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OPTIMAL CONTROLS FOR SOME IMPLUSIVE STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS DRIVEN BY ROSENBLATT PROCESS

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ABSTRACT. We study optimal control issues for a class of impulsive stochastic integrodifferential equations driven by a Rosenblatt process with non-instantaneous impulses. We begin by investigating the existence of mild solutions for the stochastic system, with the primary tools being stochastic analysis theory, the theory of resolvent operators, and a fixed point approach. Following that, we derive the optimal control findings of the stochastic system without considering the uniqueness of mild solutions. Finally, an example is provided to demonstrate the key findings of this research.

1. INTRODUCTION

In this work, we consider the following stochastic integrodifferential equations driven by Rosenblatt process with impulses

$$\begin{cases} dz(t) = \left[Az(t) + C(t)u(t) + \int_0^t \mathscr{B}(t-s)z(s)ds\right]dt + F(t,z(t))dW(t) \\ \qquad\qquad +G(t)dR_Q^{\mathsf{H}}(t), \text{ for } t \in \cup_{k=0}^m (e_k, t_{k+1}], \\ z(t) = I_k(t, z(t_k^-)),\ t \in \cup_{k=1}^m (t_k, e_k], \\ z(0) = z_0, \end{cases} \tag{1}$$

where the state $z(\cdot)$ takes values in a separable Hilbert space $\mathbb{Y}$ and $0 = e_0 = t_0 < t_1 < e_1 < t_2 < \ldots < t_m < e_m < t_{m+1} = b < \infty$, $J = [0,b]$. The operator A is a generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on $\mathbb{Y}$. $\{W(t) : t \geq 0\}$ denotes a Wiener process in a real and separable Hilbert space $\mathbb{X}_1$ and $R_Q^{\mathsf{H}} = \{R_Q^{\mathsf{H}}(t) : t \geq 0\}$ is Rosenblatt process with Hurst index $\mathsf{H} \in (1/2, 1)$ on a real and separable Hilbert space $\mathbb{X}_2$. The function $I_k(t, z(t_k^-))$ represents impulses during the intervals $(t_k, e_k]$, $k = 1, 2, \ldots, m$. Further, u is the control function which takes value in reflexive and separable Hilbert space $\mathscr{K}$ and C is linear operator from $\mathscr{K}$ into $\mathbb{Y}$. The processes R^{H} and W are independent. The functions $F : J \times \mathbb{Y} \to \mathscr{L}_2^1(\mathbb{X}_1, \mathbb{Y})$, $G : J \to \mathscr{L}_2^2(\mathbb{X}_2, \mathbb{Y})$, $I_k : (t_k, e_k] \times \mathbb{Y} \to \mathbb{Y}$, $k = 1, 2, \ldots, m$ are satisfying some suitable conditions which will be specified later.

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Stochastic differential equations have become an active area of study due to their various applications in the fields like electrical engineering, mechanics, medical biolology, economical systems etc. For more informations see [1–3, 14, 19, 25, 26, 28]. Moreover many authors investigated the existence, uniqueness, stability, controllability and others qualitative and quantitative properties of SDEs and stochastic integro-differential equations (SIEs for short) by using stochastic analysis, fixed point technique and the concept of resolvent operators in the case of SIEs. See for example [11, 13, 16, 18, 20, 25].

The theory of impulsive partial equations or inclusions appears as a natural description of many real processes subject to some perturbations whose duration is negligible in comparison with the duration of the process. It has seen a great deal of development during the last decennary [10]. Moreover, besides impulsive effects, stochastic effects also exist in real systems.Thus, impulsive stochastic differential equations describing these dynamical systems subject to both impulsive and stochastic modifications have attracted significant attention. Especially, the papers [7, 24, 31] studied the existence of mild solutions for some impulsive neutral stochastic functional integro-differential equations with infinite delay in Hilbert spaces.

Rosenblatt process with Hurst parameter $\mathsf{H} \in (0,1)$ is a centered Gaussian $\{R_Q^{\mathsf{H}}(t), t \geq 0\}$ which is employed to model numerous complex phenomena in applications as the systems have rough external forcing. There is another process with non-Gaussian character, which contributes the other properties for $\mathsf{H} > 1/2$, the long memory property.

