Electronic Journal of Mathematical Analysis and Applications
Vol. 11(1) Jan. 2023, pp. 1-15.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

QUALITATIVE ANALYSIS OF NONLINEAR DIFFERENCE EQUATIONS WITH A THIRTY-ORDER

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Abstract. The main purpose of this paper is to determine the forms of the solutions to the following thirty-order nonlinear difference equations:

$$\zeta_{r+1} = \frac{\zeta_{r-29}}{\pm 1 \pm \prod_{i=0}^{5} \zeta_{r-(5i+4)}}, \quad r = 0, 1, 2, ...$$

where the initial conditions ζ_{-29}, ζ_{-28}, ..., ζ_0 are arbitrary real numbers. Moreover, we investigate stability, boundedness, oscillation and the periodic character of these solutions. Finally, we confirm the results with some numerical examples and graphs. The MATLAB program is used to plot the presented figures.

1. Introduction

Difference equations have been successfully used to build and analyze mathematical models of biology, ecology, physics, economic processes, and other fields. Because we still know so little about nonlinear rational difference equations, research into these equations is considered as an active area. The study of global attractivity, boundedness character, periodicity, and the solution form of nonlinear difference equations has sparked a lot of interest recently. For some examples of outcomes in this field. Ahmed et al. [4] obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-14}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n = 0, 1, 2, ...,$$

where the initial conditions are arbitrary real numbers. Ahmed et al. [6] obtained the expressions of solutions of the class of difference equations

$$x_{n+1} = \frac{x_{n-2k+1}}{\pm 1 \pm \Pi_{i=1}^{k} x_{n-2i+1}}, \quad n = 0, 1, 2, ...,$$

with conditions posed on the initial values $x_{-j},\ j = 0, 1, 2, ..., 2k-1;\ k \in \{1, 2, ...\}$. Elsayed et al. [14] obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \quad n = 0, 1, 2, ...,$$

2010 *Mathematics Subject Classification.* 39A33, 39A30, 39A23, 39A22, 39A10, 39A06.
Key words and phrases. equilibrium, asymptotic stability, periodicity, exact and numerical solutions.
Submitted Feb. 17, 2022.

where the initial conditions are arbitrary real numbers. For other related papers, see [1–3, 5, 7–13, 15–20].
This work aims to analyze some dynamical properties such as equilibrium points, local and global behaviors, boundedness, and analytic solutions of the nonlinear recursive equations

$$\zeta_{r+1}=\frac{\zeta_{r-29}}{\pm 1\pm\prod_{i=0}^{5}\zeta_{r-(5i+4)}},\qquad r=0,1,2,....$$

Here, the initial values ζ_{-29}, ζ_{-28}, ..., ζ_0 are arbitrary real numbers. In this paper, we also illustrate some 2D figures with the help of MATLAB to validate the obtained results.
Definition 1 We consider $mod\,(\kappa,5)=\kappa-5\left[\frac{\kappa}{5}\right]$, where $[\Lambda]$ is the greatest integer less than or equal to the real number Λ.

2. The Difference Equation $\zeta_{r+1}=\frac{\zeta_{r-29}}{1+\prod_{i=0}^{5}\zeta_{r-(5i+4)}}$

In this section we give a specific form of the solutions of the first equation in the form

$$\zeta_{r+1}=\frac{\zeta_{r-29}}{1+\prod_{i=0}^{5}\zeta_{r-(5i+4)}},\qquad r=0,1,2,..., \tag{1}$$

with conditions posed on the initial values $\zeta_{-j},\ j=0,1,2,...,29$. Also, we investigate the stability and boundedness of these solutions.

Theorem 1. *Let $\{\zeta_r\}_{r=-29}^{\infty}$ be a solution of Eq. (1). Then, for $r=0,1,2,...$,*

$$\zeta_{30r-k}=\varepsilon_k\prod_{i=0}^{r-1}\left(\frac{1+(6i+\eta_k-1)\,\mu_k}{1+(6i+\eta_k)\,\mu_k}\right), \tag{2}$$

where $\mu_k=\prod_{j=0}^{5}\varepsilon_{mod(k,5)+5j}$, $\eta_k=6-\left[\frac{k}{5}\right]$ and $\zeta_{-k}=\varepsilon_k$, with $n\mu_k\neq-1$ such that $n\in\{1,2,3,...\},\ k=0,1,2,...,29$.

Proof. For $r=0$, the result holds. Now suppose that $r>0$ and that our assumption holds for $r-1$. That is

$$\zeta_{30r-30-k}=\varepsilon_k\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_k-1)\,\mu_k}{1+(6i+\eta_k)\,\mu_k}\right). \tag{3}$$

Now, it follows from Eq. (1) and using Eq. (3) that

$$\zeta_{30r-29}=\frac{\zeta_{30r-59}}{1+\zeta_{30r-34}\zeta_{30r-39}\zeta_{30r-44}\zeta_{30r-49}\zeta_{30r-54}\zeta_{30r-59}}$$

$$=\frac{\varepsilon_{29}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{29}-1)\mu_{29}}{1+(6i+\eta_{29})\mu_{29}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{5j+4}\prod_{i=0}^{r-2}\left(\frac{1+\left(6i+\eta_{5j+4}-1\right)\mu_{5j+4}}{1+\left(6i+\eta_{5j+4}\right)\mu_{5j+4}}\right)\right)}=\frac{\varepsilon_{29}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}{1+(6i+1)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}\right)}{1+\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}{1+(6i+6)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}\right)}.$$

Hence, we have

$$\zeta_{30r-29}=\varepsilon_{29}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}{1+(6i+1)\,\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}\right).$$

Also, it follows from Eq. (1) and using Eq. (3) that

$$\zeta_{30r-28}=\frac{\zeta_{30r-58}}{1+\zeta_{30r-33}\zeta_{30r-38}\zeta_{30r-43}\zeta_{30r-48}\zeta_{30r-53}\zeta_{30r-58}}$$

$$=\frac{\varepsilon_{28}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{28}-1)\mu_{28}}{1+(6i+\eta_{28})\mu_{28}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{5j+3}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{5j+3}-1)\mu_{5j+3}}{1+(6i+\eta_{5j+3})\mu_{5j+3}}\right)\right)}=\frac{\varepsilon_{28}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}{1+(6i+1)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}\right)}{1+\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}{1+(6i+6)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}\right)}.$$

