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VALUE DISTRIBUTION OF DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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Abstract. In this article, we establish the an inequality (Milloux inequality) about the nonlinear difference monomials of the form

$$\Psi(z) = \prod_{j=0}^{m} \left(\Delta_{\eta_j} f(z)\right)^{k_j}, \tag{1}$$

where $\eta_j \in \mathbb{C}$, at least one of the $\eta_j \neq 0$ for $j = 1, 2 \cdots m$ and $K = \sum_{j=0}^{m} k_j$, $k_0, k_1, \cdots, k_m \in \mathbb{N}$. As an application of the inequality, we also investigate the value distribution of some difference monomials and polynomials, where $f(z)$ being a transcendental meromorphic function.

1. Introduction

Throughout this article, the phrase "entire function" means that the function is analytic everywhere in $\mathbb{C}$. The fundamentals of Nevanlinna theory and standard notations can be read in ([7, 8, 11]). The notation $E = \{x : x \in \mathbb{R}^+\}$ set of positive real numbers of finite linear measure. Let $\mathcal{F} = \{f : f \ is\ non - constant\ meromorphic\ function\ in\ \mathbb{C}\}$. For $f, g \in \mathcal{F}$ and $b \in \mathbb{C} \cup \{\infty\}$, if $f - b$ and $g - b$ have the identical zeros including multiplicities then f and g share b CM (counting multiplicities), if the multiplicities are ignored, then f and g share b IM and if $1/f$ and $1/g$ share 0 CM then, f and g share ∞ CM [12]. $N(r, \frac{1}{f-b})$ denotes the counting function of f whose b-points are counted according to multiplicity and the corresponding reduced counting function when multiplicity is ignored is denoted by $\overline{N}(r, \frac{1}{f-b})$. For $\phi(z) \in \mathcal{F}$, if $T(r, \phi) = S(r, f)$ then ϕ is called a "small function" of f where $T(r, \phi)$ is the Nevanlinna characteristic function and $S(r, f) = o(T(r, f)),\ as\ r \to \infty,\ r \notin E.$

Hayman [7] established the following theorem by investigating Picard values of entire functions and their derivatives.

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Theorem 1. [7] *If $f(z)$ is a transcendental entire function, $n \geq 3$ is an integer, and $a(\neq 0)$ is a constant, then $f^{'}(z) - af^n(z)$ assumes all finite values infinitely often.*

Zheng and Chen [14] proved the following theorem for difference function for entire functions

Theorem 2. [14] *If $f(z)$ is a transcendental entire function of finite order, and let a, c be non-zero constants. Then, for any integer $n \geq 3$, $f(z+c) - af^n(z)$ assumes all finite values $b \in \mathbb{C}$ infinitely often.*

The last few years have seen many papers examining complex differences. In applying the value distribution theory of meromorphic functions, they have derived new results on differences one can refer ([1],[13],[15]). In 2018, *Renukadevi and Madhura*[4] considered general differential-difference polynomial of $f(z)$ and its shifts as follows

$$\begin{aligned} P(z,f) =& \sum_{\lambda \in I} a_\lambda(z) f(z)^{\lambda_{0,0}} f'(z)^{\lambda_{0,1}} \dots f^{(m)}(z)^{\lambda_{0,m}} \\ & \times f\left(z+c_1\right)^{\lambda_{1,0}} f'\left(z+c_1\right)^{\lambda_{1,1}} \dots f^{(m)}\left(z+c_1\right)^{\lambda_{1,m}} \\ & \dots f\left(z+c_k\right)^{\lambda_{k,0}} f'\left(z+c_k\right)^{\lambda_{k,1}} \dots f^{(m)}\left(z+c_k\right)^{\lambda_{k,m}} \\ =& \sum_{\lambda \in I} a_\lambda(z) \prod_{i=0}^{k} \prod_{j=0}^{m} f^{(j)}\left(z+c_i\right)^{\lambda_{i,j}} \end{aligned} \tag{2}$$