A lot of exciting applications of Rosenblatt process have been established in several fields such as hydrology, telecommunications, economics and finance. The observations of stock prices processes suggest that they are not self-similar.

On the other hand, the optimal control problem requires the minimization of a criterion function of the states and control inputs of the system over a set of admissible control functions.

The concept of optimal control problem plays an important role in many scientific fields such as biomedical, engineering, biology, physics, economy, etc. (see [22]).Recently, many works have been considerable interest in the study of the existence of optimal controls for different kinds of nonlinear SDEs and SIEs in infinite dimensional spaces.Yan and Lu [32] proved the existence of optimal controls for a fractional stochastic partial integrodifferential equation with infinite delay. Balasubramanian and Tamilalagan [8] investigated the solvability and optimal controls for impulsive fractional stochastic integrodifferential equations in Hilbert space. Very recently, Yan Z. and Yan X. [34] studied optimal controls of a class of impulsive partial stochastic differential equations with weighted pseudo almost periodic coefficients in Hilbert spaces. Furthermore, solvability and optimal control results of stochastic integrodifferential equations driven by the Rosenblatt process with impulses are rarely available in the literature; this fact serves as a motivation for our research work in this manuscript.

The rest of this paper is organized as follows. In Section 2, we recall some preliminaries, definitions and lemmas that are to used later to proved our main results. In Section 3, we prove existence of mild solutions for stochastic system (1). In Section 4, the existence of an optimal control pair for the proposed stochastic system is studied. Finally in Section 5, an example is provided to illustrate the main results.

2. Preliminaries

In this segment, we present some mathematical tools which are required to prove the main results. For more details, we refer the reader to [12] and the references therein. Throughout this paper, let $L^2(\Omega, \mathbb{Y})$ stands for the space of all $\mathbb{Y}$-valued random variables $\mathscr{G}$ such that $\mathbb{E} \parallel z \parallel^2 = \int_\Omega \parallel z \parallel^2 d\mathbb{P} < \infty$. Let $\mathscr{L}(\mathbb{X}_2, \mathbb{Y})$ denotes the space of all bounded linear operators from $\mathbb{X}_2$ to $\mathbb{Y}$ and $Q \in \mathscr{L}(\mathbb{X}_2, \mathbb{X}_2)$ represents a non-negative self-adjoint operator. Let $\mathscr{L}_Q^0(\mathbb{X}_2, \mathbb{Y})$ be the space of all functions $\Psi \in (\mathbb{X}_2, \mathbb{Y})$ such that $\Psi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by $\parallel \Psi \parallel^2_{\mathscr{L}_Q^0(\mathbb{X}_2,\mathbb{Y})} = \parallel \Psi Q^{\frac{1}{2}} \parallel^2 = Tr\big(\Psi Q \psi^*\big)$ and Ψ is called a Q-Hilbert-Schmidt operator from $\mathbb{X}_2$ to $\mathbb{Y}$.

2.1. **Rosenblatt process.** Now, we recall basic properties of the Rosenblatt process as well as Wiener integral with respect to it. Let $[0, b]$ denote a time interval with arbitrary fixed horizon b and let $\{R(t) : t \in [0, b]\}$ be a one-dimensional Rosenblatt process with parameter $\mathsf{H} \in (\frac{1}{2}, 1)$. Also, the Rosenblatt process with parameter $\mathsf{H} > \frac{1}{2}$ admits the following representation [27]:

$$R_{\mathsf{H}}(t) = q(\mathsf{H}) \int_0^t \int_0^t \left[\int_{z_1 \vee z_2}^t \frac{\partial N^{\mathsf{H}'}}{\partial u}(u, z_1) \frac{\partial N^{\mathsf{H}'}}{\partial u}(u, z_2) du \right] dB(z_1) dB(z_2), \tag{2}$$

where $N^{\mathsf{H}}(t,s)$ is given by

$$N^{\mathsf{H}}(t,s) = c_{\mathsf{H}} s^{\frac{1}{2}-\mathsf{H}} \int_s^t (u-s)^{\mathsf{H}-3/2} u^{\mathsf{H}-1/2} du \text{ for } t > s,$$

with

$$c_{\mathsf{H}} = \sqrt{\frac{\mathsf{H}(2\mathsf{H}-1)}{\Gamma(2-2\mathsf{H}, \mathsf{H}-\frac{1}{2})}},$$