Hence, we have

$$\zeta_{30r-28}=\varepsilon_{28}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}{1+(6i+1)\,\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}\right).$$

Also, it follows from Eq. (1) and using Eq. (3) that

$$\zeta_{30r-27}=\frac{\zeta_{30r-57}}{1+\zeta_{30r-32}\zeta_{30r-37}\zeta_{30r-42}\zeta_{30r-47}\zeta_{30r-52}\zeta_{30r-57}}$$

$$=\frac{\varepsilon_{27}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{27}-1)\mu_{27}}{1+(6i+\eta_{27})\mu_{27}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{5j+2}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{5j+2}-1)\mu_{5j+2}}{1+(6i+\eta_{5j+2})\mu_{5j+2}}\right)\right)}=\frac{\varepsilon_{27}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}{1+(6i+1)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}\right)}{1+\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}{1+(6i+6)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}\right)}.$$

Hence, we have

$$\zeta_{30r-27}=\varepsilon_{27}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}{1+(6i+1)\,\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}\right).$$

Also, it follows from Eq. (1) and using Eq. (3) that

$$\zeta_{30r-26}=\frac{\zeta_{30r-56}}{1+\zeta_{30r-31}\zeta_{30r-36}\zeta_{30r-41}\zeta_{30r-46}\zeta_{30r-51}\zeta_{30r-56}}$$

$$=\frac{\varepsilon_{26}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{26}-1)\mu_{26}}{1+(6i+\eta_{26})\mu_{26}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{5j+1}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{5j+1}-1)\mu_{5j+1}}{1+(6i+\eta_{5j+1})\mu_{5j+1}}\right)\right)}=\frac{\varepsilon_{26}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}{1+(6i+1)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}\right)}{1+\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}{1+(6i+6)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}\right)}.$$

Hence, we have

$$\zeta_{30r-26}=\varepsilon_{26}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}{1+(6i+1)\,\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}\right).$$

Also, it follows from Eq. (1) and using Eq. (3) that

$$\zeta_{30r-25}=\frac{\zeta_{30r-55}}{1+\zeta_{30r-30}\zeta_{30r-35}\zeta_{30r-40}\zeta_{30r-45}\zeta_{30r-50}\zeta_{30r-55}}$$

$$=\frac{\varepsilon_{25}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{25}-1)\mu_{25}}{1+(6i+\eta_{25})\mu_{25}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{5j}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{5j}-1)\mu_{5j}}{1+(6i+\eta_{5j})\mu_{5j}}\right)\right)}=\frac{\varepsilon_{25}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}{1+(6i+1)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}\right)}{1+\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}{1+(6i+6)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}\right)}.$$

Hence, we have

$$\zeta_{30r-25}=\varepsilon_{25}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}{1+(6i+1)\,\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2. *Assume that the initial values $\zeta_{-29}, \zeta_{-28}, ..., \zeta_0 \in [0, \infty)$, then every solution of Eq.* (1) *is bounded.*

Proof. Let $\{\zeta_r\}_{r=-29}^{\infty}$be a solution of Eq. (1). It follows from Eq. (1) that

$$0 \leq \zeta_{r+1} = \frac{\zeta_{r-29}}{1 + \zeta_{r-4}\zeta_{r-9}\zeta_{r-14}\zeta_{r-19}\zeta_{r-24}\zeta_{r-29}} \leq \zeta_{r-29} \quad \forall r \geq 0.$$

Then, the sequence $\{\zeta_{30r-i}\}_{r=0}^{\infty}$, $i = 0, 1, ..., 29$ is decreasing and so is bounded from above by $\eta = \max\{\zeta_{-29}, \zeta_{-28}, ..., \zeta_0\}$.

Theorem 3. *The only equilibrium point $\overline{\zeta}$ of Eq.* (1) *is $\overline{\zeta} = 0$.*

Proof. From Eq. (1), we can write

$$\overline{\zeta} = \frac{\overline{\zeta}}{1 + \overline{\zeta}^6}.$$

Then, we have

$$\overline{\zeta} + \overline{\zeta}^7 = \overline{\zeta},$$

or,

$$\overline{\zeta}^7 = 0.$$

Thus, the only equilibrium point of Eq. (1) is $\overline{\zeta} = 0$.

Theorem 4. *Assume that the initial values $\zeta_{-29}, \zeta_{-28}, ..., \zeta_0 \in [0, \infty)$, then the equilibrium point $\overline{\zeta} = 0$ of Eq.* (1) *is locally stable.*

Proof. Let $\epsilon > 0$, and let $\{\zeta_r\}_{r=-29}^{\infty}$be a solution of Eq. (1) such that

$$\sum_{j=0}^{29} |\zeta_{-j}| < \epsilon.$$

It suffices to show that $|\zeta_1| < \epsilon$. Now

$$0 < \zeta_1 = \frac{\zeta_{-29}}{1 + \zeta_{-4}\zeta_{-9}\zeta_{-14}\zeta_{-19}\zeta_{-24}\zeta_{-29}} \leq \zeta_{-29} < \epsilon,$$

and so the proof is completed.

Theorem 5. *Assume that the initial values $\zeta_{-29}, \zeta_{-28}, ..., \zeta_0 \in [0, \infty)$, then the equilibrium point $\overline{\zeta} = 0$ of Eq.* (1) *is globally asymptotically stable.*

Proof. We know by Theorem 4 that the equilibrium point $\overline{\zeta} = 0$ of Eq. (1) is locally stable. So let $\{\zeta_r\}_{r=-29}^{\infty}$ be a positive solution of Eq. (1). It suffices to show that $\lim_{r\to\infty} \zeta_r = \overline{\zeta} = 0$. From Theorem 2 we have $\zeta_{r+1} < \zeta_{r-29}$ for all $r \geq 0$, so the sequences $\{\zeta_{30r-i}\}_{r=0}^{\infty}$, $i = 0, 1, ..., 29$ are decreasing and bounded which implies that the sequences $\{\zeta_{30r-i}\}_{r=0}^{\infty}, i = 0, 1, ..., 29$ converge to limit (say $Z_i \geq 0$). So

$$Z_{29} = \frac{Z_{29}}{1 + Z_4 Z_9 Z_{14} Z_{19} Z_{24} Z_{29}} = 0, Z_{28} = \frac{Z_{28}}{1 + Z_3 Z_8 Z_{13} Z_{18} Z_{23} Z_{28}} = 0, ..., Z_0 = \frac{Z_0}{1 + Z_0 Z_5 Z_{10} Z_{15} Z_{20} Z_{25}} = 0,$$

which implies that $Z_0 = Z_1 = ... = Z_{29} = 0$, from which the result follows.

