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EXISTENCE OF WEAK SOLUTIONS TO A CLASS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS IN WEIGHTED SOBOLEV SPACE

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Abstract. In this work, we prove the existence of a weak solutions for the initial boundary value problem associated with the nonlinear degenerate parabolic equations

$$\frac{\partial u}{\partial t} - \operatorname{div} b(x,t,u,\nabla u) = \phi(x,t) + \operatorname{div} a(x,t,\nabla u).$$

We will use the Topological degree theory for operators of the type $\mathcal{T}+\mathcal{S}$, where $\mathcal{S}$ is a bounded demicontinuous map of type (S_+) and $\mathcal{T}$ is a linear densely defined maximal monotone map with respect to a domain of $\mathcal{T}$, and we study this problem in the space $L^p(0,T;W_0^{1,p}(\Omega,\omega))$, where Ω is a bounded domain in $\mathbb{R}^N(N\geq 2)$, $p\geq 2$ and ω is a vector of weight functions.

1. Introduction

In recent years, several models from various branches of mathematical physics, such as elastic mechanics, electrorheological fluid dynamics, and image processing, have focused on the study of partial differential equations and variational problems (see for example [7, 10, 20]).

Topological degree theory may be one of the most effective tools in solving nonlinear equations. This is introduce the first time by Leray-Schauder [14] in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Browder [6] constructed a topological degree for operators of type (S_+) in reflexive Banach spaces. For more informations about the history of this theory, the reader can refer to [1, 4, 5, 8, 11, 12].

Throughout this work, we will assume that Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$, $\Omega_T=\Omega\times(0,\ T)$ is a cylinder, and that $\Gamma=\partial\Omega\times(0,\ T)$ is its lateral surface, where $T>0$ is a fixing time.

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In this work, we consider the following nonlinear degenerate parabolic initial boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} b(x,t,u,\nabla u) = \phi(x,t) + \operatorname{div} a(x,t,\nabla u) & \text{in } \Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) \ = 0 & \text{in } \Gamma, \end{cases} \tag{1.1}$$

where, $-\operatorname{div} b(x,t,u,\nabla u)$ and $-\operatorname{div} a(x,t,\nabla u)$ are nonlinear operators from

$$\mathcal{W} := L^p(0,T;W_0^{1,p}(\Omega,\omega)), \ \ p \geq 2.$$

to its dual $\mathcal{W}^* = L^q(0,T;W^{-1,q}(\Omega,\omega^{1-q}))$ where $\dfrac{1}{p}+\dfrac{1}{q}=1$, with ϕ is assumed to belong to $\mathcal{W}^*$.

Many scholars have studied problems of the form (1.1) with different conditions in the non weighted case $\omega \equiv 1$.

In [15] Lions proved that the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) = \phi(x,t) & \text{in } \Omega_T, \\ u(x,0) = 0, & \text{in } \Omega, \\ u(x,t) \ = 0, & \text{in } \Gamma, \end{cases}$$

admits at least one solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$.

When $\omega \equiv 1$, $p=2$ and $a(x,t,\nabla u)$ is equal to zero, Asfaw [3] proved that the problem (1.1) has a weak solution $u \in \mathcal{W}$, where the author used the approach based on the topological degree theory.

This research will be combined and generalized. Thus the objective of our work is to show that the problem (1.1) admits at least one solution $u \in \mathcal{W}$ by using the Topological degree theory for operators of the type $\mathcal{T}+\mathcal{S}$, where $\mathcal{S}$ is a bounded demicontinuous map of type (S_+) and $\mathcal{T}$ is a linear densely defined maximal monotone map with respect to a domain of $\mathcal{T}$.

Let us speedily summarize the work's contents. In the following section, we review some fundamental preliminaries of weighted Sobolev spaces and introduce some classes of mappings of the (S_+) type, as well as the Berkovits and Mustonen topological degrees required for the proof of our main result. Section 4 is devoted to basic assumptions, some necessary lemmas and the main result.

2. Mathematical background

In this section, we recall some defnitions and basic properties of the weighted Sobolev spaces and the theory of topological degree that will be used throughout the paper.

2.1. **The weighted Sobolev spaces.** Assume that Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$ and ω is a weight function in Ω, that is ω measurable and strictly positive a.e. in Ω.

For $1 \leq p < \infty$, we denote by $L^p(\Omega,\omega)$ the space of measurable functions v on Ω such that

$$||v||_{L^p(\Omega,\omega)} = \left(\int_\Omega |v(x)|^p \omega(x)dx\right)^{\frac{1}{p}} < \infty,$$

It is a well-known fact that the space $L^p(\Omega,\omega)$, endowed with this norm is a Banach space. We also have that the dual space of $L^p(\Omega,\omega)$ is the space $L^q(\Omega,\omega^{1-q})$, where $q=\frac{p}{p-1}$.

Next, let $\omega = \{\omega_k(x),\ 0 \le k \le N\}$ describes the family of weight functions ω_k, and in all of our considerations, we assume that for $0 \le k \le N$

$$\omega_k \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \omega_k^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega). \tag{2.1}$$

As a consequence, under conditions (2.1), the convergence in $L^p(\Omega, \omega_k)$ implies convergence in $L^1_{loc}(\Omega)$. Moreover, every function in $L^p(\Omega, \omega_k)$ has a distributional derivatives. It thus makes sense to talk about distributional derivatives of functions in $L^p(\Omega, \omega_k)$.

