_0^1 \phi(s)d \le 1/2$ is treated a necessary condition in [14].

The proof of the main result will be based on the following fixed-point theorem.

Theorem 1. *(Schauder Fixed Point Theorem)*[13].
Let Q be a nonempty, convex, compact subset of a Banach space X, and $T: Q \to Q$ be a continuous map. Then T has at least one fixed point in Q.

Let $\mathbb{R}$ be the set of real numbers whereas $I = [0, 1]$. Let $L_1 = L_1[0, 1]$ be the class of Lebesgue integrable functions on I with the standard norm.

2. Main Theorem

Now, the coupled system (1) will be investigated under the assumptions:

(i) $a_i : I \to \mathbb{R}, \ i = 1, \ 2$ are continuous and bounded with $M_i = \sup_{t \in I} |a_i(t)|$.

(ii) $\phi_i : I \to \mathbb{R}, \ i = 1, 2$ are two functions in L_∞.

Let $C(I)$ be the class of all real functions defined and continuous on I with the norm

$$|| \ x \ || \ = \ \max\{ \ | \ x(t) \ | : \ t \ \in \ I \ \}.$$

Now, We define the Banach space $X = \{x(t) | x(t) \in C(I) \ \}$ endowed with the norm
$||x||_X = \max_{t \in I} |x(t)|, \ Y = \{y(t) | y(t) \in C(I)\}$ endowed with the norm $||y||_Y = \max_{t \in I} |y(t)|$. For $(x, y) \in X \times Y$, let $||(x, y)||_{X \times Y} = \max\{||x||_X, ||y||_Y\}$. Clearly, $(\ X \times Y, ||(x, y)||_{X \times Y} \ )$ is a Banach space.

Define the operator T by

$$T(x, y)(t) \ = \ (T_1 y(t), \ T_2 x(t)),$$

where

$$T_1 y(t) = a_1(t) + y(t) \int_0^1 \frac{\lambda_1 \, t \, \phi_1(s)}{t+s} \, y(s) \, ds, \ t \in I,$$

$$T_2 x(t) = a_2(t) + x(t) \int_0^1 \frac{\lambda_2 \, t \, \phi_2(s)}{t+s} \, x(s) \, ds, \ t \in I,$$

Theorem 2. *Let the assumptions (i) and (ii) be satisfied. If* $4 \, M_i \, \lambda_i \, ||\phi_i||_{L_\infty} \leq 1, \ i = 1, 2,$ *then the coupled system of quadratic integral equations of Chandrasekar's type (1) has at least one solution in* $X \times Y$.

Proof.
Define

$$U = \{u = (x(t), y(t)) | (x(t), y(t)) \in X \times Y : ||x||_X \leq r_2, \ ||y||_Y \leq r_1 \}.$$

Let $r = \max\{ r_1, r_2 \}.$ Then for $(x, y) \in U,$ we have

$$| T_1 y(t) | \leq |a_1(t)| + |y(t)| \int_0^1 \frac{\lambda_1 \, t| \, \phi_1(s)|}{t+s} \, |y(s)| \, ds$$

$$\leq M_1 + r_1^2 \, \lambda_1 \, ||\phi_1||_{L_\infty}, \ ||y||_Y = \max_{t \in I} |y(t)| = r_1.$$

Now

$$||T_1 y(t)|| \leq M_1 + r_1^2 \, \lambda_1 \, ||\phi_1||_{L_\infty} \leq r_1,$$

implies that $r_1 = \frac{1 + \sqrt{1 - 4 \, M_1 \lambda_1 \, ||\phi_1||_{L_\infty}}}{2 \, \lambda_1 \, ||\phi_1||_{L_\infty}}.$

By a similar way we can deduce that

$$||T_2 x(t)|| \leq M_2 + r_2^2 \, \lambda_2 \, ||\phi_2||_{L_\infty} \leq r_2$$

where $r_2 = \frac{1 + \sqrt{1 - 4 \, M_2 \lambda_2 \, ||\phi_2||_{L_\infty}}}{2 \, \lambda_2 \, ||\phi_2||_{L_\infty}}.$