3. The Difference Equation $\zeta_{r+1}=\frac{\zeta_{r-29}}{1-\prod_{i=0}^{5}\zeta_{r-(5i+4)}}$

In this section we give a specific form of the solutions of the second equation in the form

$$\zeta_{r+1}=\frac{\zeta_{r-29}}{1-\prod_{i=0}^{5}\zeta_{r-(5i+4)}},\qquad r=0,1,2,..., \tag{4}$$

with conditions posed on the initial values $\zeta_{-j},\ j=0,1,2,...,29$.

Theorem 6. *Let $\{\zeta_r\}_{r=-29}^{\infty}$ be a solution of Eq. (4). Then, for $r=0,1,2,...$*

$$\zeta_{30r-k}=\varepsilon_k\prod_{i=0}^{r-1}\left(\frac{-1+(6i+\eta_k-1)\,\mu_k}{-1+(6i+\eta_k)\,\mu_k}\right), \tag{5}$$

where $\mu_k=\prod_{j=0}^{5}\varepsilon_{mod(k,5)+5j}$, $\eta_k=6-\left[\frac{k}{5}\right]$ and $\zeta_{-k}=\varepsilon_k$, with $n\mu_k\neq 1$ such that $n\in\{1,2,3,...\},\ k=0,1,2,...,29$.

Proof. For $r=0$, the result holds. Now suppose that $r>0$ and that our assumption holds for $r-1$. That is

$$\zeta_{30r-30-k}=\varepsilon_k\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_k-1)\,\mu_k}{-1+(6i+\eta_k)\,\mu_k}\right). \tag{6}$$

Now, it follows from Eq. (4) and Eq. (6) that

$$\zeta_{30r-29}=\frac{\zeta_{30r-59}}{1-\zeta_{30r-34}\zeta_{30r-39}\zeta_{30r-44}\zeta_{30r-49}\zeta_{30r-54}\zeta_{30r-59}}$$

$$=\frac{\varepsilon_{29}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{29}-1)\mu_{29}}{-1+(6i+\eta_{29})\mu_{29}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{5j+4}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{5j+4}-1)\mu_{5j+4}}{-1+(6i+\eta_{5j+4})\mu_{5j+4}}\right)\right)}=\frac{\varepsilon_{29}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}{-1+(6i+1)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}\right)}{1-\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}{-1+(6i+6)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}\right)}.$$

Hence, we have

$$\zeta_{30r-29}=\varepsilon_{29}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}{-1+(6i+1)\,\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29}}\right).$$

Also, it follows from (4) and Eq. (6) that

$$\zeta_{30r-28}=\frac{\zeta_{30r-58}}{1-\zeta_{30r-33}\zeta_{30r-38}\zeta_{30r-43}\zeta_{30r-48}\zeta_{30r-53}\zeta_{30r-58}}$$

$$=\frac{\varepsilon_{28}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{28}-1)\mu_{28}}{-1+(6i+\eta_{28})\mu_{28}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{5j+3}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{5j+3}-1)\mu_{5j+3}}{-1+(6i+\eta_{5j+3})\mu_{5j+3}}\right)\right)}=\frac{\varepsilon_{28}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}{-1+(6i+1)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}\right)}{1-\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}{-1+(6i+6)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}\right)}.$$

Hence, we have

$$\zeta_{30r-28}=\varepsilon_{28}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}{-1+(6i+1)\,\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28}}\right).$$

Also, it follows from (4) and Eq. (6) that

$$\zeta_{30r-27}=\frac{\zeta_{30r-57}}{1-\zeta_{30r-32}\zeta_{30r-37}\zeta_{30r-42}\zeta_{30r-47}\zeta_{30r-52}\zeta_{30r-57}}$$

$$= \frac{\varepsilon_{27}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{27}-1)\mu_{27}}{-1+(6i+\eta_{27})\mu_{27}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{5j+2}\prod_{i=0}^{r-2}\left(\frac{-1+\left(6i+\eta_{5j+2}-1\right)\mu_{5j+2}}{-1+\left(6i+\eta_{5j+2}\right)\mu_{5j+2}}\right)\right)} = \frac{\varepsilon_{27}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}{-1+(6i+1)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}\right)}{1-\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}{-1+(6i+6)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}\right)}.$$

Hence, we have

$$\zeta_{30r-27} = \varepsilon_{27}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}{-1+(6i+1)\,\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27}}\right).$$

Also, it follows from (4) and Eq. (6) that

$$\zeta_{30r-26} = \frac{\zeta_{30r-56}}{1-\zeta_{30r-31}\zeta_{30r-36}\zeta_{30r-41}\zeta_{30r-46}\zeta_{30r-51}\zeta_{30r-56}}$$

$$= \frac{\varepsilon_{26}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{26}-1)\mu_{26}}{-1+(6i+\eta_{26})\mu_{26}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{5j+1}\prod_{i=0}^{r-2}\left(\frac{-1+\left(6i+\eta_{5j+1}-1\right)\mu_{5j+1}}{-1+\left(6i+\eta_{5j+1}\right)\mu_{5j+1}}\right)\right)} = \frac{\varepsilon_{26}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}{-1+(6i+1)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}\right)}{1-\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}{-1+(6i+6)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}\right)}.$$

Hence, we have

$$\zeta_{30r-26} = \varepsilon_{26}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}{-1+(6i+1)\,\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26}}\right).$$

Also, it follows from (4) and Eq. (6) that

$$\zeta_{30r-25} = \frac{\zeta_{30r-55}}{1-\zeta_{30r-30}\zeta_{30r-35}\zeta_{30r-40}\zeta_{30r-45}\zeta_{30r-50}\zeta_{30r-55}}$$

$$= \frac{\varepsilon_{25}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{25}-1)\mu_{25}}{-1+(6i+\eta_{25})\mu_{25}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{5j}\prod_{i=0}^{r-2}\left(\frac{-1+\left(6i+\eta_{5j}-1\right)\mu_{5j}}{-1+\left(6i+\eta_{5j}\right)\mu_{5j}}\right)\right)} = \frac{\varepsilon_{25}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}{-1+(6i+1)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}\right)}{1-\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}{-1+(6i+6)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}\right)}.$$

Hence, we have

$$\zeta_{30r-25} = \varepsilon_{25}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}{-1+(6i+1)\,\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}}\right).$$

Similarly, one can easily obtain the other relations. Hence, the proof is completed.