The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined as follows.

$$W^{1,p}(\Omega, \omega) = \Big\{ u \in L^p(\Omega, \omega_0) \text{ and } \partial_k u \in L^p(\Omega, \omega_k),\ k = 1, ..., N \Big\}.$$

The norm of v in $W^{1,p}(\Omega, \omega)$ is given by

$$\|v\|_{W^{1,p}(\Omega,\omega)} = \Big(\int_\Omega |v(x)|^p \omega_0(x)\, dx + \sum_{k=1}^{N} \int_\Omega |\partial_k v(x)|^p\, \omega_k(x)\, dx \Big)^{1/p}. \tag{2.2}$$

We also define $W_0^{1,p}(\Omega, \omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \omega)$ with respect to the norm (2.2). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, \omega)$ and the dual space of $W_0^{1,p}(\Omega, \omega)$ is the space $W^{-1,q}(\Omega, \omega^{1-q})$, where $q = \frac{p}{p-1}$. Furthermore, equipped by the norm (2.2), $W^{1,p}(\Omega, \omega)$ and $W_0^{1,p}(\Omega, \omega)$ are separable and reflevixe Banach spaces. For more details on weighted Sobolev spaces $W^{1,p}(\Omega, \omega)$, we propose to the reader [9, 16, 17, 18, 19].

Let us now assume that the expression

$$\|v\| = \Big(\sum_{k=1}^{N} \int_\Omega |\partial_k v(x)|^p\, \omega_k(x)\, dx \Big)^{1/p} \tag{2.3}$$

is a norm on $W_0^{1,p}(\Omega, \omega)$ and is equivalent to (2.2).

Note that $\Big(W_0^{1,p}(\Omega, \omega), \|\cdot\|\Big)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 2.1. *Suppose that* $\omega_0(x) \equiv 1$ *and there exists* $\nu \in]\frac{N}{p}, +\infty[\cap]\frac{1}{p-1}, +\infty[$ *such that*

$$\omega_j^{-\nu} \in L^1(\Omega), \text{ for all } j = 1, \cdots, N, \tag{2.4}$$

then

$$\|v\| = \Big(\sum_{j=1}^{N} \int_\Omega |\partial_j v|^p\, \omega_j(x)\, dx \Big)^{1/p} \tag{2.5}$$

is a norm defined on $W_0^{1,p}(\Omega, \omega)$ *and its equivalent to (2.2).*

In this work, we consider the following space

$$\mathcal{W} := L^p(0, T; W_0^{1,p}(\Omega, \omega)),\ p \ge 2 \text{ and } T > 0.$$

In this space, we defined the norm

$$|v|_{\mathcal{W}} = \Big(\int_0^T \|v\|^p_{W^{1,p}(\Omega,\omega)} dt \Big)^{1/p}.$$

Thanks to Poincaré inequality, the expression

$$\|v\|_{\mathcal{W}} = \Big(\int_0^T \|v\|^p dt\Big)^{1/p},$$

is a norm defined on $\mathcal{W}$ and is equivalent to the norm $|\cdot|_{\mathcal{W}}$.
Note that $\Big(\mathcal{W}, \|\cdot\|_{\mathcal{W}}\Big)$ is a separable and reflexive Banach space.

In this work, we use the following results.

Theorem 2.1. [13] *Let $1 < p < \infty$. If $u_n \longrightarrow u$ in $L^p(\Omega_T)$, then there exist a subsequence (u_k) and $\psi \in L^p(\Omega_T)$ such that*

(i): *$u_k(y) \longrightarrow u(y)$, a.e. on Ω_T.*
(ii): *$|u_k(y)| \leq \psi(y)$, a.e. on Ω_T.*

Lemma 2.1. [2] *Let $1 < p < \infty$, $(f_n)_n \subset L^p(\Omega_T, \omega)$ and $f \in L^p(\Omega_T, \omega)$ such that $\|f_n\|_{L^p(\Omega_T,\omega)} \leq C$. If $f_n(y) \to f(y)$ a.e. in Ω_T, then $f_n \rightharpoonup f$ in $L^p(\Omega_T, \omega)$, where ω is a weight function on Ω_T and $\rightharpoonup$ denotes the weak convergence.*

2.2. **Some classes of mappings and Topological degree theory.** Now we'll look at some topological degree mappings, results, and properties.

In what follows, let Y be a real reflexive and separable Banach space with dual Y^* and continuous pairing $\langle .,. \rangle$, and given a nonempty subset Ω of Y, $\partial\Omega$ and $\overline{\Omega}$ represent the boundary and the closure of Ω in Y, respectively. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 2.1. *We consider a mapping $\mathcal{T}$ defined from Y to 2^{Y^*} and its graph is given by*

$$G(\mathcal{T}) = \Big\{(u, v) \in Y \times Y^* : v \in \mathcal{T}(u)\Big\}.$$

(1) *$\mathcal{T}$ is said to be monotone if for all (u_1, v_1), (u_2, v_2) in $G(\mathcal{T})$, we have that*

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0.$$

(2) *$\mathcal{T}$ is said to be maximal monotone if it is monotone and maximal in the sense of graph inclusion among monotone mappings from Y to 2^{Y^*}, or for any $(u_1, v_1) \in Y \times Y^*$ for which $\langle v_1 - v, u_1 - u \rangle \geq 0$, for all $(u, v) \in G(\mathcal{T})$, we have $(u_1, v_1) \in G(\mathcal{T})$.*

