Electronic Journal of Mathematical Analysis and Applications
Vol. 1(1) Jan. 2013, pp. 110-137.
ISSN: 2090-792X (online)
http://ejmaa.6te.net/

DEFORMATION OF AN INFINITE, ELLIPTICAL CYLINDER OF AN ELASTIC MAGNETIZABLE MATERIAL, SUBJECTED TO AN EXTERNAL MAGNETIC FIELD AND UNIFORM BULK HEATING BY A BOUNDARY INTEGRAL METHOD

A. R. EL DHABA, A. F. GHALEB AND M. S. ABOU-DINA

Abstract. We solve the first fundamental problem of elasticity for an infinite, elliptical cylinder of a magnetizable material, subjected to an external, initially uniform magnetic field and to a uniform bulk heating. The cylinder is placed in a variable ambient temperature.

The complete solution of the uncoupled problem is obtained using a boundary integral method, previously introduced by two of the authors. The results are discussed in detail in the case of a constant ambient temperature and the combined effect of the bulk heating and the weak magnetic field is put in evidence.

1. Introduction

We obtain the deformation occurring in a long elliptic cylinder of elastic magnetizable material subjected to an external magnetic field, to bulk heating and surrounded by an ambient temperature, within the frame of the uncoupled theory. The solution for this problem is carried out within the linear uncoupled theory of magneto-thermoelasticity, using a boundary integral method previously introduced by two of the authors (AFG and MSA) in [1], [2], [3]. The dependence of the magnetic permeability of the body on magnetostriction is taken in consideration through two material parameters. An elliptic system of coordinates is used as in [4]. First, the Magnetostatic problem is solved for the vector potential everywhere in space, from which one deduces the magnetic field distribution. The results are compared with those in ([5], (p. 357)). Two important functions of position, the so-called magnetic parts of the mechanical displacements, are also obtained by line integrals. Then, the solution for the uncoupled heat problem is obtained under uniform bulk heating and radiation condition at the boundary. The thermal parts of the mechanical displacements are then calculated. Finally, the elastic problem is solved in stresses using Airy's stress function.

2000 *Mathematics Subject Classification.* 74B05, 74F05, 74F15.
Key words and phrases. Elliptical cylinder, thermomagnetoelasticity, variable ambient temperature, magnetic field, the boundary integral method.
Submitted Oct. 15, 2012. Published Jan 1, 2013.

2. Description for the domain

Let D be a two-dimensional, bounded, simply connected region occupied by the material and bounded by an infinite cylinder with boundary C having, in Cartesian coordinates, the parametric representations

$$x = x(s), \qquad y = y(s). \tag{1a}$$

where s- is the arc length as measured on C in the usual positive sense. The unit outwards normal $\mathbf{n}$ and the unit vector tangent τ to C at a general boundary point are expressed as

$$\boldsymbol{\tau} = \left(\dot{x}\,(s)\,, \dot{y}\,(s)\right), \qquad \mathbf{n} = \left(\dot{y}\,(s)\,, -\dot{x}\,(s)\right). \tag{1b}$$

3. Equations of Thermo-magnetoelasticity

We shall quote the general equations of static, linear uncoupled thermo-magnetoelasticity without proof according to [3], to be used throughout the text.

3.1. **Equation of heat conduction.** In the steady state, the temperature T, as measured from a reference temperature T_0, satisfies Poisson's equation and the heat flow vector $\mathbf{Q}$ is given by Fourier law

$$\nabla^2 T = -\frac{\zeta_0}{K}, \qquad \mathbf{Q} = -K\nabla T, \tag{2}$$

where ζ_0 is the constant rate of heat supply per unit volume and K is the coefficient of heat conduction. More extensive forms for $\mathbf{Q}$ may be found in ([6], (p. 101)). Where ζ_H represents the contribution of the magnetic field (Joule heat supply) and ζ_T is the rate of heat supply from all other sources.

The general solution of equation (2) is taken as

$$T = T_h + T_p, \tag{3}$$

where T_h is the harmonic part of T and T_p is a particular solution of Poisson's equation (2):

$$T_p = -\frac{\zeta_0}{4K}\left(x^2 + y^2\right). \tag{4}$$

3.2. **Equations of Magnetostatic.** (i) *The field equations.*

Inside the body and in the absence of volume electric charges, the field equations of Magnetostatic written in the SI system of units are:

$$curl\,\mathbf{H} = \mathbf{0}, \qquad div\,\mathbf{B} = 0, \tag{5}$$

where $\mathbf{H}$ is the magnetic field vector, $\mathbf{B}$- the magnetic induction vector.

(ii) *The magnetic constitutive relations.*

$$B_i = \mu^* \mu_{ij} H_j, \quad i, j = 1, 2, 3, \tag{6}$$

where, μ_{ij} are the components of the tensor of magnetic permeability of the body, assumed to depend linearly on strain according to the law

$$\mu_{ij} = \mu_0 \delta_{ij} + \mu_1 I_1 \delta_{ij} + \mu_2 \varepsilon_{ij} \quad i, j = 1, 2, 3, \tag{7}$$

where μ_0, μ_1 and μ_2 are constants with obvious physical meaning, I_1 is the first invariant of the strain tensor with components ε_{ij} and δ_{ij} denote the Kronecker delta symbols. Constant μ^* refers to the magnetic permeability of vacuum with value $\mu^* = 10^{-7} H.m^{-1}$.

An electrical analogue for the dielectric tensor components under isothermal conditions may be found elsewhere ([6], (p. 64)) and also [7].

Since we are assuming a quadratic dependence of strain on the magnetic field (magnetostriction), upon substitution of (7) into (6) one may neglect, as an approximation, the third and higher order terms in the magnetic field compared to the first order term and write [7]

$$\mathbf{B} = \mu^* \mu_0 \mathbf{H}. \tag{8}$$

3.3. **The magnetic vector potential.** In view of the solenoidal property (equation 5) of the magnetic induction and taking (8) into account, the magnetic field vector may be represented in the form

$$\mathbf{H} = \frac{1}{\mu^* \mu_0} \nabla \times \mathbf{A}, \tag{9}$$

where $\mathbf{A}$ is the magnetic vector potential. It is usual, for the sake of uniqueness of the solution, to impose the condition

$$\nabla \cdot \mathbf{A} = 0. \tag{10}$$

The fundamental assumption to be adopted in the sequel is that the vector potential everywhere in space is parallel to the z-axis. Hence

$$\mathbf{A} = A\,(x, y)\,\mathbf{k}. \tag{11}$$

Condition (10) is now identically satisfied, which means that function A still has some indeterminacy. In fact, it is defined up to an arbitrary additive constant.

One gets at each point of the region D

$$\nabla^2 A = 0. \tag{12}$$

In the free space surrounding the body, the equations of Magnetostatic hold with $\mu_0 = 1$ and $\mu_1 = \mu_2 = 0$. One has

$$\nabla^2 A^* = 0, \tag{13}$$

where $(*)$ refers to free space.

The solutions of equations (12) and (13) is looked for in the form

$$A = A_h, \qquad A^* = A_\infty + A_r^*, \tag{14}$$

where A_h is a harmonic function, A_r^* is the harmonic part of A^* which has a regular behavior at infinity and A_∞ is a known function which satisfies Laplace's equation but does not vanish at infinity. Function A_r^* represents the modification of the magnetic vector potential outside the body, due to the presence of the body.

For mathematical reasons concerning the proposed procedure, we assume in the sequel that

$$|A_r^*| = O\left(r^{-\delta}\right) \quad (\delta > 0) \quad \text{as} \quad r = \left(x^2 + y^2\right)^{1/2} \to \infty,$$

by which the arbitrary additive constant intervening in the definition of the magnetic vector potential has been determined.

The equations of Magnetostatic are complemented by the following two magnetic boundary conditions: The continuity of the normal component of the magnetic induction. This reduces to the condition of continuity of the vector potential, and the continuity of the tangential component of the magnetic field (in the absence of surface electric currents). These implies

$$A = A^*, \qquad \frac{1}{\mu_0}\frac{\partial A}{\partial n} = \frac{\partial A^*}{\partial n} \qquad \text{on } C. \tag{15}$$

This condition, together with the vanishing condition at infinity of A_r^* are sufficient for the complete determination of the two harmonic functions A_h and A_r^*.

If an external magnetic field

$$\mathbf{H}_0 = (H_0 \cos\alpha)\,\mathbf{i} + (H_0 \sin\alpha)\,\mathbf{j}, \tag{16}$$

is applied to the body, the corresponding expression for the magnetic vector potential far away from the body is

$$A_\infty = \mu^* H_0 \left(y\cos\alpha - x\sin\alpha\right). \tag{17}$$

Then

$$H_x = \frac{1}{\mu^*\mu_0}\frac{\partial A_h}{\partial y}, \quad H_y = -\frac{1}{\mu^*\mu_0}\frac{\partial A_h}{\partial x} \tag{18}$$

and

$$H_z = 0. \tag{19}$$

3.4. ***Equations of elasticity.*** *Equations of equilibrium*

In the absence of body forces of non-electromagnetic origin, the equations of mechanical equilibrium in the plane read

$$\nabla_j \sigma_{ij} = 0, \quad i,j = 1,2, \tag{20}$$

where σ_{ij} are the components of the "total" stress tensor and ∇_j denotes covariant differentiation.

Equations (20) are satisfied if the only identically non-vanishing stress components σ_{xx}, σ_{yy} and σ_{xy} are defined through the stress function U by the relations

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}. \tag{21}$$

Constitutive relations

The generalized Hooke's law may [[8], [9] and [6], p. 64, see also [7] for the electric analogue] where $H^2 = H_i H_i$ are

$$\begin{aligned} \sigma_{xx} &= \frac{\nu E}{(1+\nu)(1-2\nu)}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \frac{E}{1+\nu}\frac{\partial u}{\partial x} - \frac{\alpha E}{1-2\nu}T \\ &\quad + \frac{1}{2}\mu^*\left(\mu_0 - \mu_1 - \mu_2\right)H_x^2 - \frac{1}{2}\mu^*\left(\mu_0 + \mu_2\right)H_y^2, \end{aligned} \tag{22a}$$

$$\begin{aligned} \sigma_{yy} &= \frac{\nu E}{(1+\nu)(1-2\nu)}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \frac{E}{1+\nu}\frac{\partial v}{\partial y} - \frac{\alpha E}{1-2\nu}T \\ &\quad - \frac{1}{2}\mu^*\left(\mu_0 + \mu_2\right)H_x^2 + \frac{1}{2}\mu^*\left(\mu_0 - \mu_1 - \mu_2\right)H_y^2, \end{aligned} \tag{22b}$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] + \mu^*\left(\mu_0 - \frac{1}{2}\mu_1\right)H_x H_y, \tag{22c}$$

where E, ν and α are Young's modulus, Poisson's ratio and the coefficient of linear thermal expansion respectively for the considered elastic medium.

Besides the local equilibrium conditions guaranteed by the representation (20), the global equilibrium conditions of the body must also be satisfied. This requires that the resultant force and the resultant couple applied to the boundary of the body must vanish.

Kinematical relations

These are the relations between the strain tensor components ε_{ij} and the displacement vector components u_i. In Cartesian components

$$\frac{\partial u}{\partial x} = \varepsilon_{xx}, \quad \frac{\partial v}{\partial y} = \varepsilon_{yy}, \quad \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = \varepsilon_{xy}. \tag{23}$$

Compatibility condition

The condition of solvability of equations (23) is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}. \tag{24}$$

These equations are complemented with the proper boundary conditions, to be discussed in detail in subsequent sections.

3.5. **Equation for the stress function.** An equation for the stress function may be obtained from the general field equations written in covariant form [9].