Theorem 7. *Eq.* (4) *has a unique equilibrium point* $\overline{\zeta} = 0$, *which is a non hyperbolic equilibrium point.*

Proof. From Eq. (4), we can write

$$\overline{\zeta} = \frac{\overline{\zeta}}{1-\overline{\zeta}^6}.$$

Then, we have

$$\overline{\zeta} - \overline{\zeta}^7 = \overline{\zeta},$$

or,

$$\overline{\zeta}^7 = 0.$$

Thus the only equilibrium point of Eq. (4) is $\overline{\zeta} = 0$.
Define the function $f\left(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\right) = \frac{\zeta_1}{1-\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6}$ on I^6 where I is a subset of R such that $0 \in I$ and $f(I^6) \subseteq I$. Clearly, f is continuously differentiable on I^6 and we have

$$f_{\zeta_1}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6) = \frac{1}{(1-\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6)^2},\ f_{\zeta_2}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6) = \frac{\zeta_1^2\zeta_3\zeta_4\zeta_5\zeta_6}{(1-\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6)^2},$$
$$f_{\zeta_3}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6) = \frac{\zeta_1^2\zeta_2\zeta_4\zeta_5\zeta_6}{(1-\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6)^2}, f_{\zeta_4}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6) = \frac{\zeta_1^2\zeta_2\zeta_3\zeta_5\zeta_6}{(1-\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6)^2},$$
$$f_{\zeta_5}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6) = \frac{\zeta_1^2\zeta_2\zeta_3\zeta_4\zeta_6}{(1-\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6)^2}, f_{\zeta_6}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6) = \frac{\zeta_1^2\zeta_2\zeta_3\zeta_4\zeta_5}{(1-\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6)^2},$$

which implies that

$f_{\zeta_1}(\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta}) = 1,\ f_{\zeta_2}(\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta}) = f_{\zeta_3}(\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta}) = f_{\zeta_4}(\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta}) = f_{\zeta_5}(\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta}) = f_{\zeta_6}(\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta},\overline{\zeta}) = 0$. The linearized equation of Eq. (4) about the equilibrium point $\overline{\zeta} = 0$ is

$$\zeta_{r+1} = \zeta_{r-29}, \tag{7}$$

and the characteristic equation of Eq. (7) about the equilibrium point $\overline{\zeta} = 0$ is

$$\lambda^{30} - 1 = 0,$$

which implies that

$$|\lambda_i| = 1,\ \ i = 1,2,...,30.$$

So, $\overline{\zeta}$ is a non hyperbolic equilibrium point.

4. The Difference Equation $\zeta_{r+1} = \frac{\zeta_{r-29}}{-1+\prod_{i=0}^{5}\zeta_{r-(5i+4)}}$

In this section we give a specific form of the solutions of the third equation in the form

$$\zeta_{r+1} = \frac{\zeta_{r-29}}{-1+\prod_{i=0}^{5}\zeta_{r-(5i+4)}}, \quad r = 0,1,2,..., \tag{8}$$

with conditions posed on the initial values $\zeta_{-j},\ j = 0,1,2,...,29$. Also, we investigate the oscillation and periodicity of these solutions.

Theorem 8. *Let $\{\zeta_r\}_{r=-29}^{\infty}$ be a solution of the difference Eq.* (8). *Then for* $r = 0,1,2,...,$

$$\zeta_{30r-k} = \frac{\varepsilon_k}{(-1+\mu_k)^{r\alpha_k}}, \tag{9}$$

where $\mu_k = \prod_{j=0}^{5}\varepsilon_{mod(k,5)+5j}$, $\alpha_k = (-1)^{\left[\frac{k}{5}\right]+1}$ *and* $\zeta_{-k} = \varepsilon_k$, *with* $\mu_k \neq 1,\ k = 0,1,2,...,29$.

Proof. For $r = 0$, the result holds. Now suppose that $r > 0$ and that our assumption holds for $r-1$. That is

$$\zeta_{30r-30-k} = \frac{\varepsilon_k}{(-1+\mu_k)^{(r-1)\alpha_k}}. \tag{10}$$

Now, it follows from Eq. (8) and using Eq. (10) that

$$\zeta_{30r-29} = \frac{\zeta_{30r-59}}{-1+\zeta_{30r-34}\zeta_{30r-39}\zeta_{30r-44}\zeta_{30r-49}\zeta_{30r-54}\zeta_{30r-59}}$$

$$= \frac{\frac{\varepsilon_{29}}{(-1+\mu_{29})^{r-1}}}{-1+\varepsilon_4(-1+\mu_4)^{r-1}\frac{\varepsilon_9}{(-1+\mu_9)^{r-1}}\varepsilon_{14}(-1+\mu_{14})^{r-1}\frac{\varepsilon_{19}}{(-1+\mu_{19})^{r-1}}\varepsilon_{24}(-1+\mu_{24})^{r-1}\frac{\varepsilon_{29}}{(-1+\mu_{29})^{r-1}}}$$
$$= \frac{\varepsilon_{29}}{(-1+\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29})^{r-1}(-1+\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29})}.$$

Hence, we have

$$\zeta_{30r-29} = \frac{\varepsilon_{29}}{(-1+\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29})^{r}}.$$