Definition 2.2. *Let Z be a real Banach space. A operator $\mathcal{T} : \Omega \subset Y \to Z$ is said to be*

(1) *bounded, if it takes any bounded set into a bounded set.*
(2) *demicontinuous, if for any sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $\mathcal{T}(u_n) \rightharpoonup \mathcal{T}(u)$.*
(3) *compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 2.3. *A mapping $\mathcal{S} : D(\mathcal{S}) \subset Y \to Y^*$ is said to be*

(1) *of type (S_+), if for any $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty} \langle \mathcal{S}u_n, u_n - u \rangle \leq 0$, it follows that $u_n \to u$.*
(2) *quasimonotone, if for any sequence $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty} \langle \mathcal{S}u_n,\ u_n - u \rangle \geq 0$.*

In the sequel, let $\mathcal{T}$ be a linear maximal monotone map from $D(\mathcal{T}) \subset Y$ to Y^*. We consider the following classes of operators for each open and bounded subset G on Y:

$$\mathcal{F}_1(\Omega) := \Big\{ F : \Omega \to Y^* \,|\, F \text{ is bounded, demicontinuous and of type}(S_+) \Big\},$$

$$\mathcal{F}_G := \Big\{ \mathcal{T} + \mathcal{S} : \overline{G} \cap D(\mathcal{T}) \to Y^* \,|\, \mathcal{S} \text{ is bounded, demicontinuous}$$
$$\text{map of type } (S_+) \text{ with respect to } D(\mathcal{T}) \text{ from } \overline{G} \text{ to } Y^* \Big\},$$

$$\mathcal{H}_G := \Big\{ \mathcal{T} + \mathcal{S}(t) : \overline{G} \cap D(\mathcal{T}) \to Y^* \,|\, \mathcal{S}(t) \text{ is a bounded homotopy of type } (S_+)$$
$$\text{with respect to } D(\mathcal{T}) \text{ from } \overline{G} \text{ to } Y^* \Big\}.$$

Definition 2.4. *Let E be a bounded open subset of a real reflexive Banach space Y, $T \in \mathcal{F}_1(\overline{E})$ be continuous and let $F, \mathcal{S} \in \mathcal{F}_T(\overline{E})$. The affine homotopy $\Pi : [0,1] \times \overline{E} \to Y$ defined by*

$$\Pi(t,u) := (1-t)Fu + t\mathcal{S}u, \quad \textit{for all} \qquad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.2. *Note that the class $\mathcal{H}_G$ includes all affine homotopies*

$$\mathcal{L} + (1-t)\mathcal{S}_1 + t\mathcal{S}_2, \ \textit{with } (\mathcal{L} + \mathcal{S}_i) \in \mathcal{F}_G, \ i = 1,2.$$

Now, we introduce the Berkovits and Mustonen topological degree for the class $\mathcal{F}_G$, and we refer to [4, 5] for more background.

Theorem 2.2. *Let $\mathcal{L}$ a linear maximal monotone densely defined map from $D(\mathcal{L}) \subset Y$ to Y^*, and let*

$$\mathcal{E} = \Big\{ (F, G, \phi) : F \in \mathcal{F}_G, \ G \textit{ an open bounded subset in } Y, \ \phi \notin F(\partial G \cap D(\mathcal{L})) \Big\}.$$

Then, there exists a topological degree function $d : \mathcal{E} \to \mathbb{Z}$ satisfying the following properties:

(1) *(Existence) if $d(F, G, \phi) \neq 0$, then the equation $Fu = \phi$ has a solutions in $G \cap D(\mathcal{L})$.*
(2) *(Additivity) If G_1 and G_2 are two disjoint open subsets of G such that $\phi \notin F[(\overline{G} \backslash (G_1 \cup G_2)) \cap D(\mathcal{L})]$, then we have*
$$d(F, G, \phi) = d(F, G_1, \phi) + d(F, G_2, \phi).$$
(3) *(Homotopy invariance) If $F(t) \in \mathcal{H}_G$ and $f(t) \notin F(t)(\partial G \cap D(\mathcal{L}))$ for all $t \in [0,1]$, where $f(t)$ is a continuous curve in Y^*, then*
$$d(F(t), G, f(t)) = const, \ \forall t \in [0,1].$$
(4) *(Normalization) $\mathcal{L} + \mathcal{J}$ is a normalising map, where $\mathcal{J}$ is the duality mapping of Y into Y^*, that is,*
$$d(\mathcal{L} + \mathcal{J}, G, \phi) = 1, \ \textit{for all } \phi \in (\mathcal{L} + \mathcal{J})(G \cap D(\mathcal{L})).$$

Lemma 2.2. *Let $\mathcal{T} + \mathcal{S} \in \mathcal{F}_Y$ and $\phi \in Y^*$ and assume that there exists a radius $r > 0$ such that*

$$\langle \mathcal{T}u + \mathcal{S}u - \phi, u \rangle > 0, \tag{2.6}$$

for all $u \in \partial B_r(0) \cap D(\mathcal{T})$. *Then the equation* $\mathcal{T}u + \mathcal{S}u = \phi$ *has a solution* u *in* $D(\mathcal{T})$.

Proof. To show this lemma, it suffices to prove that $(\mathcal{T} + \mathcal{S})(D(\mathcal{T})) = Y^*$.