Solving (22a, 22b, 22c) for the strain components and using 21 one obtains

$$\frac{2E}{1+\nu}\varepsilon_{xx} \equiv \frac{2E}{1+\nu}\frac{\partial u}{\partial x} = (1-2\nu)\,\Delta U + \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 U}{\partial x^2} + 2\alpha ET +$$
$$(1-2\nu)\,\mu^*\left(\frac{1}{2}\mu_1 + \mu_2\right)H^2 + \mu^*\left(\mu_0 - \frac{1}{2}\mu_1\right)\left(H_y^2 - H_x^2\right), \tag{25a}$$

$$\frac{2E}{1+\nu}\varepsilon_{yy} \equiv \frac{2E}{1+\nu}\frac{\partial v}{\partial y} = (1-2\nu)\,\Delta U + \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + 2\alpha ET +$$
$$(1-2\nu)\,\mu^*\left(\frac{1}{2}\mu_1 + \mu_2\right)H^2 + \mu^*\left(\mu_0 - \frac{1}{2}\mu_1\right)\left(H_x^2 - H_y^2\right), \tag{25b}$$

$$\frac{E}{1+\nu}\varepsilon_{xy} \equiv \frac{E}{1+\nu}\frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = -\frac{\partial^2 U}{\partial x \partial y} - \mu^*\left(\mu_0 - \frac{1}{2}\mu_1\right)H_x H_y. \tag{25c}$$

Substituting from (25a, 25b, 25c) into (24) and performing some transformations using the equations of Magnetostatic and (2), one finally arrives at the following inhomogeneous biharmonic equation for the stress function U:

$$\Delta^2 U = -\frac{\alpha E}{1-\nu}\Delta T - \frac{1-2\nu}{2(1-\nu)}\mu^*\left(\frac{1}{2}\mu_1 + \mu_2\right)\Delta H^2 +$$
$$\frac{1}{1-\nu}\mu^*\left(\mu_0 - \frac{1}{2}\mu_1\right)\left(|curl\,\mathbf{H}|^2 - \mathbf{H}.curl\,curl\,\mathbf{H}\right), \tag{26a}$$

which, in view of equation (2) and the constancy of the current density, reduces to

$$\Delta^2 U = \frac{\alpha E \zeta}{(1-\nu)K} - \frac{1-2\nu}{2(1-\nu)}\mu^*\left(\frac{1}{2}\mu_1 + \mu_2\right)\Delta H^2. \tag{26b}$$

This is the same as the biharmonic equation given in [9], taking in consideration the differences in the used systems of units, but is different from that produced by Yuan [10] because different stress functions and different stresses are involved.

The solution of (26b) is sought in the form

$$U = x\Phi + y\Phi^c + \Psi + U_p, \tag{27}$$

where Φ and Ψ are two harmonic functions belonging to the class of functions $C^2(D) \cap C^1(\overline{D})$, $\overline{D}$ denotes the closure of D and superscript 'c' denotes the harmonic conjugate. Function U_p is any particular solution of the equation

$$\Delta U_p = -\frac{\alpha E}{1-\nu} T_p - \frac{1-2\nu}{2(1-\nu)} \mu^* \left(\frac{1}{2}\mu_1 + \mu_2 \right) H^2. \tag{28}$$

From 27, then

$$\Delta U = 4\frac{\partial \Phi}{\partial x} + \Delta U_P = 4\frac{\partial \Phi^c}{\partial y} + \Delta U_p. \tag{29}$$

Using equations 27 and 29 then equations (22a, 22b) may be written as

$$\frac{E}{1+\nu}\frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + 4(1-\nu)\frac{\partial \Phi}{\partial x} + \alpha E T_h + \frac{E}{1+\nu} M_H, \tag{30a}$$

$$\frac{E}{1+\nu}\frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + 4(1-\nu)\frac{\partial \Phi^c}{\partial y} + \alpha E T_h + \frac{E}{1+\nu} S_H, \tag{30b}$$

where

$$M_H = \frac{1}{2}\frac{1+\nu}{E}\mu^* \left(\mu_0 - \frac{1}{2}\mu_1 \right) \left(H_y^2 - H_x^2 \right), \tag{31a}$$

$$S_H = \frac{1}{2}\frac{1+\nu}{E}\mu^* \left(\mu_0 - \frac{1}{2}\mu_1 \right) \left(H_x^2 - H_y^2 \right). \tag{31b}$$

Let A_h^c be the harmonic conjugate function to A_h. This function is defined up to an additive arbitrary constant, which may be determined by fixing the value of the function at an arbitrarily chosen point of $\overline{D}$.

Introducing now two new functions

$$N_H = -\frac{1+\nu}{E}\mu^* \left(\mu_0 - \frac{1}{2}\mu_1 \right) H_x H_y, \tag{31c}$$

$$R_H = -\frac{1+\nu}{E}\mu^* \left(\mu_0 - \frac{1}{2}\mu_1 \right) H_x H_y, \tag{31d}$$

it can be easily verified using the equations of Magnetostatic that

$$\frac{\partial M_H}{\partial y} = \frac{\partial N_H}{\partial x}, \qquad \text{and} \qquad \frac{\partial R_H}{\partial y} = \frac{\partial S_H}{\partial x}. \tag{32}$$

Clearly, relations (32) are not affected by the addition of an arbitrary constant to the function A_h^c.

Equations (32), imply the existence of two single-valued functions u_H and v_H in D such that

$$M_H = \frac{\partial u_H}{\partial x}, \qquad N_H = \frac{\partial u_H}{\partial y}, \qquad R_H = \frac{\partial v_H}{\partial x}, \qquad S_H = \frac{\partial v_H}{\partial y}. \tag{33}$$

With these notations, equations (30a, 30b) take the form

$$\frac{E}{1+\nu}\frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + 4(1-\nu)\frac{\partial \Phi}{\partial x} + \alpha E T_h + \frac{E}{1+\nu}\frac{\partial u_H}{\partial x}, \tag{34a}$$

$$\frac{E}{1+\nu}\frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + 4(1-\nu)\frac{\partial \Phi^c}{\partial y} + \alpha E T_h + \frac{E}{1+\nu}\frac{\partial v_H}{\partial y}. \tag{34b}$$

It is to be noted that the addition of a constant to the function A_h^c amounts to adding linear terms in y to u_H and linear terms in x to v_H, which do not alter (34a, 34b).

3.6. **A representation for the mechanical displacement vector components.** Differentiating (34a) w.r.t. y and integrating the resulting equation w.r.t. x after using (32) and (33), one gets

$$\frac{E}{1+\nu}\frac{\partial u}{\partial y} = -\frac{\partial^2 U}{\partial x \partial y} + 4\,(1-\nu)\,\frac{\partial \Phi}{\partial y} - \alpha E T_h^c + \frac{E}{1+\nu}\frac{\partial u_H}{\partial y} + f(y), \tag{35a}$$

where $f(y)$ is an arbitrary function of y and T_h^c is the harmonic conjugate of function T_h. This conjugate function is defined up to an arbitrary additive constant, which may be determined by fixing the value of the function at an arbitrarily chosen point in $\overline{D}$. A similar procedure with (34b), using (32) and (33), yields

$$\frac{E}{1+\nu}\frac{\partial v}{\partial x} = -\frac{\partial^2 U}{\partial x \partial y} + 4\,(1-\nu)\,\frac{\partial \Phi^c}{\partial x} + \alpha E T_h^c + \frac{E}{1+\nu}\frac{\partial v_H}{\partial x} + g(x), \tag{35b}$$

where $g(x)$ is an arbitrary function of x.

It can be shown [3] that both $f(y)$ and $g(x)$ are constant functions and therefore may be eliminated since their contribution represents a rigid body displacement. A similar argument holds for any constant added to the expressions for A_h^c or T_h^c. For the following procedure, it will be assumed that each one of these two functions has been completely determined by assigning to it a given value at some arbitrarily chosen point in $\overline{D}$.

From (34a, 34b) and (35a, 35b), by line integrations along any path inside the region D joining an arbitrary chosen fixed point M_0 (which may be arbitrarily chosen in $\overline{D}$) to a general field point M, one obtains

$$\frac{E}{1+\nu}u = -\frac{\partial U}{\partial x} + 4(1-\nu)\Phi + \frac{E}{1+\nu}\left(u_T + u_H\right), \tag{36a}$$

$$\frac{E}{1+\nu}v = -\frac{\partial U}{\partial y} + 4(1-\nu)\Phi^c + \frac{E}{1+\nu}\left(v_T + v_H\right), \tag{36b}$$

where

$$u_T = \alpha\,(1+\nu)\int_{M_0}^{M}\left(T_h dx - T_h^c dy\right), \quad u_H = \int_{M_0}^{M}\left(M_H dx + N_H dy\right), \tag{37a}$$

$$v_T = \alpha\,(1+\nu)\int_{M_0}^{M}\left(T_h^c dx + T_h dy\right), \quad v_H = \int_{M_0}^{M}\left(R_H dx + S_H dy\right), \tag{37b}$$

the integration constants being absorbed into functions Φ and Φ^c which are yet to be determined.

The mechanical displacement components u and v given by expressions (36a, 36b) are single-valued functions in D, since the line integrals in (37a) are path independent due to the Cauchy-Riemann conditions satisfied by the functions T_h and T_h^c, and the line integrals in (37b) are path independent as well, due to relations (32).

4. Boundary integral representation of the solution

The problem now reduces to the determination of nine harmonic functions: A_h, A_h^c, A_r^*, T_h, T_h^c, Φ, Φ^c, Ψ and Ψ^c (although the conjugate function Ψ^c does not appear in the expressions given above for the stress and displacement functions, it will be required for the subsequent analysis within the proposed boundary integral method). We use the well-known integral representation of a harmonic function f at a general field point (x, y) inside the region D in terms of the boundary values of the function and its harmonic conjugate (after integrating by parts and rearranging) as

$$f(x,y) = \frac{1}{2\pi}\oint_C \left[f(s')\frac{\partial}{\partial n'}\ln R + f^c(s')\frac{\partial}{\partial s'}\ln R \right] ds', \tag{38a}$$

or, in the equivalent form

$$f(x,y) = \frac{1}{2\pi}\oint_C \left[f(s')\frac{\partial}{\partial n'}\ln R - \frac{\partial}{\partial n'}f(s')\ln R \right] ds', \tag{38b}$$

where R is the distance between the field point (x, y) in D and the current integration point (x', y') on C.

The harmonic conjugate of (38b) is

$$f^c(x,y) = \frac{1}{2\pi}\oint_C \left[f(s')\frac{\partial \Theta}{\partial n'} - \frac{\partial}{\partial n'}f(s')\Theta \right] ds', \tag{38c}$$

where

$$\Theta = \tan^{-1}\frac{y - y(s')}{x - x(s')}.$$

The representation of the conjugate function is given by

$$f^c(x,y) = \frac{1}{2\pi}\oint_C \left[f^c(s')\frac{\partial}{\partial n'}\ln R - f(s')\frac{\partial}{\partial s'}\ln R \right] ds', \tag{39a}$$

or, in the equivalent form

$$f^c(x,y) = \frac{1}{2\pi}\oint_C \left[f^c(s')\frac{\partial}{\partial n'}\ln R - \frac{\partial}{\partial n'}f^c(s')\ln R \right] ds', \tag{39b}$$

When point (x, y) tends to a boundary point, relations (39a) and (39b) are respectively replaced by

$$f(s) = \frac{1}{\pi}\oint_C \left[f(s')\frac{\partial}{\partial n'}\ln R + f^c(s')\frac{\partial}{\partial s'}\ln R \right] ds', \tag{39c}$$

$$f(s) = \frac{1}{\pi}\oint_C \left[f(s')\frac{\partial}{\partial n'}\ln R - \frac{\partial}{\partial n'}f(s')\ln R \right] ds'. \tag{39d}$$

If a function $g(x, y)$ is defined in the outer region $C\left(\overline{D}\right)$, is harmonic in this region and vanishes at infinity at least as $\left(x^2 + y^2\right)^{-(1+\delta)}$ with $\delta > 0$, it can be shown that the integral representation (39b) is replaced by

$$g(x,y) = -\frac{1}{2\pi}\oint_C \left[g(s')\frac{\partial}{\partial n'}\ln R - \frac{\partial}{\partial n'}g(s')\ln R \right] ds', \tag{40a}$$

it being understood that the boundary values $g(s')$ and $\frac{\partial}{\partial n'}g(s')$ under the integral sign on the R.H.S. are calculated at a point with parameter s' on the outer side of C. When the point (x, y) tends to a boundary point with parameter s, then (40a) is replaced by the integral relation

$$g(s) = -\frac{1}{\pi}\oint_C \left[g(s')\frac{\partial}{\partial n'}\ln R - \frac{\partial}{\partial n'}g(s')\ln R \right] ds'. \tag{40b}$$

5. Boundary condition for temperature

We shall treat the problem with heat radiation condition (Robin problem). The other cases may be treated along the same guidelines. In all cases, the function T_h^c is defined up to an additive arbitrary constant and the complete determination of this function requires the specification of its value at an arbitrarily chosen point in $\overline{D}$.