Also, it follows from (8) and using Eq. (10) that

$$\zeta_{30r-28} = \frac{\zeta_{30r-58}}{-1+\zeta_{30r-33}\zeta_{30r-38}\zeta_{30r-43}\zeta_{30r-48}\zeta_{30r-53}\zeta_{30r-58}}$$
$$= \frac{\frac{\varepsilon_{28}}{(-1+\mu_{28})^{r-1}}}{-1+\varepsilon_3(-1+\mu_3)^{r-1}\frac{\varepsilon_8}{(-1+\mu_8)^{r-1}}\varepsilon_{13}(-1+\mu_{13})^{r-1}\frac{\varepsilon_{18}}{(-1+\mu_{18})^{r-1}}\varepsilon_{23}(-1+\mu_{23})^{r-1}\frac{\varepsilon_{28}}{(-1+\mu_{28})^{r-1}}}$$
$$= \frac{\varepsilon_{28}}{(-1+\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28})^{r-1}(-1+\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28})}.$$

Hence, we have

$$\zeta_{30r-28} = \frac{\varepsilon_{28}}{(-1+\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28})^{r}}.$$

Also, it follows from (8) and using Eq. (10) that

$$\zeta_{30r-27} = \frac{\zeta_{30r-57}}{-1+\zeta_{30r-32}\zeta_{30r-37}\zeta_{30r-42}\zeta_{30r-47}\zeta_{30r-52}\zeta_{30r-57}}$$
$$= \frac{\frac{\varepsilon_{27}}{(-1+\mu_{27})^{r-1}}}{-1+\varepsilon_2(-1+\mu_2)^{r-1}\frac{\varepsilon_7}{(-1+\mu_7)^{r-1}}\varepsilon_{12}(-1+\mu_{12})^{r-1}\frac{\varepsilon_{17}}{(-1+\mu_{17})^{r-1}}\varepsilon_{22}(-1+\mu_{22})^{r-1}\frac{\varepsilon_{27}}{(-1+\mu_{27})^{r-1}}}$$
$$= \frac{\varepsilon_{27}}{(-1+\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27})^{r-1}(-1+\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27})}.$$

Hence, we have

$$\zeta_{30r-27} = \frac{\varepsilon_{27}}{(-1+\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27})^{r}}.$$

Also, it follows from (8) and using Eq. (10) that

$$\zeta_{30r-26} = \frac{\zeta_{30r-56}}{-1+\zeta_{30r-31}\zeta_{30r-36}\zeta_{30r-41}\zeta_{30r-46}\zeta_{30r-51}\zeta_{30r-56}}$$
$$= \frac{\frac{\varepsilon_{26}}{(-1+\mu_{26})^{r-1}}}{-1+\varepsilon_1(-1+\mu_1)^{r-1}\frac{\varepsilon_6}{(-1+\mu_6)^{r-1}}\varepsilon_{11}(-1+\mu_{11})^{r-1}\frac{\varepsilon_{16}}{(-1+\mu_{16})^{r-1}}\varepsilon_{21}(-1+\mu_{21})^{r-1}\frac{\varepsilon_{26}}{(-1+\mu_{26})^{r-1}}}$$
$$= \frac{\varepsilon_{26}}{(-1+\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26})^{r-1}(-1+\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26})}.$$

Hence, we have

$$\zeta_{30r-26} = \frac{\varepsilon_{26}}{(-1+\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26})^{r}}.$$

Also, it follows from (8) and using Eq. (10) that

$$\zeta_{30r-25} = \frac{\zeta_{30r-55}}{-1+\zeta_{30r-30}\zeta_{30r-35}\zeta_{30r-40}\zeta_{30r-45}\zeta_{30r-50}\zeta_{30r-55}}$$

$$= \frac{\frac{\varepsilon_{25}}{(-1+\mu_{25})^{r-1}}}{-1+\varepsilon_0(-1+\mu_0)^{r-1}\frac{\varepsilon_5}{(-1+\mu_5)^{r-1}}\varepsilon_{10}(-1+\mu_{10})^{r-1}\frac{\varepsilon_{15}}{(-1+\mu_{15})^{r-1}}\varepsilon_{20}(-1+\mu_{20})^{r-1}\frac{\varepsilon_{25}}{(-1+\mu_{25})^{r-1}}}$$

$$= \frac{\varepsilon_{25}}{(-1+\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25})^{r-1}(-1+\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25})}.$$

Hence, we have

$$\zeta_{30r-25} = \frac{\varepsilon_{25}}{(-1+\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25})^{r}}.$$

Similarly, one can easily obtain the other relations. Hence, the proof is completed.

Theorem 9. *Eq.* (8) *has three equilibrium points* 0 *and* $\pm\sqrt[6]{2}$*, which are nonhyperbolic equilibrium points.*

Proof. The proof is similar to the proof of Theorem 7, and will be omitted.

Theorem 10. *Eq.* (8) *is periodic of period 30 iff* $\mu_k = 2$; $k = 0, 1, ..., 29$ *and will be take the form*

$$\zeta_{30r-k} = \varepsilon_k, \quad k = 0, 1, ..., 29, \quad r = 0, 1, 2, ...$$

Proof. The proof follows immediately from Theorem 8.

Theorem 11. *Assume that* $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{29} \in (0, 1)$*. Then the solution* $\{\zeta_r\}_{r=-29}^{\infty}$ *oscillates about the equilibrium point* $\overline{\zeta} = 0$*, with positive semicycles of length 30, and negative semicycles of length 30.*

Proof. From Theorem 8, we have $\zeta_1, \zeta_2, ..., \zeta_{30} < 0$ and $\zeta_{31}, \zeta_{32}, ..., \zeta_{60} > 0$, and the result follows by induction.

5. The Difference Equation $\zeta_{r+1} = \frac{\zeta_{r-29}}{-1-\prod_{i=0}^{5}\zeta_{r-(5i+4)}}$

In this section we give a specific form of the solutions of the fourth equation in the form

$$\zeta_{r+1} = \frac{\zeta_{r-29}}{-1-\prod_{i=0}^{5}\zeta_{r-(5i+4)}}, \qquad r = 0, 1, 2, ..., \tag{11}$$

with conditions posed on the initial values $\zeta_{-j}, \ j = 0, 1, 2, ..., 29$. Also, we investigate the oscillation and periodicity of these solutions.