where I is a finite set of multi-indices $\lambda = (\lambda_{0,0}, \dots, \lambda_{0,m}, \lambda_{1,0}, \dots, \lambda_{1,m}, \dots, \lambda_{k,0}, \dots, \lambda_{k,m})$, $c_0(=0)$ and $c_1, \dots, c_k$ are distinct complex constants. We assume that the meromorphic coefficients $a_\lambda(z), \lambda \in I$ of $P(z,f)$ are of growth $S(r,f)$. We denote the degree of the monomial $\prod_{i=0}^{k} \prod_{j=0}^{m} f^{(j)}\left(z+c_i\right)^{\lambda_{i,j}}$ of $P(z,f)$ by $d(\lambda) = \sum_{i=0}^{k} \sum_{j=0}^{m} \lambda_{i,j}$. Then we denote the degree and the lower degree of $P(z,f)$ by

$$d(P) = \max_{\lambda \in I}\{d(\lambda)\}, \quad d^*(P) = \min_{\lambda \in I}\{d(\lambda)\}$$

respectively. In particular, we call $P(z,f)$ a homogeneous differential-difference polynomial if $d(P) = d^*(P)$. Otherwise, $P(z,f)$ is non-homogeneous. *Renukadevi and Madhura* proved the following Theorems.

Theorem 3. [4] *Let $f(z)$ be a finite order transcendental meromorphic function, let $\alpha(z)$ be a small function with respect to $f(z)$. Let $P(z,f)(\not\equiv \alpha(z))$ be a differential-difference polynomial of the form (2) and $\delta(\infty, f) > 1 - \frac{1}{2k+7}$. Then the differential-difference polynomial*

$$Q(z,f) = f(z)^n + P(z,f), \quad n \geq d(P) + 2$$

satisfies $\delta(\alpha, Q(z,f)) < 1$ and hence $Q(z,f) - \alpha(z)$ has infinitely many zeros. The condition $\delta(\infty, f) > 1 - \frac{1}{2k+7}$ in Theorem 1.1 can be relaxed by adding the condition $\bar{N}\left(r, \frac{1}{f}\right) = S(r,f)$, and we obtain the following theorem.

Theorem 4. [4] *Let $f(z)$ be a finite order transcendental meromorphic function, let $\alpha(z)$ be a small function with respect to $f(z)$. Let $P(z,f)(\not\equiv \alpha(z))$ be a differential-difference polynomial of the form (2) and let $\delta(\infty, f) > 1 - \frac{1}{2k+5}$ and $\bar{N}\left(r, \frac{1}{f}\right) =$*

$S(r,f)$, then the differential-difference polynomial

$$Q(z,f) = f(z)^n + P(z,f), \quad n \geq d(P) + 1$$

satisfies $\delta(\alpha(z), Q(z,f)) < 1$ and hence $Q(z,f) - \alpha(z)$ has infinitely many zeros.

Wu and Xu[10] obtained the following theorem by considering the non linear difference monomial

$$\Phi(z) = f^{d_1}(z+c_1)f^{d_2}(z+c_2)\cdots f^{d_m}(z+c_m), \tag{3}$$

and

$$d = d_1 + d_2 + \cdots + d_m,$$

where $c_1, c_2, \cdots, c_m$ are complex constants satisfying at least one of them is non zero and $d_1, d_2, \cdots, d_m \in \mathbb{N}$.

Theorem 5. [10] *Let $f(z)$ be a transcendental meromorphic function of finite order, and assume that $\delta(\infty, f) = 1$. Suppose that $\Psi(z)$ is a nonlinear difference monomial of the form* (3). *Then,*

(1) *for $\delta(0,f) > 0, \Psi(z)$ assumes every non-zero value α infinitely often and $\lambda(\alpha, \Psi(z)) = \sigma(f)$*
(2) *for $\delta(0,f) = 1, \Psi(z)$ assumes every non-zero value α infinitely often and*

$$T(r, \Psi(z)) \sim dT(r,f) \sim N\left(r, \frac{1}{\Psi(z) - \alpha}\right)$$

,

as $r \notin E, r \to \infty$, where E is a possible exception set of r with finite logarithmic measure.

2. Preliminary Lemmas

Lemma 2.1. [6] *Let $f(z)$ be a transcendental meromorphic function of finite order, then*

$$m\left(r, \frac{f(z+c)}{f}\right) = S(r,f).$$

Lemma 2.2. [3] *Let f be a transcendental meromorphic function of finite order. Then*

$$N(r, f(z+c)) = N(r,f) + S(r,f),$$
$$T(r, f(z+c)) = T(r,f) + S(r,f),$$

where $S(r,f) = o(T(r,f))(r \to \infty)$, possibly outside a set E of r with finite logarithmic measure.