Let the thermal boundary condition be of the form

$$\frac{\partial T}{\partial n}(s) = -\frac{Bi}{K}\left[T(s) - T_e(s)\right], \tag{41a}$$

where Bi is Biot constant and $T_e(s)$ is the external (ambient) temperature on the boundary C. Using equation (4) the radiation condition written for $T_h(s)$ as

$$\frac{\partial T_h}{\partial n}(s) = -\frac{Bi}{K}\left(T_h(s) + T_p(s) - T_e(s)\right) - \frac{\partial T_p}{\partial n}(s). \tag{41b}$$

6. Stresses and displacements in terms of harmonic functions

Having obtained the solution for the magnetic field everywhere in space and for the temperature in the region D occupied by the material, we now turn to solve the mechanical problem for the stress and the displacement components in D. The stresses are given through the stress function U from relations (21), and these may be rewritten using expression (27) in terms of the harmonic functions Φ, Φ^c, Ψ and the particular solution U_p in the form

$$\sigma_{xx} = x\frac{\partial^2 \Phi}{\partial y^2} + 2\frac{\partial \Phi^c}{\partial y} + y\frac{\partial^2 \Phi^c}{\partial y^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 U_p}{\partial y^2}, \tag{42a}$$

$$\sigma_{yy} = x\frac{\partial^2 \Phi}{\partial x^2} + 2\frac{\partial \Phi}{\partial x} + y\frac{\partial^2 \Phi^c}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 U_p}{\partial x^2}, \tag{42b}$$

$$\sigma_{xy} = -x\frac{\partial^2 \Phi}{\partial x \partial y} - y\frac{\partial^2 \Phi^c}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 U_p}{\partial x \partial y}, \tag{42c}$$

from which, using (28), one obtains

$$\sigma_{xx} + \sigma_{yy} = 4\frac{\partial \Phi}{\partial x} - \frac{\alpha E}{1-\nu}T_p - \frac{1-2\nu}{2\left(1-\nu\right)}\mu^*\left(\frac{1}{2}\mu_1 + \mu_2\right)H^2. \tag{43}$$

The mechanical displacement components are given from relations (36a, 36b), which may rewritten using (27) in terms of the harmonic functions Φ, Φ^c and Ψ as

$$\frac{E}{1+\nu}u = (3-4\nu)\Phi - x\frac{\partial \Phi}{\partial x} - y\frac{\partial \Phi^c}{\partial x} - \frac{\partial \Psi}{\partial x} - \frac{\partial U_p}{\partial x} + \frac{E}{1+\nu}\left(u_T + u_H\right), \tag{44a}$$

$$\frac{E}{1+\nu}v = (3-4\nu)\Phi^c - x\frac{\partial \Phi}{\partial y} - y\frac{\partial \Phi^c}{\partial y} - \frac{\partial \Psi}{\partial y} - \frac{\partial U_p}{\partial y} + \frac{E}{1+\nu}\left(v_T + v_H\right). \tag{44b}$$

7. Conditions for the uniqueness of the solution

We now turn to the conditions to be satisfied in order to determine the unknown harmonic functions in an unambiguous manner. This is of primordial importance for any numerical treatment of the problem, for a proper use of the solving algorithm.

7.1. **Conditions for eliminating the rigid body translation.** These are two conditions, to be applied only for the first fundamental problem. Following [1], we require that the displacement at point O vanishes, i.e.

$$u\,(0,0) = v\,(0,0) = 0.$$

From equations (44a, 44b), these two conditions may be rewritten respectively as

$$(3-4\nu)\Phi\,(0,0) - \frac{\partial \Psi}{\partial x}\,(0,0) - \frac{\partial U_p}{\partial x}\,(0,0) + \frac{E}{1+\nu}\,[u_T\,(0,0) + u_H\,(0,0)] = 0, \tag{45a}$$

$$(3-4\nu)\Phi^c\,(0,0) - \frac{\partial \Psi}{\partial y}\,(0,0) - \frac{\partial U_p}{\partial y}\,(0,0) + \frac{E}{1+\nu}\,[v_T\,(0,0) + v_H\,(0,0)] = 0. \tag{45b}$$

7.2. **Conditions for eliminating the rigid body rotation.** This condition, like the first two, is applied only for the first fundamental problem. We shall require that [1]

$$\frac{\partial u}{\partial y}\,(0,0) - \frac{\partial v}{\partial x}\,(0,0) = 0,$$

or, using (44a, 44b),

$$4(1-\nu)\frac{\partial \Phi}{\partial y}\,(0,0) - \alpha E\,T_h^c\,(0,0) + \frac{1}{2}\frac{E}{1+\nu}\,(N_H - R_H) = 0. \tag{46}$$

7.3. **Additional simplifying conditions.** We shall require the following supplementary conditions to be satisfied at the point Q_0 $(s = 0)$ of the boundary, in order to determine the totality of the arbitrary integration constants appearing throughout the solution process. These additional conditions have no physical implications on the solution of the problem:

(i) The vanishing of the function U and its first order partial derivatives at Q_0, which, in terms of the boundary values of the unknown harmonic functions, give

$$x\,(0)\,\Phi\,(0) + y\,(0)\,\Phi^c\,(0) + \Psi(0) + U_p(a,0) = 0, \tag{47a}$$

$$x\,(0)\,\dot{\Phi}\,(0) + y\,(0)\,\dot{\Phi}^c\,(0) + \dot{\Psi}(0) + \dot{x}(0)\Phi\,(0) + \dot{y}(0)\Phi^c\,(0) + \frac{\partial U_p}{\partial s}(a,0) = 0, \tag{47b}$$

$$x\,(0)\,\dot{\Phi}^c\,(0) - y\,(0)\,\dot{\Phi}\,(0) + \dot{\Psi}^c(0) + \dot{y}(0)\Phi\,(0) - \dot{x}(0)\Phi^c\,(0) + \frac{\partial U_p}{\partial n}(a,0) = 0. \tag{47c}$$

(ii) The vanishing of the combination

$$x\,(0)\,\Phi^c\,(0) - y\,(0)\,\Phi\,(0) + \Psi^c(0) = 0. \tag{47d}$$

This last additional condition amounts to determining the value of Ψ^c at Q_0 and is chosen for the uniformity of presentation as in [3], [4].

Let us finally turn to the boundary conditions related to the equations of elasticity. In what follows, we consider only the first fundamental problem of Elasticity. Let the force distribution per unit length on the boundary C. be

$$\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j} = f_\tau\boldsymbol{\tau} + f_n\mathbf{n}$$

Then, at a general boundary point Q , the stress vector satisfies the condition of continuity

$$\boldsymbol{\sigma}_n = \mathbf{f}.$$

In the absence of boundary forces of non-magnetic origin, one sets $\mathbf{f} = \mathbf{f}_H$, where $\mathbf{f}_H$ is the force due to the action of the magnetic field, per unit length of the boundary.

In components

$$\sigma_{xx}n_x + \sigma_{xy}n_y = f_x \qquad \text{and} \qquad \sigma_{xy}n_x + \sigma_{yy}n_y = f_y. \tag{48}$$

The force $\mathbf{f}_H$ may be expressed in terms of the Maxwellian stress tensor $\boldsymbol{\sigma}^*$ as

$$\mathbf{f}_H = \boldsymbol{\sigma}^*\mathbf{n}, \quad \text{with} \quad \boldsymbol{\sigma}^*_{ij} = \mu^*\left(H_i^*H_j^* - \frac{1}{2}H^{*2}\delta_{ij}\right). \tag{49}$$

Substituting equations (1b) and (21) and taking conditions (44a, 44b) into account, the last two relations yield

$$\frac{\partial U}{\partial x}(s) = -\int_0^s f_y(s')ds' = -Y(s), \quad \frac{\partial U}{\partial y}(s) = \int_0^s f_x(s')ds' = X(s), \text{ say.} \tag{50}$$

Using expressions (48), one may easily obtain the tangential and normal derivatives of the stress function U at the boundary point Q.

$$\frac{\partial U}{\partial s}(s) = -\dot{x}(s)Y(s) + \dot{y}(s)X(s), \quad \frac{\partial U}{\partial n}(s) = -\dot{y}(s)Y(s) - \dot{x}(s)X(s), \tag{51}$$

or, in terms of the unknown harmonic functions

$$x(s)\,\dot{\Phi}(s) + y(s)\,\dot{\Phi}^c(s) + \dot{\Psi}(s) + \dot{x}(s)\Phi(s) + \dot{y}(s)\Phi^c(s) = \\ -\dot{x}(s)Y(s) + \dot{y}(s)X(s) - \frac{\partial U_p}{\partial s}(s), \tag{52a}$$

$$x(s)\,\dot{\Phi}^c(s) - y(s)\,\dot{\Phi}(s) + \dot{\Psi}^c(s) + \dot{y}(s)\Phi(s) - \dot{x}(s)\Phi^c(s) = \\ -\dot{y}(s)\,Y(s) - \dot{x}(s)X(s) - \frac{\partial U_p}{\partial n}(s). \tag{52b}$$

Equations (52a, 52b), together with relation (39c) written for $\Phi(s)$ and $\Psi(s)$ form a set of four integral and differential relations, the solution of which under the set of conditions (45a, 45b), (46) and (47a, 47b, 47c, 47d) provides the boundary values of the unknown harmonic functions Φ and Ψ and their harmonic conjugates.

8. Elliptic cylinder in an external uniform magnetic field

We present herebelow the solution of problem which can be handled analytically, namely the elliptic cylinder placed in a transverse constant external magnetic field, in the presence of a uniformly distributed heat source in the bulk. The body is placed in an ambient medium of temperature T_e.

Let an elliptic cylinder of a weak magnetizable material be placed in an external, transversal constant magnetic field $\mathbf{H}_0$ perpendicular to the axis of the cylinder and inclined to the axes of the cross section with angle ϕ. A uniform heat source of intensity ζ_0 acts in the body. We assume that the cylinder is placed in an external medium with ambient temperature T_e.

Let the normal cross-section of the cylindrical body be bounded by an ellipse of major and minor semi-axes a, b respectively, centered at the origin of coordinates, with parametric equations, see [11]

$$x(s) = a\cos\theta, \qquad y(s) = b\sin\theta, \qquad -\pi < \theta \leq \pi, \tag{53a}$$

with

$$a = c\cosh\xi_0, \qquad b = c\sinh\xi_0, \qquad c = \sqrt{a^2 - b^2}, \tag{53b}$$

and θ is the angle in the associated elliptic system of coordinates (ξ, θ). The contour of the cylinder is given by the relation $\xi = \xi_0$. In elliptic system of coordinates (ξ, θ), we get

$$r \equiv r(\xi,\theta) = \sqrt{x^2+y^2} = c\sqrt{\tfrac{1}{2}(\cosh 2\xi + \cos 2\theta)}, \tag{54a}$$

$$h \equiv h(\xi,\theta) = c\sqrt{\frac{1}{2}(\cosh 2\xi - \cos 2\theta)}, \tag{54b}$$

at the boundary $(\xi = \xi_0)$, one can verify that

$$r_0 \equiv r_0(\theta) = a\sqrt{1-k^2\sin^2\theta} = b\sqrt{1+k'^2\cos^2\theta}, \tag{54c}$$

$$h_0 \equiv h_0(\theta) = b\sqrt{1+k'^2\sin^2\theta} = a\sqrt{1-k^2\cos^2\theta}, \tag{54d}$$

with

$$k^2 = \frac{c^2}{a^2}, \qquad k'^2 = \frac{c^2}{b^2}, \qquad q^2 = \frac{k^2}{k'^2} = \frac{b^2}{a^2}. \tag{55}$$

9. Solution for the equations of Magnetostatic

The solution for the magnetic vector potential components is obtained by solve equations (12, 13). In this case the magnetic field is derived from a scalar magnetic potential and A is the harmonic function in D, then

$$A = A_h, \qquad A^* = A_r^* + A_\infty^*.$$

The general solution for Laplace's equations in the elliptic cylindrical coordinates are

$$\frac{A}{a\mu^* H_0} = \alpha_0 + \sum_{n=1}^{\infty}\left[\alpha_n \frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta + \alpha_n^{'}\frac{\sinh n\xi}{\sinh n\xi_0}\sin n\theta\right], \tag{56a}$$

$$\frac{A^*}{a\mu^* H_0} = \beta_0 + \sum_{n=1}^{\infty}\left[\beta_n \cos n\theta + \beta_n^{'}\sin n\theta\right]\mathrm{e}^{-n(\xi-\xi_0)} + k(\sinh\xi\sin\theta\cos\phi - \cosh\xi\cos\theta\sin\phi), \tag{56b}$$

where A_∞^* defined by equation (25b).

Substituting equations (56a, 56b) into the boundary equations (15) and choosing $A_r^*(\xi,\theta)$ vanishing at ξ tends to infinity and using the properties of the trigonometric functions, then

$$\alpha_0 = \beta_0 = 0, \qquad \alpha_n = \beta_n = \alpha_n^{'} = \beta_n^{'} = 0, \qquad n \geq 2, \dots, \tag{57a}$$

$$\alpha_1 = -\mu_0\frac{1+q}{q+\mu_0}\sin\phi, \qquad \beta_1 = -\frac{\mu_0-1}{q+\mu_0}q\sin\phi, \tag{57b}$$

$$\alpha_1^{'} = \mu_0\frac{1+q}{1+q\mu_0}q\cos\phi, \qquad \beta_1^{'} = \frac{\mu_0-1}{1+q\mu_0}q\cos\phi. \tag{57c}$$

Then, the magnetic vector potentials are

$$\frac{A}{a\mu^* H_0} = \mu_0(1+q)k\left(\frac{\cos\phi}{1+q\mu_0}\sinh\xi\sin\theta - \frac{\sin\phi}{q+\mu_0}\cosh\xi\cos\theta\right), \tag{58a}$$

$$\frac{A^*}{a\mu^* H_0} = (\mu_0-1)q\left(\frac{\cos\phi}{1+q\mu_0}\sin\theta - \frac{\sin\phi}{q+\mu_0}\cos\theta\right)\mathrm{e}^{-(\xi-\xi_0)} + k(\sinh\xi\sin\theta\cos\phi - \cosh\xi\cos\theta\sin\phi), \tag{58b}$$

where H_0 is the intensity of the applied magnetic field. These are agreement to the solutions introduced in [5].