Theorem 12. *Let* $\{\zeta_r\}_{r=-29}^{\infty}$ *be a solution of the difference Eq.* (11)*. Then for* $r = 0, 1, 2, ...,$

$$\zeta_{30r-k} = \frac{\varepsilon_k}{(-1-\mu_k)^{r\alpha_k}}, \tag{12}$$

where $\mu_k = \prod_{j=0}^{5}\varepsilon_{mod(k,5)+5j}$, $\alpha_k = (-1)^{\left[\frac{k}{5}\right]+1}$ *and* $\zeta_{-k} = \varepsilon_k$*, with* $\mu_k \neq -1, \ k = 0, 1, 2, ..., 29.$

Proof. For $r = 0$, the result holds. Now suppose that $r > 0$ and that our assumption holds for $r - 1$. That is

$$\zeta_{30r-30-k} = \frac{\varepsilon_k}{(-1-\mu_k)^{(r-1)\alpha_k}}. \tag{13}$$

Now, it follows from Eq. (11) and using Eq. (13) that

$$\zeta_{30r-29}=\frac{\zeta_{30r-59}}{-1-\zeta_{30r-34}\zeta_{30r-39}\zeta_{30r-44}\zeta_{30r-49}\zeta_{30r-54}\zeta_{30r-59}}$$
$$=\frac{\frac{\varepsilon_{29}}{(-1-\mu_{29})^{r-1}}}{-1-\varepsilon_4(-1-\mu_4)^{r-1}\frac{\varepsilon_9}{(-1-\mu_9)^{r-1}}\varepsilon_{14}(-1-\mu_{14})^{r-1}\frac{\varepsilon_{19}}{(-1-\mu_{19})^{r-1}}\varepsilon_{24}(-1-\mu_{24})^{r-1}\frac{\varepsilon_{29}}{(-1-\mu_{29})^{r-1}}}$$
$$=\frac{\varepsilon_{29}}{(-1-\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29})^{r-1}(-1-\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29})}.$$

Hence, we have

$$\zeta_{30r-29}=\frac{\varepsilon_{29}}{(-1-\varepsilon_4\varepsilon_9\varepsilon_{14}\varepsilon_{19}\varepsilon_{24}\varepsilon_{29})^{r}}.$$

Also, it follows from Eq. (11) and using Eq. (13) that

$$\zeta_{30r-28}=\frac{\zeta_{30r-58}}{-1-\zeta_{30r-33}\zeta_{30r-38}\zeta_{30r-43}\zeta_{30r-48}\zeta_{30r-53}\zeta_{30r-58}}$$
$$=\frac{\frac{\varepsilon_{28}}{(-1-\mu_{28})^{r-1}}}{-1-\varepsilon_3(-1-\mu_3)^{r-1}\frac{\varepsilon_8}{(-1-\mu_8)^{r-1}}\varepsilon_{13}(-1-\mu_{13})^{r-1}\frac{\varepsilon_{18}}{(-1-\mu_{18})^{r-1}}\varepsilon_{23}(-1-\mu_{23})^{r-1}\frac{\varepsilon_{28}}{(-1-\mu_{28})^{r-1}}}$$
$$=\frac{\varepsilon_{28}}{(-1-\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28})^{r-1}(-1-\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28})}.$$

Hence, we have

$$\zeta_{30r-28}=\frac{\varepsilon_{28}}{(-1-\varepsilon_3\varepsilon_8\varepsilon_{13}\varepsilon_{18}\varepsilon_{23}\varepsilon_{28})^{r}}.$$

Also, it follows from Eq. (11) and using Eq. (13) that

$$\zeta_{30r-27}=\frac{\zeta_{30r-57}}{-1-\zeta_{30r-32}\zeta_{30r-37}\zeta_{30r-42}\zeta_{30r-47}\zeta_{30r-52}\zeta_{30r-57}}$$
$$=\frac{\frac{\varepsilon_{27}}{(-1-\mu_{27})^{r-1}}}{-1-\varepsilon_2(-1-\mu_2)^{r-1}\frac{\varepsilon_7}{(-1-\mu_7)^{r-1}}\varepsilon_{12}(-1-\mu_{12})^{r-1}\frac{\varepsilon_{17}}{(-1-\mu_{17})^{r-1}}\varepsilon_{22}(-1-\mu_{22})^{r-1}\frac{\varepsilon_{27}}{(-1-\mu_{27})^{r-1}}}$$
$$=\frac{\varepsilon_{27}}{(-1-\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27})^{r-1}(-1-\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27})}.$$

Hence, we have

$$\zeta_{30r-27}=\frac{\varepsilon_{27}}{(-1-\varepsilon_2\varepsilon_7\varepsilon_{12}\varepsilon_{17}\varepsilon_{22}\varepsilon_{27})^{r}}.$$

Also, it follows from Eq. (11) and using Eq. (13) that

$$\zeta_{30r-26}=\frac{\zeta_{30r-56}}{-1-\zeta_{30r-31}\zeta_{30r-36}\zeta_{30r-41}\zeta_{30r-46}\zeta_{30r-51}\zeta_{30r-56}}$$
$$=\frac{\frac{\varepsilon_{26}}{(-1-\mu_{26})^{r-1}}}{-1-\varepsilon_1(-1-\mu_1)^{r-1}\frac{\varepsilon_6}{(-1-\mu_6)^{r-1}}\varepsilon_{11}(-1-\mu_{11})^{r-1}\frac{\varepsilon_{16}}{(-1-\mu_{16})^{r-1}}\varepsilon_{21}(-1-\mu_{21})^{r-1}\frac{\varepsilon_{26}}{(-1-\mu_{26})^{r-1}}}$$
$$=\frac{\varepsilon_{26}}{(-1-\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26})^{r-1}(-1-\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26})}.$$

Hence, we have

$$\zeta_{30r-26}=\frac{\varepsilon_{26}}{(-1-\varepsilon_1\varepsilon_6\varepsilon_{11}\varepsilon_{16}\varepsilon_{21}\varepsilon_{26})^{r}}.$$

Also, it follows from Eq. (11) and using Eq. (13) that

$$\zeta_{30r-25}=\frac{\zeta_{30r-55}}{-1-\zeta_{30r-30}\zeta_{30r-35}\zeta_{30r-40}\zeta_{30r-45}\zeta_{30r-50}\zeta_{30r-55}}$$

$$=\frac{\frac{\varepsilon_{25}}{(-1-\mu_{25})^{r-1}}}{-1-\varepsilon_0(-1-\mu_0)^{r-1}\frac{\varepsilon_5}{(-1-\mu_5)^{r-1}}\varepsilon_{10}(-1-\mu_{10})^{r-1}\frac{\varepsilon_{15}}{(-1-\mu_{15})^{r-1}}\varepsilon_{20}(-1-\mu_{20})^{r-1}\frac{\varepsilon_{25}}{(-1-\mu_{25})^{r-1}}}$$

$$=\frac{\varepsilon_{25}}{\left(-1-\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}\right)^{r-1}\left(-1-\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}\right)}.$$

Hence, we have

$$\zeta_{30r-25}=\frac{\varepsilon_{25}}{\left(-1-\varepsilon_0\varepsilon_5\varepsilon_{10}\varepsilon_{15}\varepsilon_{20}\varepsilon_{25}\right)^{r}}.$$

Similarly, one can easily obtain the other relations. Hence, the proof is completed.