Since $1 - 4 \, M_i \, \lambda_i \, ||\phi_i||_{L_\infty} \geq 0,$ then $r_i, \ i = 1, 2$ are positive.
Therefore,

$$||Tu(t)|| = ||T(x, y)(t)|| = ||(T_1 y(t), T_2 x(t))|| = \max_{t \in I}\{ ||T_1 y(t)|| , ||T_2 x(t)|| \} \leq r$$

then, for every $u = (x, y) \in U$ we have $Tu \in U$ and hence $TU \subset U.$
It is clear that the set U is nonempty, bounded, closed and convex.
Also, the operator $T : U \rightarrow C(I) \times C(I)$ is a continuous operator. Now, for $u = (x, y) \in U,$ and for each $t_1, t_2 \in I$ (without loss of generality assume that $0 < t_1 < t_2,$) we get

$$(T_2 x)(t_2) - (T_2 x)(t_1) = a_2(t_2) - a_2(t_1)$$

$$+ x(t_2) \int_0^1 \frac{\lambda_2 \, t_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds - x(t_1) \int_0^1 \frac{\lambda_2 \, t_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds$$

$$+ x(t_1) \int_0^1 \frac{\lambda_2 \, t_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds - x(t_1) \int_0^1 \frac{\lambda_2 \, t_1 \, \phi_2(s)}{t_1 + s} \, x(s) \, ds$$

$$\leq a_2(t_2) - a_2(t_1) + [x(t_2) - x(t_1)] \int_0^1 \frac{\lambda_2 \, t_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds$$

$$+ x(t_1) \, [\int_0^1 \frac{\lambda_2 \, t_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds - \int_0^1 \frac{\lambda_2 \, t_1 \, \phi_2(s)}{t_1 + s} \, x(s) \, ds],$$

but $t_1 < t_2 \Rightarrow t_1 + s < t_2 + s \Rightarrow \frac{1}{t_1+s} > \frac{1}{t_2+s} \Rightarrow -\frac{1}{t_1+s} < -\frac{1}{t_2+s}$.

Then

$$\int_0^1 \frac{\lambda_2 \, t_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds - \int_0^1 \frac{\lambda_2 \, t_1 \, \phi_2(s)}{t_1 + s} \, x(s) \, ds \leq \int_0^1 \frac{\lambda_2 \, \phi_2(s) \, t_2}{t_2 + s} \, x(s) \, ds - \int_0^1 \frac{\lambda_2 \, \phi_2(s) \, t_1}{t_2 + s} \, x(s) \, ds$$

$$\leq (t_2 - t_1) \int_0^1 \frac{\lambda_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds \leq (t_2 - t_1) \, r_2 \, \lambda_2 \, ||\phi_2||_{L_\infty} \ln(t_2 + 1)$$

$$\leq (t_2 - t_1) \, r_2 \, \lambda_2 \, ||\phi_2||_{L_\infty} \ln 2$$

Then

$$|\int_0^1 \frac{\lambda_2 \, t_2 \, \phi_2(s)}{t_2 + s} \, x(s) \, ds - \int_0^1 \frac{\lambda_2 \, t_1 \, \phi_2(s)}{t_1 + s} \, x(s) \, ds| \leq r_2 \, \lambda_2 \, ||\phi_2||_{L_\infty} |t_2 - t_1| \ln 2,$$

Then we get

$$| \, (T_2 x)(t_2) - (T_2 x)(t_1) \, | \leq | \, a_2(t_2) - a_2(t_1) \, | + \lambda_2 \, ||\phi_2||_{L_\infty} \, |x(t_2) - x(t_1)| \int_0^1 \frac{t_2}{t_2 + s} \, |x(s)| \, ds$$

$$+ r_2 \, ||\phi_2||_{L_\infty} \, \lambda_2 \, |t_2 - t_1| \, |x(t_1)| \ln 2$$

i.e.,

$$| \, (T_2 x)(t_2) - (T_2 x)(t_1) \, | \leq | \, a_2(t_2) - a_2(t_1) \, | + \lambda_2 \, r_2 \, ||\phi_2||_{L_\infty} |x(t_2) - x(t_1)| + r_2^2 \, \lambda_2 \, ||\phi_2||_{L_\infty} \, |t_2 - t_1| \ln 2$$