Lemma 2.3. [5] *Let f be a transcendental meromorphic function of finite order. Then for any positive integer n, we have*

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = S(r,f).$$

Lemma 2.4. [11] *Suppose that $f(z)$ is a transcendental meromorphic function in the complex plane and $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, where $a_0 (\not\equiv 0), a_1, \cdots, a_n$ are constants. Then*

$$T(r, P(f)) = nT(r,f) + S(r,f)$$

Lemma 2.5. [9] *Let $f(z)$ be a transcendental meromorphic solution of finite order of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f), \tag{4}$$

where $U(z,f), P(z,f)$ and $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f) = n$ in $f(z)$ and its shifts and $\mathrm{deq}_f\, Q(z,f) \leq n$. Moreover, we assume that all coefficients $a_\lambda(z)$ in (4) are small in the sense that $T(r,a_\lambda) = S(r,f)$ and that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then, for each $\varepsilon > 0$, we have

$$m(r, P(z,f)) = S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.6. [2] *Let $F(r)$ and $G(r)$ be monotone increasing function such that $F(r) \leq G(r)$ outside of exceptional set E that is of finite logarithmic measure. Then for any $\alpha > 0$, there exists $r_0 > 1$ such that $F(r) \leq G(\alpha r)$ for all $r > r_0$.*

Lemma 2.7. *Let f be finite order transcendental meromorphic function and if $\Psi(z)$ as defined in (1), then for every $\alpha \in \mathbb{C} \setminus \{0\}$, we have*

$$KT(r,f) \leq KN\left(r, \frac{1}{f}\right) + 4KN(r,f) + N\left(r, \frac{1}{\Psi(z)-\alpha}\right) + S(r,f). \tag{5}$$

Proof. Since the meromorphic function $\Psi(z)$ is not constant, there is a $\beta \in \mathbb{C} \setminus \{0\}$ such that $\Delta_\beta \Psi(z) = \Delta_\beta (\Psi(z) - \alpha) \neq 0$. Observe that

$$\begin{aligned} \frac{1}{f^K} =& \frac{\Psi(z)}{\alpha f^K} - \frac{\Psi(z)-\alpha}{\alpha f^K}, \\ =& \frac{\Psi(z)}{\alpha f^K} - \frac{\Delta_\beta \Psi(z)}{\alpha f^K} \frac{\Psi(z)-\alpha}{\Delta_\beta(\Psi(z)-\alpha)}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \frac{\Delta_\beta \Psi(z)}{\alpha f^K} &= \prod_{j=0}^{m} \frac{\left(\Delta_{\eta_j} f(z+\beta)^{k_j}\right)}{\alpha f^K} - \frac{\Psi(z)}{\alpha f^K} \\ &= \frac{1}{\alpha} \prod_{j=0}^{m} \left(\frac{\Delta_{\eta_j} f(z+\beta)}{f}\right)^{k_j} - \frac{\Psi(z)}{\alpha f^K} \end{aligned}$$

and

$$\tfrac{\Psi(z)}{\alpha f^K} = \tfrac{1}{\alpha} \prod_{j=0}^{m} \left(\tfrac{\Delta_{\eta_j} f(z)}{f}\right)^{k_j},$$

the definition of proximity function and lemma 2.3, it follows that

$$\begin{aligned} m\left(r, \frac{\Psi(z)}{\alpha f^K}\right) &= m\left(r, \prod_{j=0}^{m} \left(\frac{\Delta_{\eta_j} f(z)}{f}\right)^{k_j}\right) \\ &\leq \sum_{j=0}^{m} k_j m\left(r, \frac{\Delta_{\eta_j} f}{f}\right) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{7}$$