Theorem 13. *Eq.* (11) *has a unique equilibrium point* $\overline{\zeta}=0$*, which is a nonhyperbolic equilibrium point.*

Proof. The proof is similar to the proof of Theorem 7, and will be omitted.

Theorem 14. *Eq.* (11) *is periodic of period 30 iff* $\mu_k=-2$, $k=0,1,...,29$ *and will be take the form*

$$\zeta_{30r-k}=\varepsilon_k,\ \ k=0,1,...,29,\ \ r=0,1,2,...$$

Proof. The proof follows immediately from Theorem 12.

Theorem 15. *Assume that* $\varepsilon_0,\varepsilon_1,...,\varepsilon_{29}\in(0,\infty)$*. Then the solution* $\{\zeta_r\}_{r=-29}^{\infty}$ *oscillates about the equilibrium point* $\overline{\zeta}=0$*, with positive semicycles of length 30, and negative semicycles of length 30.*

Proof. From Theorem 12, we have $\zeta_1,\zeta_2,...,\zeta_{30}<0$ and $\zeta_{31},\zeta_{32},...,\zeta_{60}>0$, and the result follows by induction.

6. Numerical investigation

This section is included to ensure that the constructed results are accurate. Under specified initial conditions, we provide various 2D examples.

Example 16. *The graph of Eq.* (1) *and the case when* $\zeta_{-29}=2$, $\zeta_{-28}=1.5$, $\zeta_{-27}=3.5$, $\zeta_{-26}=1.4$, $\zeta_{-25}=2.2$, $\zeta_{-24}=5.2$, $\zeta_{-23}=0.5$, $\zeta_{-22}=2.1$, $\zeta_{-21}=7.5$, $\zeta_{-20}=5.1$, $\zeta_{-19}=4$, $\zeta_{-18}=6.2$, $\zeta_{-17}=7.3$, $\zeta_{-16}=1$, $\zeta_{-15}=6$, $\zeta_{-14}=0$, $\zeta_{-13}=3$, $\zeta_{-12}=9.5$, $\zeta_{-11}=0.4$, $\zeta_{-10}=10.2$, $\zeta_{-9}=2.3$, $\zeta_{-8}=0.5$, $\zeta_{-7}=1.7$, $\zeta_{-6}=0.7$, $\zeta_{-5}=0.5$, $\zeta_{-4}=1.7$, $\zeta_{-3}=5.2$, $\zeta_{-2}=5.3$, $\zeta_{-1}=5$ *and* $\zeta_0=1$ *is shown in Figure 1.*

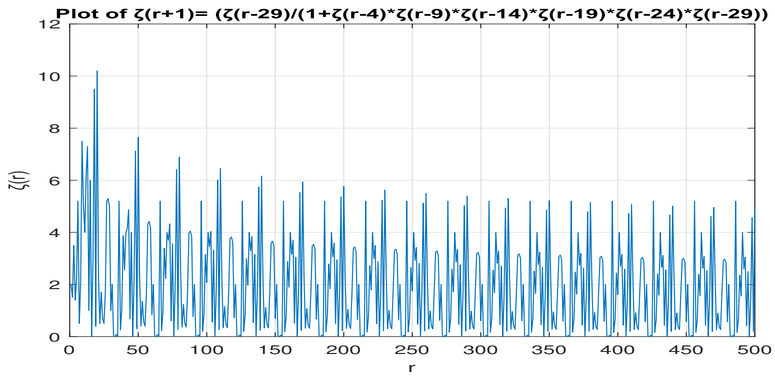

FIGURE 1. The behavior of Eq. (1).

Example 17. *The graph of Eq.* (4) *and the case when* $\zeta_{-29} = 1.5$, $\zeta_{-28} = 2.1$, $\zeta_{-27} = 3.2$, $\zeta_{-26} = 4.3$, $\zeta_{-25} = 5.25$, $\zeta_{-24} = 6.3$, $\zeta_{-23} = 7.2$, $\zeta_{-22} = 2.15$, $\zeta_{-21} = 4.05$, $\zeta_{-20} = 1.01$, $\zeta_{-19} = 3.4$, $\zeta_{-18} = 0.2$, $\zeta_{-17} = 0.3$, $\zeta_{-16} = 4.1$, $\zeta_{-15} = 6.6$, $\zeta_{-14} = 7.5$, $\zeta_{-13} = 8.9$, $\zeta_{-12} = 4.2$, $\zeta_{-11} = 0.4$, $\zeta_{-10} = 0.2$, $\zeta_{-9} = 2.13$, $\zeta_{-8} = 5$, $\zeta_{-7} = 0.17$, $\zeta_{-6} = 0.27$, $\zeta_{-5} = 35$, $\zeta_{-4} = 0.47$, $\zeta_{-3} = 1$, $\zeta_{-2} = 0.6$, $\zeta_{-1} = 0.75$ *and* $\zeta_0 = 5$ *is shown in Figure 2.*

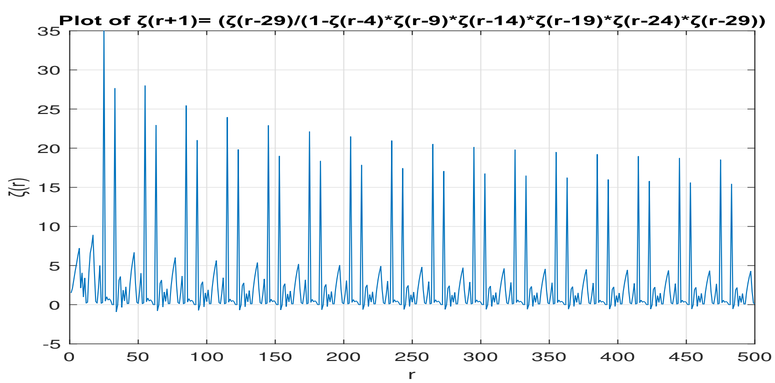

FIGURE 2. The behavior of Eq. (4).