As done above we can obtain

$$| \, (T_1 y)(t_2) - (T_1 y)(t_1) \, | \leq | \, a_1(t_2) - a_1(t_1) \, | + \lambda_1 \, r_1 \, ||\phi_1||_{L_\infty} \, |y(t_2) - y(t_1)| + r_1^2 \, \lambda_1 \, ||\phi_1||_{L_\infty} \, |t_2 - t_1| \ln 2$$

Now, from the definition of the operator T, we get

$$\begin{aligned} Tu(t_2) - Tu(t_1) &= T(x,y)(t_2) - T(x,y)(t_1) \\ &= (T_1 y(t_2), \, T_2 x(t_2)) - (T_1 y(t_1), \, T_2 x(t_1)) \\ &= (T_1 y(t_2) - T_1 y(t_1), \, T_2 x(t_2) - T_2 x(t_1)), \end{aligned}$$

and

$$||Tu(t_2) - Tu(t_1)|| = \max_{t_1,t_2 \in I} \{||T_1 y(t_2) - T_1 y(t_1)||, \, ||T_2 x(t_2) - T_2 x(t_1)||\}$$

$$\leq | \, a_1(t_2) - a_1(t_1) \, | + | \, a_2(t_2) - a_2(t_1) \, | + \lambda_2 \, r_2 \, ||\phi_2||_{L_\infty} \, |x(t_2) - x(t_1)| + r_2^2 \, ||\phi_2||_{L_\infty} \, \lambda_2 \, |t_2 - t_1| \ln 2$$

$$+ \lambda_1 \, r_1 \, ||\phi_1||_{L_\infty} \, |y(t_2) - y(t_1)| + r_1^2 \, \lambda_1 \, ||\phi_1||_{L_\infty} \, |t_2 - t_1| \ln 2.$$

Hence

$$| \, t_2 - t_1 \, | < \delta \Longrightarrow || \, Tu(t_2) - Tu(t_1) \, || < \varepsilon(\delta),$$

This means that the functions of TU are equi-continuous on I. Then by the Arzela-Ascoli Theorem [13] the closure of TU is compact subset of $X \times Y$.

Since all conditions of the Schauder Fixed-point Theorem 1 hold, then T has a fixed point in U.

It remains to prove the continuity of the solution $u = (x,y)(t)$ of the coupled system (1). For $t, \, t_0 \in I$ we have

$$\begin{aligned} u(t) - u(t_0) &= (x,y)(t) - (x,y)(t_0), \\ &= (x(t), \, y(t)) - (x(t_0), \, y(t_0)) \\ &= (x(t) - x(t_0), \, y(t) - y(t_0)), \end{aligned}$$

and

$$||u(t) - u(t_0)|| = \max_{t_0,t \in I} \{||y(t) - y(t_0)||, \, ||x(t) - x(t_0)||\}$$

$$\leq | \, a_1(t) - a_1(t_0) \, | + | \, a_2(t) - a_2(t_0) \, | + \lambda_2 \, r_2 \, ||\phi_2||_{L_\infty} \, |x(t) - x(t_0)| + r_2^2 \, ||\phi_2||_{L_\infty} \, \lambda_2 \, |t - t_0| \ln 2$$

$$+ \lambda_1 \ r_1 \ ||\phi_1||_{L_\infty} \ |y(t) - y(t_0)| + \ r_1^2 \ \lambda_1 \ ||\phi_1||_{L_\infty} \ |t - t_0| \ln 2.$$

Hence

$$| \ t - t_0 \ | < \delta \implies \ || \ u(t) \ - \ u(t_0) \ || < \varepsilon(\delta),$$

This proves that the solution of the coupled system (1) is continuous. which completes the proof. ∎

3. Applications

As a particular case of Theorem 2 we can obtain an existence theorem for the coupled system of the well-known Chandrasekar's integral equations

$$x(t) \ = \ 1 + y(t) \ \int_0^1 \frac{t \ \lambda_1 \ \phi_1(s)}{t + s} \ y(s) \ ds, \ t \ \in \ I,$$

$$(2)$$

$$y(t) \ = \ 1 + x(t) \ \int_0^1 \frac{t \ \lambda_2 \ \phi_2(s)}{t + s} \ x(s) \ ds, \ t \ \in \ I,$$