Similarly, we get

$$m\left(r, \frac{\Delta_\beta \Psi(z)}{\alpha f^K}\right) = S(r, f). \tag{8}$$

From (7), (8) and (6), we get
$m\left(r, \frac{1}{f^K}\right) \leq m\left(r, \frac{\Psi(z)-\alpha}{\Delta_\beta(\Psi-\alpha)}\right) + S(r, f)$.
Thus

$$\begin{aligned} T\left(r, \frac{1}{f^K}\right) &\leq m\left(r, \frac{\Psi(z)-\alpha}{\Delta_\beta(\Psi-\alpha)}\right) + N\left(r, \frac{1}{f^K}\right) + S(r, f) \\ &= KN\left(r, \frac{1}{f}\right) + m\left(r, \frac{\Psi(z)-\alpha}{\Delta_\beta(\Psi-\alpha)}\right) + S(r, f). \end{aligned} \tag{9}$$

Also from (1), we get

$$T(r, \Psi) \leq 2KT(r, f) + S(r, f). \tag{10}$$

Thus

$$\rho(\Psi) \leq \rho(f) \quad and \quad S(r, \Psi) = S(r, f). \tag{11}$$

Nevanlinna's first fundamental theorem gives us

$$m\left(r, \frac{\Psi(z)-\alpha}{\Delta_\beta(\Psi-\alpha)}\right) \leq m\left(r, \frac{\Delta_\beta(\Psi-\alpha)}{\Psi(z)-\alpha}\right) + N\left(r, \frac{\Delta_\beta(\Psi-\alpha)}{\Psi(z)-\alpha}\right) + O(1). \tag{12}$$

Combining above inequality with lemma 2.3, we get

$$m\left(r, \frac{\Psi(z)-\alpha}{\Delta_\beta(\Psi-\alpha)}\right) = S(r, f). \tag{13}$$

Since $\Delta_\beta\left(\Psi(z)-\alpha\right) = \Delta_\beta \Psi(z)$, it follows that

$$\begin{aligned} N\left(r, \frac{\Delta_\beta(\Psi(z)-\alpha)}{\Psi(z)-\alpha}\right) &\leq N\left(r, \frac{1}{\Psi-\alpha}\right) + N\left(r, \Delta_\beta(\Psi(z)-\alpha)\right) \\ &\leq N\left(r, \frac{1}{\Psi-\alpha}\right) + N(r, \Psi(z+\beta)) + N(r, \Psi(z)) + S(r, f) \\ &= N\left(r, \frac{1}{\Psi-\alpha}\right) + 4KN(r, f) + S(r, f) \end{aligned} \tag{14}$$

From 2.4,(9) and (12)-(14), we get

$$\begin{aligned} KT(r, f) =& T\left(r, \frac{1}{f^K}\right) + S(r, f) \\ &= KN\left(r, \frac{1}{f}\right) + 4KN(r, f) + N\left(r, \frac{1}{\Psi(z)-\alpha}\right) + S(r, f). \end{aligned} \tag{15}$$

□

3. Main Results

In this paper we consider the difference polynomial of the form

$$Q(z,f)=\sum_{\lambda\in I}a_{\lambda}(z)\prod_{j=1}^{m}\left(\Delta_{c}^{j}f(z)\right)^{l_{\lambda,j}}, \tag{16}$$

where I is the finite index set and the coefficients $a_{\lambda}(z)$ are small functions of f, $l_j=\max\limits_{\lambda\in I}\{l_{\lambda j}\}$ and $d(p)=\sum\limits_{j=1}^{m}l_j$ be the degree of $Q(z,f)$, and obtained the following theorem.

Theorem 1 Let $f(z)$ be a finite order transcendental meromorphic function and $Q(z,f)(\not\equiv q(z))$ be difference polynomial defined in (16) and $q(z)$ be a small function of f. assume that $\delta(\infty,f)>1-\frac{1}{(m^2+m+5)}$ and the difference polynomial

$$P(z,f)=f^n+Q(z,f),\quad n\geq d(p)+2, \tag{17}$$

satisfying $\delta(q,P(z,f))<1$ and $P(z,f)-q(z)$ has infinitely many zeros.

Proof. Since $\delta(0,f)>0$, we claim that $\Psi(z)$ is a transcendental meromorphic function. Otherwise, then there is a rational function $q(z)$ such that $q(z)\Psi(z)\equiv 1$,

$$\begin{aligned}\frac{1}{f^K}&=q(z)\frac{\Psi(z)}{f^K}\\&=q(z)\prod_{j=0}^{m}\left(\frac{\Delta_{\eta_j}f(z)}{f}\right)^{k_j}.\end{aligned}$$

Since f is transcendental and from lemma 2.3, we get $m\left(r,\frac{1}{f^K}\right)=S(r,f)$, from first fundamental theorem of nevanlinna, we get

$$\begin{aligned}KT(r,f)=&T\left(r,f^K\right)=T\left(r,\frac{1}{f^K}\right)+S(r,f)\\&\leq N\left(r,\frac{1}{f^K}\right)+S(r,f)\\KT(r,f)&\leq KN\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}$$

Which is contradiction to $\delta(0,f)>0$. Hence $\Psi(z)$ is transcendental meromorphic function.