The conjugate function using the Cauchy-Riemann relation and choosing A_h^c to vanish at the origin, one gets

$$\frac{A_h^c}{a\mu^* H_0} = -\mu_0 (1+q) k \left(\frac{\cos\phi}{1+q\mu_0} \cosh\xi \cos\theta + \frac{\sin\phi}{q+\mu_0} \sinh\xi \sin\theta \right). \tag{58c}$$

To determine the magnetic field vector we must express about equation (9) in elliptic Coordinates as follows

$$H_\xi = \frac{1}{\mu^*\mu_0} \frac{1}{h} \frac{\partial A}{\partial \theta}, \qquad H_\theta = -\frac{1}{\mu^*\mu_0} \frac{1}{h} \frac{\partial A}{\partial \xi}, \tag{59a}$$

$$H_\xi^* = \frac{1}{\mu^*} \frac{1}{h} \frac{\partial A^*}{\partial \theta}, \qquad H_\theta^* = -\frac{1}{\mu^*} \frac{1}{h} \frac{\partial A^*}{\partial \xi}. \tag{59b}$$

Substituting equations (58a, 58b) into equations (59a, 59b), then, the magnetic field are:

$$\frac{1}{H_0} H_\xi = \frac{c}{h} (1+q) \left(\frac{\cos\phi}{1+q\mu_0} \sinh\xi \cos\theta + \frac{\sin\phi}{q+\mu_0} \cosh\xi \sin\theta \right), \tag{60a}$$

$$\frac{1}{H_0} H_\theta = -\frac{c}{h} (1+q) \left(\frac{\cos\phi}{1+q\mu_0} \cosh\xi \sin\theta - \frac{\sin\phi}{q+\mu_0} \sinh\xi \cos\theta \right), \tag{60b}$$

and

$$\begin{aligned} \frac{1}{H_0} H_\xi^* = {} & \frac{1}{k} \frac{c}{h} \left[(\mu_0 - 1) q \left(\frac{\cos\phi}{1+q\mu_0} \cos\theta + \frac{\sin\phi}{q+\mu_0} \sin\theta \right) \mathrm{e}^{-(\xi-\xi_0)} \right. \\ & \left. + k (\sinh\xi \cos\theta \cos\phi + \cosh\xi \sin\theta \sin\phi) \right], \end{aligned} \tag{61a}$$

$$\begin{aligned} \frac{1}{H_0} H_\theta^* = {} & -\frac{1}{k} \frac{c}{h} \left[-(\mu_0 - 1) q \left(\frac{\cos\phi}{1+q\mu_0} \sin\theta - \frac{\sin\phi}{q+\mu_0} \cos\theta \right) \mathrm{e}^{-(\xi-\xi_0)} \right. \\ & \left. + k (\cosh\xi \sin\theta \cos\phi - \sinh\xi \cos\theta \sin\phi) \right]. \end{aligned} \tag{61b}$$

9.1. **Displacement due to the magnetic field.** Now, we calculate the expressions M_H, S_H, N_H and S_H. The magnetic field vector in the x and y directions are expressed in terms of the components in the elliptic coordinates as follows:

$$H_x = \frac{c}{h} (H_\xi \sinh\xi \cos\theta - H_\theta \cosh\xi \sin\theta), \tag{62a}$$

$$H_y = \frac{c}{h} (H_\xi \cosh\xi \sin\theta + H_\theta \sinh\xi \cos\theta). \tag{62b}$$

Substituting from equations (60a, 60b) into equations (62a, 62b), one gets

$$\frac{1}{H_0} H_x = (1+q) \frac{\cos\phi}{1+q\mu_0}, \qquad \frac{1}{H_0} H_y = (1+q) \frac{\sin\phi}{q+\mu_0}, \tag{63}$$

then

$$M_H = -(1+q)^2 W_1 Q_1, \qquad S_H = (1+q)^2 W_1 Q_1, \tag{64a}$$

$$N_H = -(1+q)^2 W_1 Q_2, \qquad R_H = -(1+q)^2 W_1 Q_2, \tag{64b}$$

with

$$Q_1 = \frac{\cos^2\phi}{(1+q\mu_0)^2} - \frac{\sin^2\phi}{(q+\mu_0)^2}, \quad Q_2 = \frac{\sin 2\phi}{(q+\mu_0)(1+q\mu_0)}, \tag{64c}$$

$$W_1 = \frac{\gamma_0}{2} (1+\nu) \left(\mu_0 - \frac{1}{2}\mu_1 \right), \qquad \gamma_0 = \frac{\mu^* H_0^2}{E}, \tag{64d}$$

γ_0 being a dimensionless parameter.

The displacements due to the magnetic field are

$$\frac{u_H}{a} = -k\left(1+q\right)^2 W_1 \left(Q_2 \sinh\xi \sin\theta + Q_1 \cosh\xi \cos\theta\right), \quad (65a)$$

$$\frac{v_H}{a} = k\left(1+q\right)^2 W_1 \left(Q_1 \sinh\xi \sin\theta - Q_2 \cosh\xi \cos\theta\right). \quad (65b)$$

It is easy to verify that

$$u_H^2 + v_H^2 = \left(1+q\right)^4 W_1^2 Q_3^2 r^2, \quad (65c)$$

with

$$Q_3 = \frac{\cos^2\phi}{\left(1+q\mu_0\right)^2} + \frac{\sin^2\phi}{\left(q+\mu_0\right)^2}, \quad (65d)$$

which shows that the $u_H^2 + v_H^2$ quantity depends on the radial coordinate r only.

9.2. **The Maxwellian stress tensor components.** To calculate the Maxwellian stress tensor components, set equations (61a, 61b) into equation (49) with some manipulations. One obtains

$$\frac{\boldsymbol{\sigma}^*_{\xi\xi}}{E} = \frac{\gamma_0}{2k^2}\frac{c^2}{h^2}\left[\mathcal{H}_1 + \mathcal{H}_2\cos 2\theta + \mathcal{H}_3 \sin 2\theta\right], \quad (66a)$$

$$\frac{\boldsymbol{\sigma}^*_{\theta\theta}}{E} = -\frac{\gamma_0}{2k^2}\frac{c^2}{h^2}\left[\mathcal{H}_1 + \mathcal{H}_2\cos 2\theta + \mathcal{H}_3 \sin 2\theta\right], \quad (66b)$$

$$\frac{\boldsymbol{\sigma}^*_{\xi\theta}}{E} = -\frac{\gamma_0}{2k^2}\frac{c^2}{h^2}\left[\mathcal{H}_4 + \mathcal{H}_5\cos 2\theta + \mathcal{H}_6 \sin 2\theta\right], \quad (66c)$$

with

$$\begin{aligned}
\mathcal{H}_1 &= \left(\mu_0 - 1\right) q\left(1+q\right) Q_4 - \frac{1}{2}k^2 \cos 2\phi, \\
\mathcal{H}_2 &= q\left(1+q\right)\left(\mu_0 - 1\right)\left[\frac{q}{k^2}\left(1+q\right)\left(\mu_0-1\right) Q_1 - Q_4\right] \mathrm{e}^{-2\xi} + \frac{1}{2}k^2 \cosh 2\xi \cos 2\phi, \\
\mathcal{H}_3 &= q\left(1+q\right)\left(\mu_0 - 1\right)^2\left[\frac{q}{k^2}\left(1+q\right) + \tfrac{1}{2}\left(1-q\right)\right] Q_2\, \mathrm{e}^{-2\xi} + \frac{1}{2}k^2 \sinh 2\xi \sin 2\phi, \\
\mathcal{H}_4 &= -\frac{1}{2}q\left(\mu_0 - 1\right)\left[k^2\left(\mu_0-1\right)\left(1+\mathrm{e}^{-2\xi}\right) + \left(1+q\right)^2\left(\mu_0+1\right)\mathrm{e}^{-2\xi}\right] Q_2 + \frac{1}{2}k^2 \sin 2\phi, \\
\mathcal{H}_5 &= q\left(1+q\right)\left(\mu_0 - 1\right)\left[\frac{q}{k^2}\left(\mu_0-1\right) - \tfrac{1}{2}\left(1+q\right)\left(1+\mu_0\right)\right] Q_2\, \mathrm{e}^{-2\xi} - \frac{1}{2}k^2 \cosh 2\xi \sin 2\phi, \\
\mathcal{H}_6 &= -q\left(1+q\right)\left(\mu_0 - 1\right)\left[\frac{q}{k^2}\left(\mu_0-1\right) Q_1 - Q_4\right] \mathrm{e}^{-2\xi} + \frac{1}{2}k^2 \sinh 2\xi \cos 2\phi,
\end{aligned} \quad (67a)$$

with

$$Q_4 = \frac{\cos^2\phi}{1+q\mu_0} + \frac{\sin^2\phi}{q+\mu_0}. \quad (67b)$$

10. Solution for temperature

To determinate the temperature of the medium, the thermal radiation boundary condition (41b) takes the form at $\xi = \xi_0$

$$\frac{1}{h_0}\frac{\partial}{\partial\xi}\left(T_h + T_p\right) = -\frac{Bi}{K}\left(T_h + T_p - T_e\right). \quad (68)$$

Where the external boundary temperature $T_e(\theta)$ is taken to be an even function of the angle θ in the form:

$$T_e(\theta) = T_0 + \sum_{n=2}^{\infty} T_n \cos n\theta, \qquad -\pi < \theta \leq \pi. \tag{69a}$$

As will be noted below, for reasons of global equilibrium, the following condition must hold $T_1 = 0$. The particular solution, equation (4) provides

$$T_p = -\frac{a^2\zeta_0}{8K} k^2 (\cosh 2\xi + \cos 2\theta). \tag{69b}$$

The temperature function T_h be expanded in Fourier series in the form:

$$T_h(\theta) = A_0 + \sum_{n=2}^{\infty} A_n \cos n\theta. \tag{70a}$$

Substituting equation (70a) into equation (38a) and integrate w. r. t θ along the boundary of the ellipse one can verify that

$$\varepsilon_n = \tanh n\xi_0, \qquad n = 1, 2, \dots . \tag{70b}$$

From the general representation of harmonic functions in elliptic coordinates, one can find the general solution for Laplace's equation inside the cylinder is

$$T_h(\xi, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \frac{\cosh n\xi}{\cosh n\xi_0} \cos n\theta. \tag{70c}$$

For later use, let us expand the term $\frac{c}{h(\theta)}$ in Fourier series as follows:

$$\frac{c}{h(\theta)} = \frac{\kappa_0}{2} + \sum_{r=1}^{\infty} \kappa_r \cos r\theta, \qquad -\pi < \theta \leq \pi, \tag{71a}$$

with

$$\kappa_0 = \frac{2}{\pi}\int_0^{\pi} \frac{c}{h(\theta)} d\theta, \quad \kappa_r = \frac{2}{\pi}\int_0^{\pi} \frac{c}{h(\theta)} \cos r\theta d\theta, \quad r = 1, 2, \dots \tag{71b}$$

Noting that the function $\frac{c}{h(\theta)}$, as well as all its derivatives, are continuous functions for $\xi_0 > 0$, its Fourier series is rapidly convergent. More precisely, the behavior of the general term κ_r as $r \to \infty$ is like $\frac{1}{r^m}$, for any integer m. Moreover, since this is an even function of its argument, one finds that

$$\kappa_{2r-1} = 0, \quad r = 1, 2, \dots .$$

Then the analytical formulae for the Fourier constants calculated in [4] and use [12]as

$$\kappa_0 = 2kF\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

$$\kappa_{2r} = \frac{2k}{\pi} \sum_{s=0}^{r} (-1)^{s+r} \binom{2r}{2s} B(s + \frac{1}{2}, r - s + \frac{1}{2}) F(s + \frac{1}{2}, \frac{1}{2}; r + 1; k^2), \quad r = 1, 2, ..$$

where where $B(x, y)$ and $F(\alpha, \beta; \gamma; z)$ are the Beta and the Hypergeometric Function respectively, with expressions [[13], p. 425, 388].