Let $F_\varepsilon(t,\ u) = \mathcal{T}u + (1-t)\mathcal{J}u + t(\mathcal{S}u + \varepsilon\mathcal{J}u - \phi)$, for all $\varepsilon > 0$ and $t \in [0,1]$.

From (2.6) and since $0 \in \mathcal{T}(0)$, we obtain

$$\begin{aligned}
\langle F_\varepsilon(t,u), u\rangle &= \langle t(\mathcal{T}u + \mathcal{S}u - \phi, u\rangle + \langle (1-t)\mathcal{T}u + (1-t+\varepsilon)\mathcal{J}u, u\rangle \\
&\geq \langle (1-t)\mathcal{T}u + (1-t+\varepsilon)\mathcal{J}u, u\rangle \\
&= (1-t)\langle \mathcal{T}u,\ u\rangle + (1-t+\varepsilon)\langle \mathcal{J}u, u\rangle \\
&\geq (1-t+\varepsilon)\|u\|^2 \\
&= (1-t+\varepsilon)r^2 > 0.
\end{aligned}$$

This implies $0 \notin F_\varepsilon(t,\ u)$.

Or $\mathcal{J}$ and $\mathcal{S} + \varepsilon\mathcal{J}$ are continuous, bounded and of type (S_+), then $\{F_\varepsilon(t,\ \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Therefore, applying the homotopy invariance and normalisation property of the degree d stated in Theorem 2.2, we obtain

$$d(F_\varepsilon(t,\cdot), B_r(0), 0) = d(\mathcal{T} + \mathcal{J}, B_r(0), 0) = 1 \neq 0.$$

Consequently, by existence property of the degree d there exists a point $u_\varepsilon \in D(\mathcal{T})$ such that $0 \in F_\varepsilon(t,\ \cdot)$. In particular, by setting $\varepsilon \to 0^+$ and $t = 1$, we get $\phi \in (\mathcal{T} + \mathcal{S})(D(\mathcal{T}))$ for some $u \in D(\mathcal{T})$. Since $\phi \in Y^*$ is arbitrary, we deduce that $(\mathcal{T} + \mathcal{S})(D(\mathcal{T})) = Y^*$. □

3. Main result

3.1. **Hypotheses and technical Lemmas.** In this subsection, we focus our attention on the basic assumptions and the operators associated with our problem to prove the existence results, and we introduce some useful technical lemmas to prove existence results.

Throughout this paper, we assume that the operators $a : \Omega_T \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $b : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ are Carathéodory's functions satisfying the following assumptions:

(A_1) There exists c_1, c_2 positive consts and $k_1, k_2 \in L^q(\Omega_T)$ such that

$$|a_i(x,t,\zeta)| \leq c_1\omega_i^{1/p}\Big(k_1(x,t) + \sum_{i=1}^N \omega_i^{1/q}|\zeta_i|^{p-1}\Big),$$

$$|b_i(x,t,\eta,\zeta)| \leq c_2\omega_i^{1/p}\Big(k_2(x,t) + \sum_{i=1}^N \omega_i^{1/q}|\zeta_i|^{p-1}\Big),$$

for all $i \in \{1,\cdots,N\}$.

$(A_2)$$(a(x,t,\zeta) - a(x,t,\zeta'))(\zeta - \zeta') > 0$, $(b(x,t,\eta,\zeta) - b(x,t,\eta,\zeta'))(\zeta - \zeta') > 0$.

(A_3) There exists α_1, α_2 positive constants such that

$$\sum_{i=1}^N a_i(x,t,\zeta)\zeta_i \geq \alpha_1\sum_{i=1}^N \omega_i|\zeta_i|^p,\ \sum_{i=1}^N b_i(x,t,\eta,\zeta)\zeta_i \geq \alpha_2\sum_{i=1}^N \omega_i|\zeta_i|^p,$$

for all $(x,\ t) \in \Omega_T$, $\eta \in \mathbb{R}$ and $(\zeta',\zeta) \in \mathbb{R}^N \times \mathbb{R}^N$ with $\zeta' \neq \zeta$.

Now, we give the property of the related operator which will be used later.

Lemma 3.1. *Assume that the assumptions* $(A_1) - (A_3)$ *hold. Then the operator* $\mathcal{S}$ *defined from* $\mathcal{W}$ *to* $\mathcal{W}^*$ *by*

$$\langle \mathcal{S}u, v\rangle = \sum_{i=1}^{N} \int_{\Omega_T} \Big(a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u)\Big)\partial_i v dx dt, \quad u, v \in \mathcal{W}$$

is bounded, continuous and of type (S_+).

Proof. Firstly, let us show that the operator $\mathcal{S}$ is bounded.