$\Gamma(\cdot,\cdot)$ denotes the Gamma function, $N^{\mathsf{H}}(t,s) = 0$ when $t \le s$, $\{B(t), t \in [0,b]\}$ is a Brownian motion, $\mathsf{H}' = \frac{\mathsf{H}+1}{2}$ and $q(\mathsf{H}) = \frac{1}{\mathsf{H}+1} \sqrt{\frac{\mathsf{H}}{2(2\mathsf{H}-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{R_{\mathsf{H}}(t), t \in [0,b]\}$ is

$$\mathbb{E}(R_{\mathsf{H}}(t) R_{\mathsf{H}}(s)) = \frac{1}{2} \left(s^{2\mathsf{H}} + t^{2\mathsf{H}} - |s-t|^{2\mathsf{H}} \right).$$

The covariance structure of the Rosenblatt process allows to construct Wiener integral with respect to it. We refer to Maejima and Tudor [5] for the definition of Wiener integral with respect to general Hermite processes and to Kruk, Russo, and Tudor [4] for a more general context (see also Tudor [6]).

Notice that

$$R_{\mathsf{H}}(t) = \int_0^b \int_0^b I(1_{[0,t]})(z_1, z_2) dB(z_1) dB(z_2),$$

where the operator I is defined on the set of functions $\mathscr{G} : [0,b] \to \mathbb{R}$, which takes its values in the set of functions $\mathscr{G} : [0,b]^2 \to \mathbb{R}^2$ and is given by

$$I(\mathscr{G})(z_1, z_2) = q(\mathsf{H}) \int_{z_1 \vee z_2}^b \mathscr{G}(u) \frac{\partial N^{\mathsf{H}'}}{\partial u}(u, z_1) \frac{\partial N^{\mathsf{H}'}}{\partial u}(u, z_2) du.$$

Let $\mathscr{G}$ be an element of the set $\mathscr{E}$ of step functions on $[0,b]$ of the form

$$\mathscr{G}=\sum_{i=0}^{n-1} a_i 1_{(t_i,t_{i+1}]},\ t_i\in[0,b].$$

Then, define its Wiener integral with respect to R_{H} as

$$\int_0^b \mathscr{G}(u)dN_{\mathsf{H}}(u):=\sum_{i=0}^{n-1} a_i(R_{\mathsf{H}}(t_{i+1})-R_{\mathsf{H}}(t_i))=\int_0^b\int_0^b I(\mathscr{G})(z_1,z_2)dB(z_1)dB(z_2).$$

Let $\mathscr{H}$ be the set of functions $\mathscr{G}$ such that

$$\|\mathscr{G}\|^2_{\mathscr{H}}:=2\int_0^b\int_0^b (I(\mathscr{G})(z_1,z_2))^2dz_1dz_2<\infty.$$

It follows from [6] that

$$\|\mathscr{G}\|^2_{\mathscr{H}}=\mathsf{H}(2\mathsf{H}-1)\int_0^b\int_0^b \mathscr{G}(u)\mathscr{G}(v)|u-v|^{2\mathsf{H}-2}dudv.$$

It has been proved in [5] that the mapping

$$\mathscr{G}\to\int_0^b \mathscr{G}(u)dR_{\mathsf{H}}(u)$$

defines an isometry from $\mathscr{E}$ to $L^2(\Omega)$ and it can be extended continuously to an isometry from $\mathscr{H}$ to $L^2(\Omega)$ because $\mathscr{E}$ is dense in $\mathscr{H}$. We call this extension as the Wiener integral of $\mathscr{G}\in\mathscr{H}$ with respect to R_{H}.
Notice that the space $\mathscr{H}$ contains not only functions but its elements could be also distributions.
Therefore it is suitable to know subspaces $|\mathscr{H}|$ of $\mathscr{H}$:

$$|\mathscr{H}|=\left\{\mathscr{G}:[0,b]\to\mathbb{R}\Big|\int_0^b\int_0^b|\mathscr{G}(u)||\mathscr{G}(v)|u-v|^{2\mathsf{H}-2}dudv<\infty\right\}.$$