Substituting equations (70c) and (69a, 69b) into equation (68) and take in account equation (71b) and using the orthogonal properties of trigonometric functions, one get

$$A_0 = T_0 + \frac{a^2\zeta_0}{4K} \frac{1}{Bk'} \kappa_0 + \frac{a^2\zeta_0}{8K} (1 + q^2) - \frac{1}{Bk} \sum_{n=2}^{\infty} \frac{n}{2} \kappa_n \varepsilon_n A_n, \tag{72a}$$

and the remaining constants in the matrix form

$$\sum_{j=1}^{\infty} M_{ij} A_j = C_i, \qquad i,j = 1,2,...N, \tag{72b}$$

with

$$\begin{aligned} M_{ii} &= Bk + \frac{i}{2}\varepsilon_i \left(\kappa_0 + \delta^*_{ij}\kappa_{2i}\right), \qquad 2i \le N && (73a)\\ M_{ij} &= \frac{j}{2}\varepsilon_j \left(\kappa_{j-i} + \delta^*_{ij}\kappa_{i+j}\right), \qquad j \le N-i && (73b)\\ M_{ji} &= \frac{i}{2}\varepsilon_i \left(\kappa_{j-i} + \delta^*_{ij}\kappa_{i+j}\right), \qquad j \le N-i && (73c)\end{aligned}$$

$$C_i = \frac{ab\zeta_0}{2K}\kappa_i + Bk\left(T_i + \frac{a^2\zeta_0}{8K}k^2\delta_{i2}\right), \tag{73d}$$

and

$$\delta^*_{ij} = \begin{cases} 1, & j \le N-i \\ 0, & \text{otherwise} \end{cases}$$

where $B\left(=\frac{aBi}{k}\right)$ being the dimensionless Biot constant.

The complex conjugate T_h^c (chosen to vanish at the origin O) satisfying the thermal boundary condition (68) and Cauchy-Reimman condition is

$$T_h^c(\xi,\theta) = \sum_{n=1}^{\infty} A_n \frac{\sinh n\xi}{\cosh n\xi_0} \sin n\theta. \tag{74}$$

The temperature of the medium may finally be written as

$$T(\xi,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta - \frac{a^2\zeta_0}{8K}k^2\left(\cosh 2\theta + \cos 2\theta\right). \tag{75}$$

10.1. **Displacement due to the temperature.** Inserting expressions (70c) and (74) into (37a) and performing the integrations along rays starting at the origin $O(\xi = 0, \theta = \frac{\pi}{2})$ to a general point) $P(\xi,\theta)$ inside the ellipse see [4], we get the following expressions for the displacement vector components due to the temperature u_T and v_T:

$$\begin{aligned}\frac{u_T}{a} = (1+\nu)\,k\alpha A_0 \cosh\xi\cos\theta &- \frac{k}{2}\sum_{n=2}^{\infty}\frac{1+\nu}{n-1}\alpha A_n \frac{\cosh(n-1)\xi}{\cosh n\xi_0}\cos(n-1)\theta \\ &+ \frac{k}{2}\sum_{n=2}^{\infty}\frac{1+\nu}{n+1}\alpha A_n \frac{\cosh(n+1)\xi}{\cosh n\xi_0}\cos(n+1)\theta, \end{aligned} \tag{76a}$$

and

$$\begin{aligned}\frac{v_T}{a} = (1+\nu)\,k\alpha A_0 \sinh\xi\sin\theta &- \frac{k}{2}\sum_{n=2}^{\infty}\frac{1+\nu}{n-1}\alpha A_n \frac{\sinh(n-1)\xi}{\cosh n\xi_0}\sin(n-1)\theta \\ &+ \frac{k}{2}\sum_{n=2}^{\infty}\frac{1+\nu}{n+1}\alpha A_n \frac{\sinh(n+1)\xi}{\cosh n\xi_0}\sin(n+1)\theta. \end{aligned} \tag{76b}$$

11. The elastic solution

Having obtained the magnetic field everywhere in space and the temperature inside the body, one turns to the solution of the elastic problem, through the determination of the stress function U.

Let us chose the particular solution U_p of equation (26b) as

$$\frac{1}{a^2E}U_p = D_1\left(\frac{r}{a}\right)^2 + D_2\left(\frac{r}{a}\right)^4, \tag{77}$$

where we have introduced the dimensionless parameters

$$D_1 = -\frac{\gamma_0}{8}\frac{1-2\nu}{1-\nu}\left(\frac{1}{2}\mu_1+\mu_2\right)(1+q)^2 Q_3, \quad D_2 = \frac{\beta_0}{16\,(1-\nu)}, \quad \beta_0 = \frac{\alpha a^2\zeta_0}{4K}. \tag{78}$$

The tangential and normal derivatives of U_p in elliptic coordinates are calculated from the expressions

$$\frac{\partial U_p}{\partial s} = \nabla U_p.\boldsymbol{\tau} = \frac{1}{h}\frac{\partial U_p}{\partial \theta}, \qquad \frac{\partial U_p}{\partial n} = \nabla U_p.\mathbf{n} = \frac{1}{h}\frac{\partial U_p}{\partial \xi}.$$

Then

$$\frac{1}{a^2E}\frac{\partial U_p}{\partial s}\frac{ds}{d\theta} = -k^2\left[D_1 + D_2\left(1+q^2+k^2\cos 2\theta\right)\right]\sin 2\theta, \tag{79a}$$

$$\frac{1}{a^2E}\frac{\partial U_p}{\partial n}\frac{ds}{d\theta} = 2q\left[D_1 + D_2\left(1+q^2+k^2\cos 2\theta\right)\right]. \tag{79b}$$

The restrictions of the functions Φ and Ψ and of their complex conjugates to the boundary are expressed as Fourier expansions in the angle θ of the system of elliptic coordinates (ξ,θ) in the form

$$\frac{1}{aE}\Phi(\theta) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\theta + \overline{a}_n\overline{\varepsilon}_n\sin n\theta\right), \tag{80a}$$

$$\frac{1}{aE}\Phi^c(\theta) = b_0 + \sum_{n=1}^{\infty}\left(\overline{a}_n\cos n\theta + a_n\varepsilon_n\sin n\theta\right), \tag{80b}$$

$$\frac{1}{a^2E}\Psi(\theta) = c_0 + \sum_{n=1}^{\infty}\left(c_n\cos n\theta + \overline{c}_n\overline{\varepsilon}_n\sin n\theta\right), \tag{80c}$$

$$\frac{1}{a^2E}\Psi^c(\theta) = d_0 + \sum_{n=1}^{\infty}\left(\overline{c}_n\cos n\theta + c_n\varepsilon_n\sin n\theta\right). \tag{80d}$$

Substituting these expressions into the integral relations (39c), one finds

$$\overline{\varepsilon}_n = -\tanh n\xi_0 = -\varepsilon_n, \qquad n = 1,2,\ldots .$$

Also, substitution of equations (76a, 76b), (65a, 65b), (53a), (79a, 79b), (A.1.2a, b) and (80a, 80b) into equations (52a, 52b) and using properties of the trigonometric functions yields

$$a_1 = \frac{1}{8q}\frac{\gamma_0}{k^3}\left(-\mathcal{G}_2' + 2k\mathcal{G}_4' - q\mathcal{G}_8'\right) - \left(D_1 + D_2\left(1+q^2\right)\right),$$

$$a_{2n} = 0, \qquad ,\overline{a}_{2n} = 0, \qquad n = 1,2,\ldots ,$$

$$c_{2n+1} = 0, \qquad ,\overline{c}_{2n+1} = 0, \qquad n = 1,2,\ldots ,$$

$$\begin{aligned} a_3 &= \frac{1}{2(q+\varepsilon_3)-(1+q\varepsilon_3)\varepsilon_2}\left[-2qk^2D_2+\frac{\gamma_0}{6}\frac{\mathcal{G}_4'}{1+q^2}\right],\\ \overline{a}_3 &= \frac{1+3q^2}{192q^3}\frac{\gamma_0}{k}\left[-3q\mathcal{G}_3'+4k\mathcal{G}_5'-3\mathcal{G}_7'\right], \end{aligned}$$

$$\begin{aligned} c_2 &= -\frac{1}{2}(1+q\varepsilon_3)a_3+\frac{1}{16q}\frac{\gamma_0}{k^3}\left(1+q^2\right)\mathcal{G}_2'-\frac{\gamma_0}{24}\frac{3+2q+3q^2}{q(1+q)^2}\mathcal{G}_4'-\frac{1}{16}\frac{\gamma_0}{k^3}\left(1+q^2\right)\mathcal{G}_8',\\ \overline{c}_2 &= -(1+q\varepsilon_3)\overline{a}_3-\frac{1}{8}\frac{\gamma_0}{k^3}\mathcal{G}_3'+\frac{\gamma_0}{6}\frac{\mathcal{G}_5'}{(1+q)^2}+\frac{1}{8}q\frac{\gamma_0}{k^3}\mathcal{G}_7', \end{aligned}$$

$$\begin{aligned} a_5 &= \frac{1}{3(q+\varepsilon_5)-2(1+q\varepsilon_5)\varepsilon_4}\left[\frac{k^4D_2}{2}+\left[2(1-q\varepsilon_3)\varepsilon_4+(q-\varepsilon_3)\right]a_3-\frac{\gamma_0\mathcal{G}_4'}{15}\frac{(1+\varepsilon_4)(1-q)}{(1+q)^3}\right],\\ \overline{a}_5 &= \frac{1}{2(q+\varepsilon_5)-3(1+q\varepsilon_5)\varepsilon_4}\left[\left[2(q-\varepsilon_3)+(1-q\varepsilon_3)\varepsilon_4\right]\overline{a}_3-\frac{\gamma_0\mathcal{G}_5'}{15}\frac{(1+\varepsilon_4)(1-q)}{(1+q)^3}\right], \end{aligned}$$

$$\begin{aligned} c_4 &= -\frac{1}{2}(1-q\varepsilon_3)a_3-\frac{1}{2}(1+q\varepsilon_5)a_5-\frac{k^4}{8}D_2+\frac{\gamma_0\mathcal{G}_4'}{60}\frac{1-q}{(1+q)^3},\\ \overline{c}_4 &= -\frac{1}{4}(1-q\varepsilon_3)\overline{a}_3-\frac{3}{4}(1+q\varepsilon_5)\overline{a}_5+\frac{\gamma_0\mathcal{G}_5'}{60}\frac{1-q}{(1+q)^3}, \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \frac{\gamma_0}{2k^3}(\varepsilon_n+4)\frac{\mathcal{R}_n}{\mathcal{S}_n}\mathcal{G}_4'+\frac{\mathcal{L}_n}{\mathcal{S}_n}a_{n-1},\\ \overline{a}_{n+1} &= \frac{\gamma_0}{2k^3}(1+4\varepsilon_n)\frac{\mathcal{R}_n'}{\mathcal{S}_n'}\mathcal{G}_5'+\frac{\mathcal{L}_n'}{\mathcal{S}_n'}\overline{a}_{n-1}, \end{aligned}$$

$$\begin{aligned} c_n &= -\frac{\gamma_0\mathcal{G}_4'}{4k^3}\frac{\mathcal{R}_n}{n}-\frac{1}{2}(1+q\varepsilon_{n+1})a_{n+1}-\frac{1}{2}(1-q\varepsilon_{n-1})a_{n-1}\\ \overline{c}_n &= -\frac{\gamma_0\mathcal{G}_5'}{k^3}\frac{\mathcal{R}_n'}{n}-2\frac{n+2}{n}(1+q\varepsilon_{n+1})\overline{a}_{n+1}-2\frac{n-2}{n}(1-q\varepsilon_{n-1})\overline{a}_{n-1} \end{aligned}$$

$$\begin{aligned} \mathcal{R}_n &= \frac{q-1}{n-1}\left(1-(-)^{n-1}\right)\left(\frac{k}{1+q}\right)^{n-1}+\frac{q+1}{n+1}\left(1-(-)^{n+1}\right)\left(\frac{k}{1+q}\right)^{n+1},\\ \mathcal{S}_n &= 4(n+2)(q+\varepsilon_{n+1})-n(1+q\varepsilon_{n+1})\varepsilon_n,\\ \mathcal{L}_n &= 4(n-2)(q-\varepsilon_{n-1})+n(1-q\varepsilon_{n-1})\varepsilon_n,\\ \mathcal{S}_n' &= n(q+\varepsilon_{n+1})-4(n+2)(1+q\varepsilon_{n+1})\varepsilon_n,\\ \mathcal{L}_n' &= n(q-\varepsilon_{n-1})+4(n-2)(1-q\varepsilon_{n-1})\varepsilon_n, \qquad n=5,6,\ldots\,. \end{aligned}$$