Let $u, v \in \mathcal{W}$. By using the Hölder's inequality, we get

$$\begin{aligned}
&|\langle \mathcal{S}u, v\rangle| \\
&\le \int_0^T \Big[\sum_{i=1}^{N} \int_\Omega |a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u)|\omega_i^{-1/p}|\partial_i v|\omega_i^{1/p} dx\Big] dt \\
&\le \int_0^T \Big[\sum_{i=1}^{N} \int_\Omega |a_i(x,t,\nabla u)|\omega_i^{-1/p}|\partial_i v|\omega_i^{1/p} dx\Big] dt \\
&\quad + \int_0^T \Big[\sum_{i=1}^{N} \int_\Omega |b_i(x,t,u,\nabla u)|\omega_i^{-1/p}|\partial_i v|\omega_i^{1/p} dx\Big] dt \\
&\le \int_0^T \Big[\sum_{i=1}^{N} \Big(\int_\Omega \big|a_i(x,t,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q} \Big(\int_\Omega |\partial_i v|^p \omega_i dx\Big)^{1/p}\Big] dt \\
&\quad + \int_0^T \Big[\sum_{i=1}^{N} \Big(\int_\Omega \big|b_i(x,t,u,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q} \Big(\int_\Omega |\partial_i v|^p \omega_i dx\Big)^{1/p}\Big] dt \\
&\le \int_0^T \Big[\Big(\sum_{i=1}^{N} \int_\Omega \big|a_i(x,t,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q} \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p \omega_i dx\Big)^{1/p}\Big] dt \\
&\quad + \int_0^T \Big[\Big(\sum_{i=1}^{N} \int_\Omega \big|b_i(x,t,u,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q} \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p \omega_i dx\Big)^{1/p}\Big] dt \\
&= \int_0^T \Big[\Big(\sum_{i=1}^{N} \int_\Omega |a_i(x,t,\nabla u)|^q \omega_i^{1-q} dx\Big)^{1/q} \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p \omega_i dx\Big)^{1/p}\Big] dt \\
&\quad + \int_0^T \Big[\Big(\sum_{i=1}^{N} \int_\Omega |b_i(x,t,u,\nabla u)|^q \omega_i^{1-q} dx\Big)^{1/q} \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p \omega_i dx\Big)^{1/p}\Big] dt \\
&= \int_0^T \sum_{i=1}^{N} \Big[\|a_i(x,t,\nabla u)\|_{L^q(\Omega,\omega_i^{1-q})} + \|b_i(x,t,u,\nabla u)\|_{L^q(\Omega,\omega_i^{1-q})}\Big]\|v\| dt.
\end{aligned}$$

Thanks to (A_1) and for all $i \in \{1, ..., N\}$, we can easily prove that $\|a_i(x,t,\nabla u)\|_{L^q(\Omega,\omega_i^{1-q})}$ and $\|b_i(x,t,u,\nabla u)\|_{L^q(\Omega,\omega_i^{1-q})}$ are bounded for all $u \in W_0^{1,p}(\Omega,\omega)$. Therefore

$$|\langle \mathcal{S}u,\ v\rangle| \le const \int_0^T \|v\| dt = const\|v\|_{L^1(0,T;W_0^{1,p}(\Omega,\omega))}.$$

From the continuous embedding $\mathcal{W} \hookrightarrow L^1(0,T;W_0^{1,p}(\Omega,\omega))$, we concludes that

$$|\langle \mathcal{S}u,\ v\rangle| \le const\|v\|_{\mathcal{W}}.$$

Hence, the operator $\mathcal{S}$ is bounded.

Secondly, we show that $\mathcal{S}$ is continuous. Let $u_n \to u$ in $\mathcal{W}$. We need to show that $\mathcal{S}u_n \to \mathcal{S}u$. By using the Hölder's inequality, we have for all $v \in \mathcal{W}$

$$
\begin{aligned}
|\langle \mathcal{S}u_n - \mathcal{S}u, v\rangle| \leq & \int_0^T \Big(\int_\Omega |a(x,t,\nabla u_n) - a(x,t,\nabla u)|\omega^{-1/p} \cdot |\nabla v|\omega^{1/p} dx\Big) dt \\
& + \int_0^T \Big(\int_\Omega |b(x,t,u,\nabla u_n) - b(x,t,u,\nabla u)|\omega^{-1/p} \cdot |\nabla v|\omega^{1/p} dx\Big) dt \\
\leq & \int_0^T \|a(x,t,\nabla u_n) - a(x,t,\nabla u)\|_{L^q(\Omega,\omega^{1-q})} \|\nabla v\|_{L^p(\Omega,\omega)} dt \\
& + \int_0^T \|b(x,t,u_n,\nabla u_n) - b(x,t,u,\nabla u)\|_{L^q(\Omega,\omega^{1-q})} \|\nabla v\|_{L^p(\Omega,\omega)} dt \\
\leq & \Big[\|a(x,t,\nabla u_n) - a(x,t,\nabla u)\|_{L^q(\Omega_T,\omega^{1-q})} \\
& \quad + \|b(x,t,u_n,\nabla u_n) - b(x,t,u,\nabla u)\|_{L^q(\Omega_T,\omega^{1-q})} \Big] \|v\|_{\mathcal{W}},
\end{aligned}
$$

so, we need to show that

$$\|a(x,t,\nabla u_n) - a(x,t,\nabla u)\|_{L^q(\Omega_T,\omega^{1-q})} \to 0,$$

and

$$\|b(x,t,u_n,\nabla u_n) - b(x,t,u,\nabla u)\|_{L^q(\Omega_T,\omega^{1-q})} \to 0.$$

On the other hand, note that if $u_n \to u$ in $\mathcal{W}$, then $\nabla u_n \to \nabla u$ in $\prod_{i=1}^{N} L^p(\Omega_T, \omega_i)$. Hence, by Theorem 2.1, there exist a subsequence (u_k) and functions φ in $L^p(\Omega_T, \omega_0)$ and ψ in $\prod_{i=1}^{N} L^p(\Omega_T, \omega_i)$ such that

$$
\begin{aligned}
& u_k \to u \text{ and } \nabla u_k \to \nabla u, \\
& |u_k(x,t)| \leq \varphi(x,t) \text{ and } |\nabla u_k(x,t)| \leq |\psi(x,t)|, \qquad (3.1)
\end{aligned}
$$

for a.e. $(x,t) \in \Omega_T$ and all $k \in \mathbb{N}$.