(1) Since $\delta(0,f)>0$ and $\delta(\infty,f)=1$, in this case a positive number $\kappa<1$ exists such that

$$N\left(r,\frac{1}{f}\right)<\kappa T(r,f), \tag{18}$$

$$N(r,f)=o(1)T(r,f). \tag{19}$$

From lemma 2.7, (18) and (19), we get

$$K(1-4(o(1))-\kappa)T(r,f)\leq N\left(r,\frac{1}{\Psi-\alpha}\right)+S(r,f), \tag{20}$$

$r\notin E, r\to\infty$, where E is a possible exceptional set with finite logarithmic measure. Since f is transcendental, combining lemma 2.6 and (20), we can get that $\Psi(z)$

assumes every non-zero value α infinitely often and $\lambda(\alpha, \Psi(z)) = \rho(f)$.
(2) Since $\delta(0, f) = 1$ and $\delta(\infty, f) = 1$,

$$N\left(r, \frac{1}{f}\right) = S(r, f), \tag{21}$$

$$N(r, f) = S(r, f). \tag{22}$$

From (1) and lemma 2.3, we have

$$\begin{aligned} T(r, \Psi) &= m(r, \Psi) + N(r, \Psi) \\ &\leq m\left(r, \frac{\Psi}{\alpha f^K}\right) + m\left(r, \alpha f^K\right) + N(r, \Psi) + S(r, f) \\ &= Km(r, f) + 2KN(r, f) + S(r, f) \\ T(r, f) &\leq KT(r, f) + S(r, f) \end{aligned} \tag{23}$$

From (21)-(23) and lemma 2.7, we get

$$\begin{aligned} KT(r, f) &\leq N\left(r, \frac{1}{\Psi - \alpha}\right) + S(r, f), \\ &\leq N(r, \Psi) + S(r, f) \\ &\leq KT(r, f) + S(r, f). \end{aligned} \tag{24}$$

Since f is transcendental, (24) means that $\Psi(z)$ assumes every non-zero value α infinitely often and

$$T(r, \Psi(z)) \sim KT(r, f) \sim N\left(r, \tfrac{1}{\Psi - \alpha}\right)$$

as $r \notin E, r \to \infty$, where E is a possible exceptional set of r with finite logarithmic measure.

Theorem 2 Let $\Psi(z)$ be a non linear difference monomial of the form (1), If $\delta(\infty, f) = 1$ and $f(z)$ be a finite-order transcendental meromorphic function, then

(1) for $\delta(0, f) > 0, \Psi(z)$ assumes every non-zero value α infinitely often and $\lambda(\alpha, \Psi(z)) = \sigma(f)$

(2) for $\delta(0, f) = 1, \Psi(z)$ assumes every non-zero value α infinitely often and

$$T(r, \Psi(z)) \sim dT(r, f) \sim N\left(r, \frac{1}{\Psi(z) - \alpha}\right)$$

,

as $r \notin E, r \to \infty$, where E is a possible exception set of r with finite logarithmic measure.

Proof. First, we prove that $P(z, f)$ cannot be reduced to any constant. Suppose, assume that $P(z, f) = d$, where d is some constant. Then from (17), we get
$f^n = d - Q(z, f)$, also it can be written as

$$f^{n-1} f = d - Q(z, f). \tag{25}$$

Since $n \geq d(p) + 2$, then from (25) and lemma 2.5, we get

$$m(r, f) = S(r, f) \tag{26}$$

Since $\delta(\infty, f) = \delta > 0$, then for any given $\epsilon_0 (0 < \epsilon_0 < \delta)$ and sufficiently large r, we have

$$\delta(\infty, f) = \liminf_{r \to \infty} \frac{m(r, f)}{T(r, f)} = \delta > 0,$$