Conditions (45a, 45b) and (46) applied at the point $\left(0,\frac{\pi}{2}\right)$ and conditions (47a, 47b, 47c, 47d) applied at the point $(\xi_0,0)$ give

$$\begin{aligned}
a_0 &= -\frac{1}{1-\nu}\frac{\gamma_0}{8k^3}\mathcal{G}_6', \qquad b_0=\frac{1}{1-\nu}\frac{\gamma_0}{8k^3}\mathcal{G}_1',\\
c_1 &= -\frac{3-4\nu}{1-\nu}\frac{\gamma_0}{8k^3}\mathcal{G}_6', \qquad \overline{c}_1=-\frac{3-4\nu}{1-\nu}\frac{\gamma_0}{8k^3}\mathcal{G}_1',\\
\overline{a}_1 &= -\tfrac{1}{k}\sum_{n=2}^{\infty} n\frac{\overline{a}_n}{\cosh n\xi_0}\sin n\frac{\pi}{2},\\
c_0 &= -a_0-\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}c_n-D_1-D_2,\\
d_0 &= -b_0-\sum_{n=1}^{\infty}\overline{a}_n-\sum_{n=1}^{\infty}\overline{c}_n.
\end{aligned}$$

The expressions of the harmonic functions inside the body are

$$\frac{1}{aE}\Phi(\xi,\theta) = a_0+\sum_{n=1}^{\infty}\left(a_n\frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta+\overline{a}_n\overline{\varepsilon}_n\frac{\sinh n\xi}{\sinh n\xi_0}\sin n\theta\right), \quad (81a)$$

$$\frac{1}{aE}\Phi^c(\xi,\theta) = b_0+\sum_{n=1}^{\infty}\left(\overline{a}_n\frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta+a_n\varepsilon_n\frac{\sinh n\xi}{\sinh n\xi_0}\sin n\theta\right), \quad (81b)$$

$$\frac{1}{a^2E}\Psi(\xi,\theta) = c_0+\sum_{n=1}^{\infty}\left(c_n\frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta+\overline{c}_n\overline{\varepsilon}_n\frac{\sinh n\xi}{\sinh n\xi_0}\sin n\theta\right), \quad (81c)$$

$$\frac{1}{a^2E}\Psi^c(\xi,\theta) = d_0+\sum_{n=1}^{\infty}\left(\overline{c}_n\frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta+c_n\varepsilon_n\frac{\sinh n\xi}{\sinh n\xi_0}\sin n\theta\right), \quad (81d)$$

11.1. **The stress function.** Substituting equations (53a), (81a, 81b, 81c) and (77) into equation (27), the stress function U inside the domain D is obtained as

$$\begin{aligned}
\frac{U}{a^2E} &= c_0+ka_0\cosh\xi\cos\theta+kb_0\sinh\xi\sin\theta+D_1\left(\frac{r}{a}\right)^2+D_2\left(\frac{r}{a}\right)^4\\
&+\frac{k}{2}\sum_{n=1}^{\infty}a_n\frac{\cosh(n-1)\xi}{\cosh n\xi_0}\cos(n+1)\theta+\frac{k}{2}\sum_{n=1}^{\infty}a_n\frac{\cosh(n+1)\xi}{\cosh n\xi_0}\cos(n-1)\theta\\
&-\frac{k}{2}\sum_{n=1}^{\infty}\overline{a}_n\frac{\sinh(n+1)\xi}{\cosh n\xi_0}\sin(n-1)\theta-\frac{k}{2}\sum_{n=1}^{\infty}\overline{a}_n\frac{\sinh(n-1)\xi}{\cosh n\xi_0}\sin(n+1)\theta\\
&+\sum_{n=1}^{\infty}c_n\frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta-\sum_{n=1}^{\infty}\overline{c}_n\frac{\sinh n\xi}{\cosh n\xi_0}\sin n\theta. \qquad (82)
\end{aligned}$$

11.2. **Stress tensor components.** The components of the stress tensor σ_{kl} in the system of elliptic coordinates (ξ,θ) are [14]

$$\begin{aligned}
\frac{\sigma_{\xi\xi}}{E} &= \frac{1}{k^2}\frac{c^2}{h^2}\left(\frac{1}{a^2E}\frac{\partial^2U}{\partial\theta^2}\right)+\frac{1}{k^2}\frac{c^4}{2h^4}\left[\left(\frac{1}{a^2E}\frac{\partial U}{\partial\xi}\right)\sinh 2\xi-\left(\frac{1}{a^2E}\frac{\partial U}{\partial\theta}\right)\sin 2\theta\right],\\
\frac{\sigma_{\theta\theta}}{E} &= \frac{1}{k^2}\frac{c^2}{h^2}\left(\frac{1}{a^2E}\frac{\partial^2U}{\partial\xi^2}\right)-\frac{1}{k^2}\frac{c^4}{2h^4}\left[\left(\frac{1}{a^2E}\frac{\partial U}{\partial\xi}\right)\sinh 2\xi-\left(\frac{1}{a^2E}\frac{\partial U}{\partial\theta}\right)\sin 2\theta\right],\\
\frac{\sigma_{\xi\theta}}{E} &= -\frac{1}{k^2}\frac{c^2}{h^2}\left(\frac{1}{a^2E}\frac{\partial^2U}{\partial\xi\partial\theta}\right)+\frac{1}{k^2}\frac{c^4}{2h^4}\left[\left(\frac{1}{a^2E}\frac{\partial U}{\partial\xi}\right)\sin 2\theta+\left(\frac{1}{a^2E}\frac{\partial U}{\partial\theta}\right)\sinh 2\xi\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\sigma_{\xi\xi}}{E} &= \frac{1}{k^2}\frac{c^2}{h^2}\left\{2D_1\left(\frac{r}{a}\right)^2 - 2k^2D_1\cos 2\theta + 4D_2\left(\frac{r}{a}\right)^4 - 2k^4D_2\left(\cosh 2\xi\cos 2\theta + \cos 4\theta\right)\right\} \\
&+ \frac{1}{k^2}\frac{c^2}{h^2}\left\{\sum_{n=1}^{\infty} n^2\left[\overline{c}_n\frac{\sinh n\xi}{\cosh n\xi_0}\sin n\theta - c_n\frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta\right]\right. \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n+1)^2\left[\overline{a}_n\frac{\sinh(n-1)\xi}{\cosh n\xi_0}\sin(n+1)\theta - a_n\frac{\cosh(n-1)\xi}{\cosh n\xi_0}\cos(n+1)\theta\right] \\
&\left.+\frac{k}{2}\sum_{n=1}^{\infty}(n-1)^2\left[\overline{a}_n\frac{\sinh(n+1)\xi}{\cosh n\xi_0}\sin(n-1)\theta - a_n\frac{\cosh(n+1)\xi}{\cosh n\xi_0}\cos(n-1)\theta\right]\right\} \\
&+ \frac{1}{2k^2}\frac{c^4}{h^4}\left\{\sum_{n=1}^{\infty} n\left[\frac{\sinh n\xi}{\cosh n\xi_0}\sinh 2\xi\cos n\theta + \frac{\cosh n\xi}{\cosh n\xi_0}\sin n\theta\sin 2\theta\right]c_n\right. \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n-1)\left[\frac{\sinh(n-1)\xi}{\cosh n\xi_0}\sinh 2\xi\cos(n+1)\theta + \frac{\cosh(n+1)\xi}{\cosh n\xi_0}\sin(n-1)\theta\sin 2\theta\right]a_n \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n+1)\left[\frac{\sinh(n+1)\xi}{\cosh n\xi_0}\sinh 2\xi\cos(n-1)\theta + \frac{\cosh(n-1)\xi}{\cosh n\xi_0}\sin(n+1)\theta\sin 2\theta\right]a_n \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n+1)\left[\frac{\sinh(n-1)\xi}{\cosh n\xi_0}\cos(n+1)\theta\sin 2\theta - \frac{\cosh(n+1)\xi}{\cosh n\xi_0}\sinh 2\xi\sin(n-1)\theta\right]\overline{a}_n \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n-1)\left[\frac{\sinh(n+1)\xi}{\cosh n\xi_0}\cos(n-1)\theta\sin 2\theta - \tfrac{\cosh(n-1)\xi}{\cosh n\xi_0}\sinh 2\xi\sin(n+1)\theta\right]\overline{a}_n \\
&\left.+ \sum_{n=1}^{\infty} n\left[\frac{\sinh n\xi}{\cosh n\xi_0}\cos n\theta\sin 2\theta - \frac{\cosh n\xi}{\cosh n\xi_0}\sinh 2\xi\sin n\theta\right]\overline{c}_n\right\}, \qquad (83a)
\end{aligned}$$

$$\begin{aligned}
\frac{\sigma_{\theta\theta}}{E} &= -\frac{1}{k^2}\frac{c^2}{h^2}\left\{2D_1\left(\frac{r}{a}\right)^2 - 2k^2D_1\cosh 2\xi + 4D_2\left(\frac{r}{a}\right)^4 - 2k^4D_2\left(\cosh 4\xi + \cosh 2\xi\cos 2\theta\right)\right\} \\
&- \frac{1}{k^2}\frac{c^2}{h^2}\left\{\sum_{n=1}^{\infty} n^2\left[\overline{c}_n\frac{\sinh n\xi}{\cosh n\xi_0}\sin n\theta - c_n\frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta\right]\right. \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n-1)^2\left[\overline{a}_n\frac{\sinh(n-1)\xi}{\cosh n\xi_0}\sin(n+1)\theta - a_n\frac{\cosh(n-1)\xi}{\cosh n\xi_0}\cos(n+1)\theta\right] \\
&\left.+\frac{k}{2}\sum_{n=1}^{\infty}(n+1)^2\left[\overline{a}_n\frac{\sinh(n+1)\xi}{\cosh n\xi_0}\sin(n-1)\theta - a_n\frac{\cosh(n+1)\xi}{\cosh n\xi_0}\cos(n-1)\theta\right]\right\} \\
&+ \frac{1}{2k^2}\frac{c^4}{h^4}\left\{\sum_{n=1}^{\infty} n\left[\frac{\cosh n\xi}{\cosh n\xi_0}\sinh 2\xi\sin n\theta - \frac{\sinh n\xi}{\cosh n\xi_0}\cos n\theta\sin 2\theta\right]\overline{c}_n\right. \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n+1)\left[\frac{\cosh(n+1)\xi}{\cosh n\xi_0}\sinh 2\xi\sin(n-1)\theta - \frac{\sinh(n-1)\xi}{\cosh n\xi_0}\cos(n+1)\theta\sin 2\theta\right]\overline{a}_n \\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n-1)\left[\frac{\cosh(n-1)\xi}{\cosh n\xi_0}\sinh 2\xi\sin(n+1)\theta - \frac{\sinh(n+1)\xi}{\cosh n\xi_0}\cos(n-1)\theta\sin 2\theta\right]\overline{a}_n \\
&- \frac{k}{2}\sum_{n=1}^{\infty}(n+1)\left[\frac{\cosh(n-1)\xi}{\cosh n\xi_0}\sin(n+1)\theta\sin 2\theta + \frac{\sinh(n+1)\xi}{\cosh n\xi_0}\sinh 2\xi\cos(n-1)\theta\right]a_n \\
&- \frac{k}{2}\sum_{n=1}^{\infty}(n-1)\left[\frac{\cosh(n+1)\xi}{\cosh n\xi_0}\sin(n-1)\theta\sin 2\theta - \frac{\sinh(n-1)\xi}{\cosh n\xi_0}\sinh 2\xi\cos(n+1)\theta\right]a_n \\
&\left.- \sum_{n=1}^{\infty} n\left[\frac{\cosh n\xi}{\cosh n\xi_0}\sin n\theta\sin 2\theta + \frac{\sinh n\xi}{\cosh n\xi_0}\sinh 2\xi\cos n\theta\right]c_n\right\}, \qquad (83b)
\end{aligned}$$