Then, in the light of the operators a and b are Carathéodory functions, we deduce that

$$a(x,t,\nabla u_k(x,t)) \to a(x,t,\nabla u(x,t)) \text{ a.e. } (x,t) \in \Omega_T, \qquad (3.2)$$

$$b(x,t,u_k,\nabla u_k(x,t)) \to b(x,t,u,\nabla u(x,t)) \text{ a.e. } (x,t) \in \Omega_T. \qquad (3.3)$$

On another side, in view of (A_1), we get for all $i = 1, \cdots, N$

$$|a_i(x,t,\zeta)| \leq c_1 \omega_i^{1/p} \Big(k_1(x,t) + \sum_{i=1}^{N} \omega_i^{1/q} |\psi_i(x,t)|^{p-1} \Big),$$

$$|b_i(x,t,\eta,\zeta)| \leq c_2 \omega_i^{1/p} \Big(k_2(x,t) + \sum_{i=1}^{N} \omega_i^{1/q} |\psi_i(x,t)|^{p-1} \Big),$$

for a.e. $(x,\ t) \in \Omega_T$.

As

$$c_1 \omega_i^{1/p} \Big(k_1(x,t) + \sum_{i=1}^{N} \omega_i^{1/q} |\psi_i(x,t)|^{p-1} \Big) \in \prod_{i=1}^{N} L^q(\Omega_T, \omega_i^{1-q}),$$

and

$$c_2\omega_i^{1/p}\Big(k_2(x,t)+\sum_{i=1}^{N}\omega_i^{1/q}|\psi_i(x,t)|^{p-1}\Big)\in\prod_{i=1}^{N}L^q(\Omega_T,\omega_i^{1-q}),$$

therefore, thanks to (3.2), (3.3) and the dominated convergence theorem, we obtain

$$a(x,t,\nabla u_k(x,t))\to a(x,t,\nabla u(x,t)) \text{ in } L^q(\Omega_T,\omega^{1-q}),$$

$$b(x,t,u_k,\nabla u_k(x,t))\to b(x,t,u,\nabla u(x,t)) \text{ in } L^q(\Omega_T,\omega^{1-q}).$$

Thus, in view to convergence principle in Banach spaces, we conclude that

$$a(x,t,\nabla u_n(x,t))\to a(x,t,\nabla u(x,t)) \text{ in } L^q(\Omega_T,\omega^{1-q}), \tag{3.4}$$

$$b(x,t,u_n,\nabla u_n(x,t))\to b(x,t,u,\nabla u(x,t)) \text{ in } L^q(\Omega_T,\omega^{1-q}). \tag{3.5}$$

According to (3.4) and (3.5), we deduce that

$$\langle \mathcal{S}u_n-\mathcal{S}u,v\rangle\to 0, \text{ for all } v\in\mathcal{W},$$

that means, the operator $\mathcal{S}$ is continuous.

Next, we prove that the operator $\mathcal{S}$ is of type (S_+). Let $(u_n)_n\subset\mathcal{W}$ such that

$$\begin{cases} u_n\rightharpoonup u \text{ in } \mathcal{W},\\ \limsup\limits_{n\to\infty}\langle \mathcal{S}u_n,\ u_n-u\rangle\le 0.\end{cases} \tag{3.6}$$

We will prove that

$$u_n\to u \text{ in } \mathcal{W}.$$

Since $u_n\rightharpoonup u$ in $\mathcal{W}$, then $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega)$, then there exist a subsequence still denoted by (u_n) such that $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega)$,

$$u_n\to u \quad \text{ in } L^p(\Omega,\omega_0) \quad \text{and a.e in } \Omega.$$

On the other hand, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \mathcal{S}u_n,\ u_n-u\rangle\\
&\quad=\limsup_{n\to\infty}\langle \mathcal{S}u_n-\mathcal{S}u,\ u_n-u\rangle\\
&\quad=\limsup_{n\to\infty}\Big[\int_{\Omega_T}\Big(a(x,t,\nabla u_n(x,t))-a(x,t,\nabla u(x,t))\Big).\Big(\nabla u_n-\nabla u\Big)dxdt\\
&\qquad+\int_{\Omega_T}\Big(b(x,t,u_n,\nabla u_n(x,t))-b(x,t,u,\nabla u(x,t))\Big).\Big(\nabla u_n-\nabla u\Big)dxdt\Big]\\
&\quad\le 0.
\end{aligned}$$

From (A_2) and (3.6), we obtain

$$\lim_{n\to\infty}\langle \mathcal{S}u_n,\ u_n-u\rangle=\lim_{n\to\infty}\langle \mathcal{S}u_n-\mathcal{S}u,\ u_n-u\rangle=0. \tag{3.7}$$

Let

$$\Theta_n(x,t)=\Big(a(x,t,\nabla u_n)-a(x,t,\nabla u)\Big).\Big(\nabla u_n-\nabla u\Big),$$

Under (3.7), we have

$$\Theta_n\to 0 \text{ in } L^1(\Omega_T) \text{ and a.e. in } \Omega_T,$$

Since $\Theta_n\to 0$ a.e in Ω_T, then there exists a subset B of Ω_T ($mes(B)=0$) such that for all $(x,t)\in\Omega\backslash B$,

$$|u(x,t)|<\infty,\quad |\nabla u(x.t)|<\infty,\quad u_n\to u,\quad \Theta_n\to 0.$$