Example 18. *The graph of Eq.* (8) *and the case when* $\zeta_{-29} = 0.2$, $\zeta_{-28} = 0.5$, $\zeta_{-27} = 0.5$, $\zeta_{-26} = 0.4$, $\zeta_{-25} = 0.2$, $\zeta_{-24} = 0.2$, $\zeta_{-23} = 0.5$, $\zeta_{-22} = 0.1$, $\zeta_{-21} = 0.5$, $\zeta_{-20} = 0.1$, $\zeta_{-19} = 0.4$, $\zeta_{-18} = 0.2$, $\zeta_{-17} = 0.3$, $\zeta_{-16} = 0.1$, $\zeta_{-15} = 0.6$, $\zeta_{-14} = 0.5$, $\zeta_{-13} = 0.3$, $\zeta_{-12} = 0.5$, $\zeta_{-11} = 0.4$, $\zeta_{-10} = 0.2$, $\zeta_{-9} = 0.3$, $\zeta_{-8} = 0.5$, $\zeta_{-7} = 0.7$, $\zeta_{-6} = 0.7$, $\zeta_{-5} = 0.5$, $\zeta_{-4} = 0.7$, $\zeta_{-3} = 0.2$, $\zeta_{-2} = 0.3$, $\zeta_{-1} = 0.5$ *and* $\zeta_0 = 0.1$ *is shown in Figure 3.*

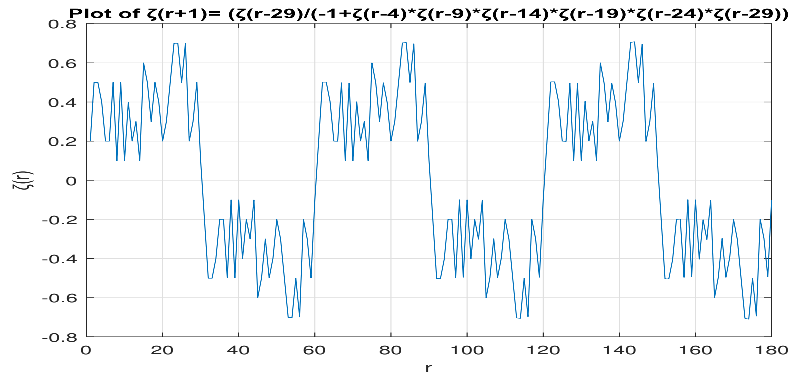

FIGURE 3. The behavior of Eq. (8).

Example 19. *The graph of Eq.* (11) *and the case when* $\zeta_{-29} = 0.5$, $\zeta_{-28} = 0.1$, $\zeta_{-27} = 0.2$, $\zeta_{-26} = 0.3$, $\zeta_{-25} = 0.25$, $\zeta_{-24} = 0.3$, $\zeta_{-23} = 0.2$, $\zeta_{-22} = 0.15$, $\zeta_{-21} = 0.05$, $\zeta_{-20} = 0.01$, $\zeta_{-19} = 0.4$, $\zeta_{-18} = 0.2$, $\zeta_{-17} = 0.3$, $\zeta_{-16} = 0.1$, $\zeta_{-15} = 0.6$, $\zeta_{-14} = 0.5$, $\zeta_{-13} = 0.9$, $\zeta_{-12} = 0.2$, $\zeta_{-11} = 0.4$, $\zeta_{-10} = 0.2$, $\zeta_{-9} = 0.13$, $\zeta_{-8} = 0.5$, $\zeta_{-7} = 0.17$, $\zeta_{-6} = 0.27$, $\zeta_{-5} = 0.35$, $\zeta_{-4} = 0.47$, $\zeta_{-3} = 0.52$, $\zeta_{-2} = 0.63$, $\zeta_{-1} = 0.75$ *and* $\zeta_0 = 0.81$ *is shown in Figure 4.*

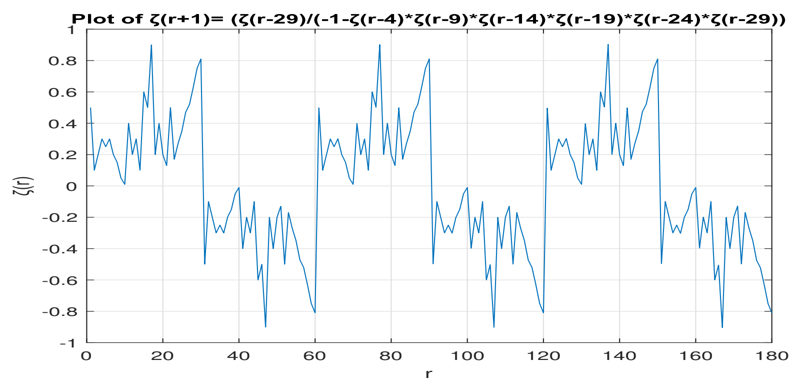

FIGURE 4. The behavior of Eq. (11).

7. CONCLUSION

To summarize, this paper has investigated four main rational difference equations with thirty-order. We have introduced the solutions of the considered equations using modulus operator. In Theorem 1, we have presented and proved the solutions of Eq. (1), while Theorem 2 has shown the boundedness of the solutions of Eq. (1). It has been proved that the fixed point of Eq. (1) is globally stable. Theorem 10 has presented that Eq. (8) is periodic of period 30 if and only if $\mu_\kappa = 2$. Furthermore, in Theorem 14, we have explored the solutions of Eq. (11) which are periodic of period 30 if and only if $\mu_\kappa = -2$. We have also plotted the periodicity of Eq. (8). and Eq. (11) in Figures 3 and 4, respectively. Finally, the used approaches can be simply applied for other nonlinear equations.

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