and

$$\begin{aligned}\frac{\sigma_{\xi\theta}}{E} &= \frac{1}{k^2}\frac{c^2}{h^2}\left\{2k^4 D_2 \sinh 2\xi \sin 2\theta + \sum_{n=1}^{\infty} n^2 \left[\overline{c}_n \frac{\cosh n\xi}{\cosh n\xi_0}\cos n\theta + c_n \frac{\sinh n\xi}{\cosh n\xi_0}\sin n\theta\right]\right.\\
&+ \frac{k}{2}\sum_{n=1}^{\infty}\left(n^2-1\right)\left[\overline{a}_n \frac{\cosh\left(n+1\right)\xi}{\cosh n\xi_0}\cos\left(n-1\right)\theta + a_n \frac{\sinh\left(n+1\right)\xi}{\cosh n\xi_0}\sin\left(n-1\right)\theta\right]\\
&\left.+\frac{k}{2}\sum_{n=1}^{\infty}\left(n^2-1\right)\left[\overline{a}_n \frac{\cosh\left(n-1\right)\xi}{\cosh n\xi_0}\cos(n+1)\theta + a_n \frac{\sinh\left(n-1\right)\xi}{\cosh n\xi_0}\sin(n+1)\theta\right]\right\}\\
&+ \frac{1}{k^2}\frac{c^4}{2h^4}\left\{\sum_{n=1}^{\infty} n\left[\frac{\sinh n\xi}{\cosh n\xi_0}\cos n\theta \sin 2\theta - \frac{\cosh n\xi}{\cosh n\xi_0}\sinh 2\xi \sin n\theta\right] c_n\right.\\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n-1)\left[\frac{\sinh\left(n-1\right)\xi}{\cosh n\xi_0}\cos(n+1)\theta\sin 2\theta - \frac{\cosh\left(n+1\right)\xi}{\cosh n\xi_0}\sinh 2\xi \sin\left(n-1\right)\theta\right] a_n\\
&+ \frac{k}{2}\sum_{n=1}^{\infty}(n+1)\left[\frac{\sinh\left(n+1\right)\xi}{\cosh n\xi_0}\cos(n-1)\theta\sin 2\theta - \frac{\cosh\left(n-1\right)\xi}{\cosh n\xi_0}\sinh 2\xi \sin(n+1)\theta\right] a_n\\
&- \frac{k}{2}\sum_{n=1}^{\infty}(n+1)\left[\frac{\cosh\left(n+1\right)\xi}{\cosh n\xi_0}\sin(n-1)\theta\sin 2\theta + \frac{\sinh\left(n-1\right)\xi}{\cosh n\xi_0}\sinh 2\xi \cos(n+1)\theta\right] \overline{a}_n\\
&- \frac{k}{2}\sum_{n=1}^{\infty}(n-1)\left[\frac{\cosh\left(n-1\right)\xi}{\cosh n\xi_0}\sin(n+1)\theta\sin 2\theta + \frac{\sinh\left(n+1\right)\xi}{\cosh n\xi_0}\sinh 2\xi \cos\left(n-1\right)\theta\right] \overline{a}_n\\
&\left.- \sum_{n=1}^{\infty} n\left[\frac{\cosh n\xi}{\cosh n\xi_0}\sin n\theta \sin 2\theta + \frac{\sinh n\xi}{\cosh n\xi_0}\sinh 2\xi \cos n\theta\right]\overline{c}_n\right\}. \qquad (83c)\end{aligned}$$

11.3. **Displacement vector components.** Finally, the displacement vector components in (ξ,θ) −coordinates are

$$\begin{aligned}\frac{1}{1+\nu}\frac{u_\xi}{a} &= \frac{a}{h}\left\{k\left(3-4\nu\right)a_0 \sinh\xi\cos\theta + \left(3-4\nu\right)kb_0\cosh\xi\sin\theta + \sum_{n=1}^{\infty} n\overline{c}_n \frac{\cosh n\xi}{\cosh n\xi_0}\sin n\theta\right.\\
&+ \left[\frac{k^2}{2}\alpha A_0 - k^2 D_1 - 2k^2 D_2\left(\frac{r}{a}\right)^2\right]\sinh 2\xi - \sum_{n=1}^{\infty} nc_n \frac{\sinh n\xi}{\cosh n\xi_0}\cos n\theta\\
&+ \frac{k}{2}\sum_{n=1}^{\infty}\left(n+3-4\nu\right)\left[\overline{a}_n \frac{\cosh\left(n-1\right)\xi}{\cosh n\xi_0}\sin(n+1)\theta - a_n \frac{\sinh\left(n-1\right)\xi}{\cosh n\xi_0}\cos(n+1)\theta\right]\\
&+ \frac{k}{2}\sum_{n=1}^{\infty}\left(n-3+4\nu\right)\left[\overline{a}_n \frac{\cosh\left(n+1\right)\xi}{\cosh n\xi_0}\sin(n-1)\theta - a_n \frac{\sinh\left(n+1\right)\xi}{\cosh n\xi_0}\cos(n-1)\theta\right]\\
&+ \frac{k^2}{4}\sum_{n=2}^{\infty}\frac{1}{n-1}\left[\frac{\sinh(n-2)\xi}{\cosh n\xi_0}\cos n\theta - \frac{\sinh n\xi}{\cosh n\xi_0}\cos(n-2)\theta\right]\alpha A_n\\
&+ \frac{k^2}{4}\sum_{n=2}^{\infty}\frac{1}{n+1}\left[\frac{\sinh(n+2)\xi}{\cosh n\xi_0}\cos n\theta - \frac{\sinh n\xi}{\cosh n\xi_0}\cos(n+2)\theta\right]\alpha A_n\\
&\left.-\frac{k^2}{2}\left(1+q\right)^2\frac{W_1}{1+\nu}\left[Q_1\sinh 2\xi\cos 2\theta + Q_2\cosh 2\xi \sin 2\theta\right]\right\}, \qquad (84a)\end{aligned}$$

$$\frac{1}{1+\nu}\frac{v_\theta}{a} = \frac{a}{h}\left\{-(3-4\nu)\,ka_0\cosh\xi\sin\theta + (3-4\nu)\,kb_0\sinh\xi\cos\theta + \sum_{n=1}^{\infty} nc_n\frac{\cosh n\xi}{\cosh n\xi_0}\sin n\theta\right.$$
$$+\left[k^2D_1 + 2k^2D_2\left(\frac{r}{a}\right)^2 - \frac{k^2}{2}\alpha A_0\right]\sin 2\theta + \sum_{n=1}^{\infty} n\bar{c}_n\frac{\sinh n\xi}{\cosh n\xi_0}\cos n\theta$$
$$+\frac{k}{2}\sum_{n=1}^{\infty}(n+3-4\nu)\left[\bar{a}_n\frac{\sinh(n+1)\xi}{\cosh n\xi_0}\cos(n-1)\theta + a_n\frac{\cosh(n+1)\xi}{\cosh n\xi_0}\sin(n-1)\theta\right]$$
$$+\frac{k}{2}\sum_{n=1}^{\infty}(n-3+4\nu)\left[\bar{a}_n\frac{\sinh(n-1)\xi}{\cosh n\xi_0}\cos(n+1)\theta + a_n\frac{\cosh(n-1)\xi}{\cosh n\xi_0}\sin(n+1)\theta\right]$$
$$+\frac{k^2}{4}\sum_{n=2}^{\infty}\frac{1}{n-1}\left[\frac{\cosh(n-2)\xi}{\cosh n\xi_0}\sin n\theta - \frac{\cosh n\xi}{\cosh n\xi_0}\sin(n-2)\theta\right]\alpha A_n$$
$$+\frac{k^2}{4}\sum_{n=2}^{\infty}\frac{1}{n+1}\left[\frac{\cosh(n+2)\xi}{\cosh n\xi_0}\sin n\theta - \frac{\cosh n\xi}{\cosh n\xi_0}\sin(n+2)\theta\right]\alpha A_n$$
$$\left.+\frac{k^2}{2}(1+q)^2\frac{W_1}{1+\nu}\left[Q_1\cosh 2\xi\sin 2\theta - Q_2\sinh 2\xi\cos 2\theta\right]\right\}. \tag{84b}$$

12. Numerical results and conclusions

As an illustration, consider the concrete case for which

$$\begin{aligned} a &= 5, \quad b = 3, \quad B = 1.5, \quad \nu = 0.25, \\ \mu^* &= 10^{-7}, \quad \mu_0 = 0.9999, \quad \mu_1 = \mu_2 = 0.5. \end{aligned}$$

We study the effects of two factors γ_0 and β_0.

12.1. **Effect of the magnetic field orientation.**

Here we study the effect of the magnetic field orientation upon the stress tensor and displacement vector components. The angle ϕ of inclination of the magnetic field vector on the major axis of the ellipse takes on the following values:

$$\phi = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}.$$

Also, let

$$\gamma_0 = 0.5$$

and

$$\beta_0 = 0, \quad \alpha T_0 = 0 \quad , \alpha T_2 = 0,$$

so that all thermal effects are disregarded.

Figs. $(1-3)$ show the stress components distributions on the boundary for the chosen values of angle ϕ, corresponding to traces 1, 2, 3, 4 and 5 respectively.

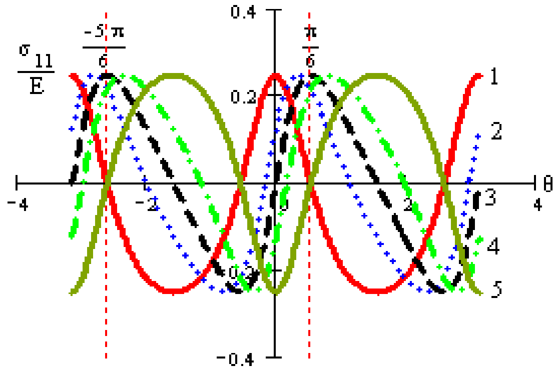

Fig (1): $\frac{\sigma_{11}}{E}$ on the boundary changing ϕ, $\beta_0 = 0$.

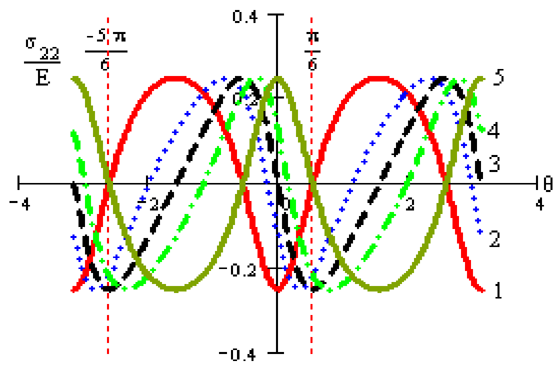

Fig (2): $\frac{\sigma_{22}}{E}$ on the boundary changing ϕ, $\beta_0 = 0$.

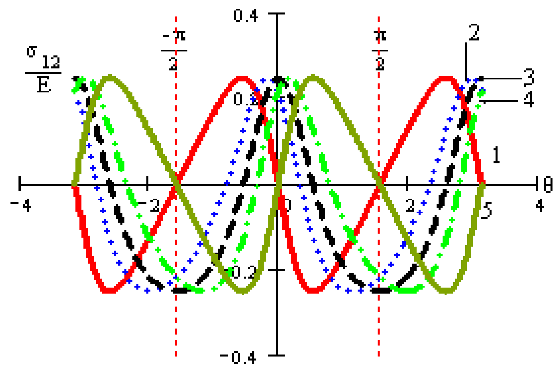

Fig (3): $\frac{\sigma_{12}}{E}$ on the boundary changing ϕ, $\beta_0 = 0$.

It is noted that the stress components in the two cases $\phi = 0$ and $\phi = \pi/2$ are very nearly reflections to each other with respect to the θ-axis. Also, as the angle ϕ increases, the curves are shifted to the right.

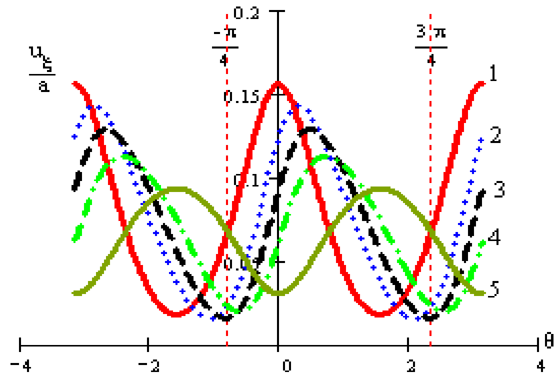

Fig (4): $\frac{u_\xi}{a}$ on the boundary changing ϕ, $\beta_0 = 0$.

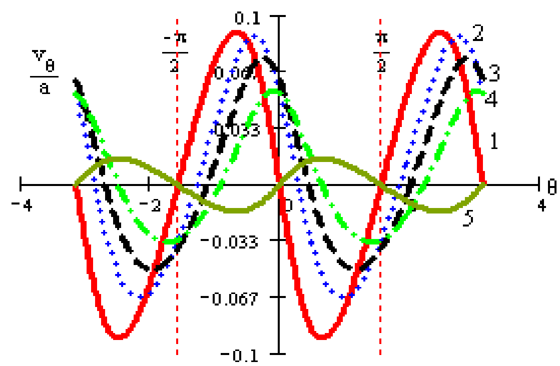

Fig (5): $\frac{v_\theta}{a}$ on the boundary changing ϕ, $\beta_0 = 0$.

Figures (4-5) show the effect of the magnetic field on the mechanical displacement, when the angle ϕ take different values. We note that u_ξ is always positive, a fact that reflects the well-known inflation phenomenon of the cylinder. As angle ϕ increases, the two displacement components become more uniform on the boundary.