Thanks to (A_1) and (A_3), if we pose $\zeta_n = \nabla u_n$ and $\zeta = \nabla u$, we get

$$\begin{aligned}\Theta_n(x,t) &= \Big(a(x,t,\zeta_n) - a(x,t,\zeta)\Big).\Big(\zeta_n - \zeta\Big) \\ &= a(x,t,\zeta_n).\zeta_n + a(x,t,\zeta).\zeta - a(x,t,\zeta_n).\zeta - a(x,t,\zeta).\zeta_n \\ &\geq \alpha_1 \sum_{i=1}^{N} \omega_i |\zeta_n^i|^p + \alpha_1 \sum_{i=1}^{N} \omega_i |\zeta^i|^p \\ &\quad - \sum_{i=1}^{N} c_1 \omega_i^{1/p} \Big(k_1(x,t) + \sum_{j=1}^{N} \omega_j^{1/q} |\zeta_n^j|^{p-1}\Big) |\zeta_n^i| \\ &\quad - \sum_{i=1}^{N} c_1 \omega_i^{1/p} \Big(k_1(x,t) + \sum_{j=1}^{N} \omega_j^{1/q} |\zeta_n^j|^{p-1}\Big) |\zeta^i| \\ &\geq \alpha_1 \sum_{i=1}^{N} \omega_i |\zeta_n^i|^p - C\Big(1 + \sum_{i=1}^{N} \omega_i^{1/q} |\zeta_n^i|^{p-1} + \sum_{i=1}^{N} \omega_i^{1/q} |\zeta_n^i|\Big)\end{aligned}$$

where C is a const which depends only on x.

Then by a standard argument $(\zeta_n)_n$ is bounded a.e. Ω_T, we deduce that

$$\Theta_n(x,t) \geq \sum_{i=1}^{N} |\zeta_n^i|^p \Big(\alpha_1 \omega_i - \frac{C}{N|\zeta_n^i|^p} - \frac{C\omega_i^{1/q}}{|\zeta_n^i|} - \frac{C\omega_i^{1/q}}{|\zeta_n^i|^{p-1}}\Big).$$

Hence, if $|\zeta_n| \to \infty$, then $\Theta_n \to \infty$; what is contradiction with $\Theta_n \to 0$ in $L^1(\Omega_T)$..

Next, for ζ^* be an adherent point of ζ_n, we have $|\zeta^*| < \infty$ and the continuity of a, with respect to the last two variables, we will obtain

$$\Big(a(x,t,\zeta_n) - a(x,t,\zeta)\Big)\big(\zeta^* - \zeta\big) = 0. \tag{3.8}$$

Analogously, if we choose

$$\Lambda_n(x,t) = \Big(b(x,t,u_n,\nabla u_n) - b(x,t,u,\nabla u)\Big).\Big(\nabla u_n - \nabla u\Big),$$

and we take $\zeta_n = \nabla u_n$ and $\zeta = \nabla u$, then, by the same arguments used above, we obtain

$$\Big(b(x,t,\eta,\zeta_n) - b(x,t,\eta,\zeta)\Big)\big(\zeta^* - \zeta\big) = 0. \tag{3.9}$$

Then, according to (3.8), (3.9) and (A_2) we get $\zeta^* = \zeta$. Hence, by the uniqueness of the adherent point, we deduce that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega_T. \tag{3.10}$$

On the other hand, seeing that $a(x,t,\nabla u_n)$ and $b(x,t,u_n,\nabla u_n)$ are bounded in $\prod_{i=1}^{N} L^q(\Omega, \omega^{1-q})$, and

$$a(x,t,\nabla u_n) \longrightarrow a(x,t,\nabla u) \text{ a.e. in } \Omega_T,$$

$$b(x,t,u_n,\nabla u_n) \longrightarrow b(x,t,u,\nabla u) \text{ a.e. in } \Omega_T,$$

then, by Lemma 2.1, we have

$$a(x,t,\nabla u_n) \rightharpoonup a(x,t,\nabla u) \quad \text{in } \prod_{i=1}^{N} L^q(\Omega, \omega_i^{1-q}),$$

$$b(x,t,u,\nabla u_n) \rightharpoonup b(x,t,u,\nabla u) \quad \text{in} \quad \prod_{i=1}^{N} L^q(\Omega, \omega_i^{1-q}).$$

If we pose

$$\overline{\rho}_n = \Big(a(x,t,\nabla u_n) + b(x,t,u_n,\nabla u_n) \Big) \cdot \nabla u_n,$$

$$\overline{\rho} = \Big(a(x,t,\nabla u) + b(x,t,u,\nabla u) \Big) \cdot \nabla u,$$

we can write

$$\overline{\rho}_n \to \overline{\rho} \quad \text{in} \;\; L^1(\Omega_T).$$

Thanks to (A_3), we obtain

$$\overline{\rho}_n \geq (\alpha_1 + \alpha_2) \sum_{i=1}^{N} \omega_i |\partial_i u_n|^p \quad \text{and} \quad \overline{\rho} \geq (\alpha_1 + \alpha_2) \sum_{i=1}^{N} \omega_i |\partial_i u|^p.$$