which implies that

$$(\delta - \epsilon_0)T(r, f) \leq m(r, f). \tag{27}$$

From (26) and (27), we get $T(r, f) = S(r, f)$. Which is contradiction. Hence $P(z, f)$ can never reduce to constant. set $h_j(z) = \frac{\Delta_c^j f}{f}$ and substituting this in (17), we get

$$P(z, f) = f^n + \sum_{t=0}^{d(p)} b_t(z) f^t(z), \tag{28}$$

where $b_t(z) = \sum_{l_\lambda = t} a_\lambda(z) \prod_{j=1}^{m} (h_j(z))^{l_{\lambda j}}$, $t = 0, 1, \cdots, d(p)$. Clearly from lemma 2.1, we get

$$m(r, b_t) = S(r, f), \tag{29}$$

using (28),(29) and lemma 2.4, we obtain

$$m(r, P) \leq nm(r, f) + S(r, f). \tag{30}$$

Also from (17) and lemma 2.2, we get

$$\begin{aligned} N(r, P(z, f)) &\leq N(r, f^n) + N(r, Q(z, f)) + S(r, f) \\ &\leq nN(r, f) + d(p) \sum_{j=1}^{m} N(r, \Delta_c^j f) + S(r, f) \\ &\leq nN(r, f) + d(p) \sum_{j=1}^{m} N(r, \sum_{i=0}^{j} (-1)^i \binom{j}{i} f(z + (j - i)c) \\ &\leq nN(r, f) + d(p) \sum_{j=1}^{m} \sum_{i=0}^{j} N(r, (-1)^i \binom{j}{i} f(z + (j - i)c) \\ &= nN(r, f) + d(p) \sum_{j=1}^{m} (j + 1) N(r, f) + S(r, f) \\ &= nN(r, f) + d(p) \left(\frac{m^2 + m - 2}{2} \right) N(r, f) + S(r, f). \end{aligned} \tag{31}$$

From (30) and (31), implies

$$T(r, P(z, f)) \leq \left(n + \left(\frac{m^2 + m - 2}{2} \right) d(p) \right) T(r, f) + S(r, f) \ \ and \ \ S(r, P) = S(r, f). \tag{32}$$

And also from (17) and lemma 2.2, we have

$$\begin{aligned}\overline{N}(r,P) &\leq \overline{N}(r,f)+\overline{N}(r,P(z,f))+S(r,f)\\ &= \overline{N}(r,f)+\sum_{j=1}^{m}\overline{N}(r,\Delta_c^j f)+S(r,f)\\ &\leq \overline{N}(r,f)+\sum_{j=1}^{m}(j+1)\overline{N}(r,f)+S(r,f)\\ &= \overline{N}(r,f)+\left(\frac{m^2+m-2}{2}\right)\overline{N}(r,f)+S(r,f)\\ \overline{N}(r,P) &\leq \left(\frac{m(m+1)}{2}\right)\overline{N}(r,f)+S(r,f) \qquad (33)\end{aligned}$$

From (17), it is possible to write

$$P(z,f)-q(z)=f^n+P(z,f)-q(z). \qquad (34)$$

Differentiating (34), on simple calculation, we get

$$f^{n-1}\left(nf^{'}-f\frac{(P-q)^{'}}{P-q}\right)=\frac{(P-q)^{'}}{P-q}Q(z,f)-Q^{'}(z,f)-\frac{(P-q)^{'}}{P-q}q+q^{'}, \qquad (35)$$

and

$$f^{n-2}\left(f\left(nf^{'}-f\frac{(P-q)^{'}}{P-q}\right)\right)=\frac{(P-q)^{'}}{P-q}Q(z,f)-Q^{'}(z,f)-\frac{(P-q)^{'}}{P-q}q+q^{'}. \qquad (36)$$