12.2. **Effect of weak bulk heating.** In this subsection, we study the effect of a weak bulk heating on the stress tensor and displacement vector components. This is conditioned by the fact that heat effects are usually predominant over magnetic effects and in order to be able to discuss the combined effect, we must restrict our considerations to weak heating. We let β_0 vary over the values

$$\beta_0 = 0, 0.02, 0.1, 0.6$$

and for one value $\phi = \frac{\pi}{4}$. As before, we set

$$\gamma_0 = 0.5, \quad \alpha T_0 = 0 \quad , \alpha T_2 = 0.$$

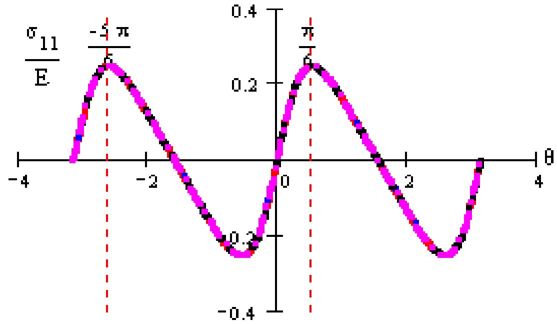

Fig (6): $\frac{\sigma_{11}}{E}$ on the boundary changing β_0, $\phi = \frac{\pi}{4}$.

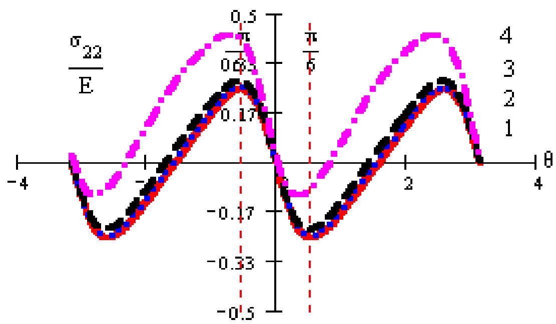

Fig (7): $\frac{\sigma_{22}}{E}$ on the boundary changing β_0, $\phi = \frac{\pi}{4}$.

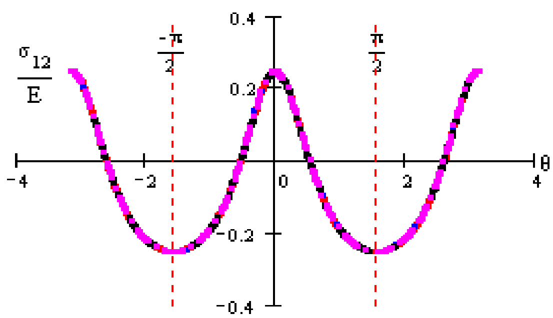

Fig (8): $\frac{\sigma_{12}}{E}$ on the boundary changing β_0, $\phi = \frac{\pi}{4}$.

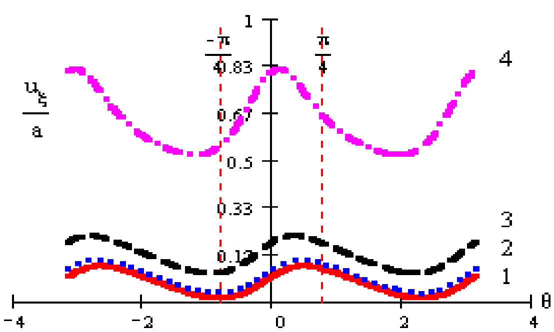

Fig (9): $\frac{u_\xi}{a}$ on the boundary changing β_0, $\phi = \frac{\pi}{4}$.

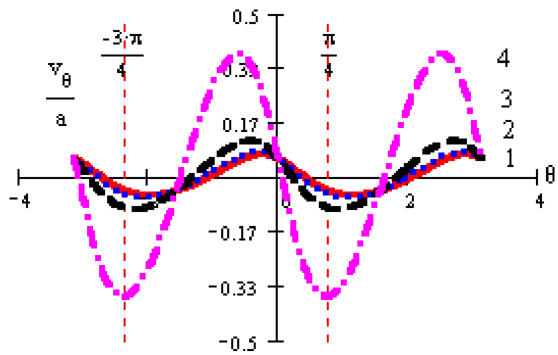

Fig (10): $\frac{v_\theta}{a}$ on the boundary changing β_0, $\phi = \frac{\pi}{4}$.

Figures (6-10) show the effect of the bulk heating on the stresses and displacements when the magnetic field is taken inclined at an angle $\frac{\pi}{4}$ to the major axis of the ellipse. Similar results were obtained for other values of the angle ϕ. We note that changing β_0 does not affect the stresses appreciably (for the stress components σ_{11} and σ_{12} the different curves are almost indistinguishably), as long as $\beta_0 < \gamma_0$. But if $\beta_0 > \gamma_0$, the changes become more appreciable, as expected.

Appendix A. Appendix

To calculate the resultant external force acting on the boundary due to the magnetic field where, the components of the resultant external forces in (x, y) are

$$X(\theta) = \int_0^\theta f_x(s') \frac{ds'}{d\theta'} d\theta', \quad Y(\theta) = \int_0^\theta f_y(s') \frac{ds'}{d\theta'} d\theta'.$$

The forces in the Cartesian coordinates can be written in elliptic coordinates at the boundary as where $h_0 = \frac{ds}{d\theta}$, then

$$X(\theta) = \int_0^\theta \left(b\cos\theta' f_\xi - a\sin\theta' f_\theta\right) d\theta', \; Y(\theta) = \int_0^\theta \left(a\sin\theta' f_\xi + b\cos\theta' f_\theta\right) d\theta'. \quad \text{(A.1.1)}$$

From equation (49) the magnetic force $\mathbf{f}$ can be written in matrix form in elliptic cylindrical coordinates as

$$\begin{bmatrix} f_\xi \\ f_\theta \end{bmatrix} = \begin{bmatrix} \sigma^*_{\xi\xi} & \sigma^*_{\xi\theta} \\ \sigma^*_{\xi\theta} & \sigma^*_{\theta\theta} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{i.e.} \quad f_\xi = \sigma^*_{\xi\xi}, \qquad f_\theta = \sigma^*_{\xi\theta},$$

where f_ξ and f_θ are the components of the magnetic forces in (ξ, θ), $\mathbf{n} = (1, 0)$ is the unit vector normal.

Use equations (66a, 66c), then, the external forces are

$$\frac{2}{\gamma_0}\frac{X(\theta)}{aE} = (q\mathcal{H}_3+\mathcal{H}_4)\int_0^\theta \frac{\sin\theta}{1-k^2\cos^2\theta}d\theta + (q\mathcal{H}_3+\mathcal{H}_5)\int_0^\theta \frac{\sin\theta}{1-k^2\cos^2\theta}\cos 2\theta d\theta,$$

$$+ (q\mathcal{H}_1+\mathcal{H}_6)\int_0^\theta \frac{\cos\theta}{1-k^2\cos^2\theta}d\theta + (q\mathcal{H}_2-\mathcal{H}_6)\int_0^\theta \frac{\cos\theta}{1-k^2\cos^2\theta}\cos 2\theta d\theta$$

$$\frac{2}{\gamma_0}\frac{Y(\theta)}{aE} = (\mathcal{H}_1-q\mathcal{H}_6)\int_0^\theta \frac{\sin\theta}{1-k^2\cos^2\theta}d\theta + (\mathcal{H}_2-q\mathcal{H}_6)\int_0^\theta \frac{\sin\theta}{1-k^2\cos^2\theta}\cos 2\theta d\theta$$

$$+ (\mathcal{H}_3-q\mathcal{H}_4)\int_0^\theta \frac{\cos\theta}{1-k^2\cos^2\theta}d\theta - (\mathcal{H}_3+q\mathcal{H}_5)\int_0^\theta \frac{\cos\theta}{1-k^2\cos^2\theta}\cos 2\theta d\theta,$$

then

$$\frac{X(\theta)}{aE} = \frac{\gamma_0}{2k^3}\left\{\mathcal{G}_1' - \mathcal{G}_2'\sin\theta + \mathcal{G}_3'\cos\theta + \mathcal{G}_4'\operatorname{atan}(k'\sin\theta) - \mathcal{G}_5'\operatorname{atanh}(k\cos\theta)\right\}, \tag{A.1.2a}$$

$$\frac{Y(\theta)}{aE} = \frac{\gamma_0}{2k^3}\left\{\mathcal{G}_6' + \mathcal{G}_7'\sin\theta + \mathcal{G}_8'\cos\theta - \mathcal{G}_4'\operatorname{atanh}(k\cos\theta) - \mathcal{G}_5'\operatorname{atan}(k'\sin\theta)\right\}, \tag{A.1.2b}$$

with

$$\begin{aligned}
\mathcal{G}_1' &= -2qk\mathcal{H}_3 - 2k\mathcal{H}_5 + \left[2q\mathcal{H}_3 + k^2\mathcal{H}_4 + \left(1+q^2\right)\mathcal{H}_5\right]\operatorname{atanh} k,\\
\mathcal{G}_2' &= 2k\left[q\mathcal{H}_2 - \mathcal{H}_6\right], \qquad \mathcal{G}_4' = \left[k^2\mathcal{H}_1 + \left(1+q^2\right)\mathcal{H}_2 - 2q\mathcal{H}_6\right]\\
\mathcal{G}_3' &= 2k\left[q\mathcal{H}_3 + \mathcal{H}_5\right], \qquad \mathcal{G}_5' = \left[2q\mathcal{H}_3 + k^2\mathcal{H}_4 + \left(1+q^2\right)\mathcal{H}_5\right],\\
\mathcal{G}_6' &= -2k\mathcal{H}_2 + 2qk\mathcal{H}_6 + \left[k^2\mathcal{H}_1 - 2q\mathcal{H}_6 + \left(1+q^2\right)\mathcal{H}_2\right]\operatorname{atanh} k,\\
\mathcal{G}_7' &= 2k\left[\mathcal{H}_3 + q\mathcal{H}_5\right], \qquad ,\mathcal{G}_8' = 2k\left[\mathcal{H}_2 - q\mathcal{H}_6\right].
\end{aligned}$$

References

[1] M. S. Abou-Dina and A. F. Ghaleb, *On the boundary integral formulation of the plane theory of elasticity with applications (Analytical Aspects)*. J. Comp. and Appl. Math. 106 n°1, 55-70 (1999).

[2] M. S. Abou-Dina and A. F. Ghaleb, *On the boundary integral formulation of the plane theory of thermoelasticity (Analytical Aspects)*.J. Thermal Stresses **25,** n°1, 1-29 (2002) .

[3] M. S. Abou-Dina and A. F. Ghaleb, *Boundary integral formulation of the plane theory of thermo-magnetoelasticity.* I. J. of Appl. Electromagnetics and Mechanics. n°11, 185-201 (2000).

[4] A. R. El Dhaba, M. S. Abou-Dina and A. F. Ghaleb, *A problem of plane, uncoupled linear thermoelasticity for an infinite, elliptical cylinder by a boundary integral method.* Journal of Thermal Stresses, Vol. 26, Num. 2, pp. 93-121 (2003).

[5] V. C. A. Ferraro, Electromagnetic theory, University of London, The Athlone Press (1970).

[6] L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of continuous media, 2^{nd}* ed., Pergamon Press (1984).

[7] R. J. Knops, *Two-dimensional electrostriction,* Quart. J. Mech. Appl. Math., **16**, n°3, 377-388 (1963).

[8] A. F. Ghaleb, *Deformation of an infinite, hollow, circular cylindrical conductor carrying a current,* Proc. Math. Phys. Soc. Egypt, **45**, 63-68 (1978).

[9] M. M. Ayad *et al.*, *Deformation of an infinite elliptic cylindrical conductor carrying a uniform axial current,* mechanics of materials, **17**, 351-361 (1994).

[10] K. Yuan, *Magnetothermoelastic stresses in an infinitely long cylindrical conductor carrying a uniformly distributed axial current,* Appl. Sci. Res., **26**, n°3,4, 307-314 (1972).

[11] Morse, P. M. and Feshbach, H. *Methods of theoretical physics*, Part I. McGraw-Hill, New York:, p. 657, 1953.

[12] I. S. Gradshtein and I. M. Ryzhik, *Table of integrals, series, and products.* Corrected and Enlarged Edition. New York, Toronto, (1980).
[13] Philip M. Morse, Herman Feshbach, *Methods of theoretical physics*, New York, Toronto, Part II (1953).
[14] A. E. H. Love *The mathematical theory of elasticity,* Dover Publ., New York (1944).

A. R. El Dhaba, Faculty of Science, Damanhour University, Damanhour, Egypt
E-mail address: amrramadaneg@yahoo.com

A. F. Ghaleb, Cairo University, Giza, Egypt
E-mail address: afghaleb@gmail.com

M. S. Abou-Dina, Faculty of Science, Cairo University, Giza, Egypt
E-mail address: moustafa_aboudina@hotmail.com