In view of $\tau_n = \sum_{i=1}^{N} \omega_i |\partial_i u_n|^p$, $\tau = \sum_{i=1}^{N} \omega_i |\partial_i u|^p$, $\rho_n = \dfrac{\overline{\rho}_n}{(\alpha_1 + \alpha_2)}$ and $\rho = \dfrac{\overline{\rho}}{(\alpha_1 + \alpha_2)}$, we have

$$\rho_n \geq \tau_n \quad \text{and} \quad \rho \geq \tau.$$

Then by Fatou's lemma, we get

$$\int_{\Omega_T} 2\,\rho\, dxdt \leq \liminf_{n \to \infty} \int_{\Omega_T} \rho + \rho_n - |\tau_n - \tau| dxdt,$$

i.e.,

$$0 \leq - \limsup_{n \to \infty} \int_{\Omega_T} |\tau_n - \tau|\, dxdt.$$

So

$$0 \leq \liminf_{n \to \infty} \int_{\Omega_T} |\tau_n - \tau|\, dxdt \leq \limsup_{n \to \infty} \int_{\Omega_T} |\tau_n - \tau|\, dxdt \leq 0,$$

consequently

$$\nabla u_n \longrightarrow \nabla u \quad \text{in} \quad \prod_{i=1}^{N} L^p(\Omega, \omega_i). \tag{3.11}$$

According to (3.10) and (3.11), we have

$$u_n \longrightarrow u \text{ in } W_0^{1,p}(\Omega, \omega),$$

this implies

$$u_n \to u \text{ in } \mathcal{W},$$

what implies that $\mathcal{S}$ is of type (S_+), which completes the proof. □

3.2. **Main result.** First, let us recall that the definition of a weak solution for problem (1.1) can be stated as follows.

Definition 3.1. *We say that the function $u \in \mathcal{W}$ is a weak solution of* (1.1) *if*

$$-\int_{\Omega_T} u v_t dxdt + \sum_{i=1}^{N} \int_{\Omega_T} \Big(a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u) \Big) \partial_i v dxdt = \int_{\Omega_T} \phi v dxdt,$$

for all $v \in \mathcal{W}$.

We are now in the position to get existence result of weak solution for (1.1).

Theorem 3.1. *Let $\phi \in \mathcal{W}^*$, $u_0 \in L^2(\Omega)$ and assume that the assumptions $(A_1)-(A_3)$ hold. Then, the problem* (1.1) *admits at least one weak solution $u \in D(\mathcal{T})$, where $D(\mathcal{T}) = \{v \in \mathcal{W} : v' \in \mathcal{W}^*,\ v(0) = 0\}$.*

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be the operators defined from $D(\mathcal{T}) \subset \mathcal{W}$ to $\mathcal{W}^*$, by

$$\langle \mathcal{S}u, v \rangle = \sum_{i=1}^{N} \int_{\Omega_T} \Big(a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u) \Big) \partial_i v dxdt,$$

$$\langle \mathcal{T}u,\ v \rangle = -\int_{\Omega_T} u\, v_t\, dx\, dt,$$

for all $u \in D(\mathcal{T}),\ v \in \mathcal{W}$. Then $u \in D(\mathcal{T})$ is a weak solution for (1.1) if and only if

$$\mathcal{T}u + \mathcal{S}u = \phi \text{ for all } u \in D(\mathcal{T}).$$

One can verify, as in Zeidler [21], that the operator $\mathcal{T}$ is linear densely defined and maximal monotone [21, Theorem 32.L, pp.897-899].
Next, it follows from Lemma 3.1 that $\mathcal{S}$ is bounded, continuous and of type (S_+). Let $u \in \mathcal{W}$. Using the monotonicity of $\mathcal{T}$ $\big(\langle \mathcal{L}u, u \rangle \geq 0$ for all $u \in D(\mathcal{L}) \big)$ and the assumption (A_2), we deduce that

$$\begin{aligned}
\langle \mathcal{T}u + \mathcal{S}u, u \rangle &\geq \langle \mathcal{S}u, u \rangle \\
&= \int_{\Omega_T} \Big(a(x,t,\nabla u) + b(x,t,u,\nabla u) \Big) \cdot \nabla v dxdt \\
&\geq \int_{\Omega_T} \alpha_1 \sum_{i=1}^{N} \omega_i |\nabla u|^p dxdt + \int_{\Omega_T} \alpha_2 \sum_{i=1}^{N} \omega_i |\nabla u|^p dxdt \\
&\geq \min(\alpha_1, \alpha_2) \int_{\Omega_T} \sum_{i=1}^{N} \omega_i |\nabla u|^p dxdt \\
&= \min(\alpha_1, \alpha_2) \int_0^T \|u\|^p dt \\
&= \min(\alpha_1, \alpha_2) \|u\|_{\mathcal{W}}^p.
\end{aligned}$$

Because the right-hand side of the previous inequality approximates to ∞ when $\|u\|_{\mathcal{W}} \to \infty$, then for every $\phi \in \mathcal{W}^*$ there is a radius $r = r(\phi) > 0$ such that

$$\langle \mathcal{T}u + \mathcal{S}u - \phi, u \rangle > 0, \quad \text{for each} \quad u \in B_r(0) \cap D(\mathcal{T}).$$

So, all the conditions of Lemma 2.2 are satisfied. Consequently, Lemma 2.2 leads us to the conclusion that the equation $\mathcal{T}u + \mathcal{S}u = \phi$ has a weak solution in $D(\mathcal{T})$, which implies that the problem (1.1) admits at least one weak solution. This completes the proof. □

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