Next, we assert that $nf^{'}-f\frac{(P-q)^{'}}{P-q}\not\equiv 0$. If not, $n\frac{f^{'}}{f}=\frac{(P-q)^{'}}{P-q}$, integrating this, we get $P-q=Cf^n$, for some constant C, substituting this in (34), we get

$$(C-1)f^n=Q(z,f)-q(z). \qquad (37)$$

Since $Q(z,f)\not\equiv q(z)$, observe that $C\neq 1$. Also we have $n>d(p)$, from (37) and lemma 2.5, we get $m(r,f)=S(r,f)$. Since $\delta(\infty,f)>0$, from (27), we conclude that $T(r,f)=S(r,f)$. which is a contradiction. Since $d(p)\leq n-2$, from (36), (37) and lemma 2.5, we obtain

$$m\left(r,nf^{'}-f\frac{(P-q)^{'}}{P-q}\right)=S(r,f). \qquad (38)$$

$$m\left(r,f\left(nf^{'}-f\frac{(P-q)^{'}}{P-q}\right)\right)=S(r,f). \qquad (39)$$

Let us consider
$N\left(r,nf^{'}-\frac{(P-q)^{'}}{P-q}f\right)\leq N(r,f)+\overline{N}(r,f)+\overline{N}(r,P-q)+\overline{N}\left(r,\frac{1}{P-q}\right)+O(1)$. From (33), above inequality implies

$$N\left(r,nf^{'}-\frac{(P-q)^{'}}{P-q}f\right)\leq N(r,f)+\left(\frac{m^2+m+2}{2}\right)\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{P-q}\right)+S(r,f). \qquad (40)$$

Similarly,

$$N\left(r, nf^{'} - \frac{(P-q)^{'}}{P-q}f\right) \leq 2N(r,f)+\left(\frac{m^2+m+2}{2}\right)\overline{N}(r,f)+\overline{N}\left(r, \frac{1}{P-q}\right)+S(r,f). \tag{41}$$

From (38) and (40), we obtain

$$T\left(r, nf^{'} - \frac{(P-q)^{'}}{P-q}f\right) \leq N(r,f)+\left(\frac{m^2+m+2}{2}\right)\overline{N}(r,f)+\overline{N}\left(r, \frac{1}{P-q}\right)+S(r,f). \tag{42}$$

Also from (39) and (41), we have

$$T\left(r, nf^{'} - \frac{(P-q)^{'}}{P-q}f\right) \leq 2N(r,f)+\left(\frac{m^2+m+2}{2}\right)\overline{N}(r,f)+\overline{N}\left(r, \frac{1}{P-q}\right)+S(r,f). \tag{43}$$

Thus, from (42) and (43), we get

$$\begin{aligned} T(r,f) &\leq T\left(r, f\left(nf^{'} - \frac{(P-q)^{'}}{P-q}f\right)\right) + T\left(r, nf^{'} - \frac{(P-q)^{'}}{P-q}f\right) + S(r,f) \\ &= 3N(r,f) + \left(m^2+m+2\right)\overline{N}(r,f) + 2\overline{N}\left(r, \frac{1}{P-q}\right) + S(r,f), \end{aligned}$$

which implies

$$T(r,f) \leq \left(m^2+m+5\right)N(r,f) + 2\overline{N}\left(r, \frac{1}{P-q}\right) + S(r,f). \tag{44}$$

For any given $\epsilon \left(< \epsilon < \delta - 1 + \frac{1}{m^2+m+5}\right)$, It is possible to write (44) as

$$T(r,f) \leq \left(m^2+m+5\right)(1-\delta+\epsilon)N(r,f) + 2\overline{N}\left(r, \tfrac{1}{P-q}\right) + S(r,f)$$

or

$$\left(1-(m^2+m+5)(1-\delta+\epsilon)+O(1)\right)T(r,f) \leq 2\overline{N}\left(r, \frac{1}{P-q}\right). \tag{45}$$

From (32) and (45), we get
$\left(1-(m^2+m+5)(1-\delta+\epsilon)+O(1)\right)T(r,P) \leq (2n+(m^2+m-2)d(p))\overline{N}\left(r, \frac{1}{P-q}\right).$

Thus, $\limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{P-q}\right)}{T(r,P)} \geq \frac{1-(m^2+m+5)(1-\delta+\epsilon)}{2n+(m^2+m-2)d(p)}$
choose $(m^2+m+5)(1-\delta+\epsilon) < 1$, then from the above inequality, we have

$$\limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{P-q}\right)}{T(r,P)} > 0,$$

therefore

$$\delta(q,P) < 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{P-q}\right)}{T(r,P)} < 1.$$

Hence, $P(z,f) = f^n + Q(z,f)$ assumes $q(z)$ infinitely often.

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