Theorem 3. *Let ϕ_i, $i = 1, 2$ be two functions in L_∞. If $4 \ \lambda_i \ ||\phi_i||_{L_\infty} \leq 1$, $i = 1, 2$, then the coupled system of Chandrasekar's integral equations (2) has at least one solution in $X \times Y$.*

Proof:

The proof straight forward as done in the proof of theorem 2 where

$$a_1(t) = a_2(t) = 1 \Rightarrow M_1 = M_2 = 1.$$

Then $r_i = \frac{1 + \sqrt{1 - 4 \ \lambda_i \ ||\phi_i||_{L_\infty}}}{2 \ \lambda_i \ ||\phi_i||_{L_\infty}}$, $i = 1, 2$ and hence $r = \max\{r_1, \ r_2\}$.

Since $1 - 4 \ \lambda_i \ ||\phi_i||_{L_\infty} \geq 0$, then r_i, $i = 1, 2$ are positive.

Remark:

In case of $\lambda_i = 1$, $i = 1, 2$. Then $||\phi_i||_{L_\infty} \leq \frac{1}{4}$ and $r_i \leq 4$, $i = 1, 2$.

Therefore, the coupled system (1) with ($M_i = 1$ and $\lambda_i = 1$, $i = 1, 2$)

$$x(t) \ = \ 1 + y(t) \ \int_0^1 \frac{t \ \phi_1(s)}{t + s} \ y(s) \ ds, \ t \ \in \ I,$$

$$y(t) \ = \ 1 + x(t) \ \int_0^1 \frac{t \ \phi_2(s)}{t + s} \ x(s) \ ds, \ t \ \in \ I,$$

has at least one solution in

$$\{u = (x(t), y(t)) | (x(t), y(t)) \in X \times Y : \ ||(x, y)||_{X \times Y} \leq 4\}.$$

Example 1:

Consider the following coupled system of Chandrasekar's integral equations

$$x(t) \ = \ 1 + \ \frac{1}{10} \ y(t) \ \int_0^1 \frac{s \ t}{t + s} \ y(s) \ ds, \ t \ \in \ I$$

$$(3)$$

$$y(t) \ = \ 1 + \ \frac{3}{20} \ y(t) \ \int_0^1 \frac{s^{\frac{-1}{6}} \ t}{t + s} \ x(s) \ ds, \ t \ \in \ I,$$

where $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{2}$, $\phi_1(s) = \frac{s}{5}$ and $\phi_2(s) = \frac{s^{\frac{-1}{6}}}{10}$.

Also, $||\phi_1||_{L_\infty} = \frac{1}{10}$ and $||\phi_2||_{L_\infty} = \frac{1}{8}$, then the condition $4 \lambda_i ||\phi_i||_{L_\infty} \leq 1$, $i = 1, 2$ is satisfied and $r = 9$.
Therefore, the coupled system (3) has at least one solution in

$$\{u = (x(t), y(t)) | (x(t), y(t)) \in X \times Y : \ ||(x, y)||_{X \times Y} \leq 9\}.$$

Example 2:
Consider the following coupled system of Chandrasekar's integral equations

$$x(t) \ = \ 1 + \ \frac{1}{5} \ y(t) \ \int_0^1 \frac{exp(-s) \ t}{t + s} \ y(s) \ ds, \ t \ \in \ I$$

$$\tag{4}$$

$$y(t) \ = \ 1 + \ \frac{1}{3} \ y(t) \ \int_0^1 \frac{t}{(s + 1)(t + s)} \ x(s) \ ds, \ t \ \in \ I,$$

where $\lambda_1 = \frac{1}{5}$, $\lambda_2 = 1$, $\phi_1(s) = \frac{1}{3} exp(-s)$ and $\phi_2(s) = \frac{1}{s+1}$.

Also, $||\phi_1||_{L_\infty} \leq 1$ and $||\phi_2||_{L_\infty} = \frac{\ln 2}{3}$, then the condition $4 \lambda_i ||\phi_i||_{L_\infty} \leq 1$, $i = 1, 2$ is satisfied and $r = 9$.
Therefore, the coupled system (4) has at least one solution in

$$\{u = (x(t), y(t)) | (x(t), y(t)) \in X \times Y : \ ||(x, y)||_{X \times Y} \leq 9\}.$$

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