The space $|\mathscr{H}|$ is not complete with respect to the norm $\|.\|_{\mathscr{H}}$ but it is a Banach space with respect to the norm

$$\|\mathscr{G}\|^2_{|\mathscr{H}|}=\mathsf{H}(2\mathsf{H}-1)\int_0^b\int_0^b|\mathscr{G}(u)||\mathscr{G}(v)|u-v|^{2\mathsf{H}-2}dudv.$$

As a consequence, we have

$$L^2([0,b])\subset L^{1/\mathsf{H}}([0,b])\subset|\mathscr{H}|\subset\mathscr{H}.$$

For any $\mathscr{G}\in L^2([0,b])$, we have

$$\|\mathscr{G}\|^2_{|\mathscr{H}|}\le 2\mathsf{H}b^{2\mathsf{H}-1}\int_0^b|\mathscr{G}(s)|^2ds$$

and

$$\|\mathscr{G}\|^2_{|\mathscr{H}|}\le c_{\mathsf{H}}\|\mathscr{G}\|^2_{L^{1/\mathsf{H}}([0,b])},\tag{3}$$

for some constant $c_{\mathsf{H}}>0$. Let $c_{\mathsf{H}}>0$ stand for a positive constant depending only on H and its value may be different in different appearances.

Define the linear operator N^*_{H} from $\mathscr{E}$ to $L^2([0,b])$ by

$$(N^*_{\mathsf{H}}\mathscr{G})(z_1,z_2)=\int_{z_1\vee z_2}^b\mathscr{G}(t)\frac{\partial\mathscr{N}}{\partial t}(t,z_1,z_2)dt,$$

where $\mathscr{N}$ is the kernel of Rosenblatt process in representation (2)

$$\mathscr{N}(t,z_1,z_2) = 1_{[0,t]}(z_1)1_{[0,t]}(z_2)\int_{z_1\vee z_2}^{t}\frac{\partial N^{\mathsf{H}'}}{\partial u}(u,z_1)\frac{\partial N^{\mathsf{H}'}}{\partial u}(u,z_2)du.$$

Note that $(N_{\mathsf{H}}^{*}1_{[0,t]})(z_1,z_2) = \mathscr{N}(t,z_1,z_2)1_{[0,t]}(z_1)1_{[0,t]}(z_2)$. The operator N_{H}^{*} is an isometry between $\mathscr{E}$ to $L^2([0,b])$, which can be extended to the Hilbert space $\mathscr{H}$. Also, for any $s,t\in[0,b]$ we have

$$\begin{aligned}
\left\langle N_{\mathsf{H}}^{*}1_{[0,t]}, N_{\mathsf{H}}^{*}1_{[0,s]}\right\rangle_{L^2([0,b])} &= \left\langle \mathscr{N}(t,.,.)1_{[0,t]}, \mathscr{N}(s,.,.)1_{[0,s]}\right\rangle_{L^2([0,b])}\\
&= \int_0^{t\wedge s}\int_0^{t\wedge s}\mathscr{N}(t,z_1,z_2)\mathscr{N}(s,z_1,z_2)dz_1dz_2\\
&= \mathsf{H}(2\mathsf{H}-1)\int_0^t\int_0^s|u-v|^{2\mathsf{H}-2}dudv\\
&= \left\langle 1_{[0,t]}, 1_{[0,s]}\right\rangle_{\mathscr{H}}.
\end{aligned}$$

Further, for $\mathscr{G}\in\mathscr{H}$, we have

$$R_{\mathsf{H}}(\mathscr{G}) = \int_0^b\int_0^b (N_{\mathsf{H}}^{*}\mathscr{G})(z_1,z_2)dB(z_1)dB(z_2).$$

Let $\left(\kappa_n(t)\right)_{n\in\mathbb{N}}$ be a sequence of two-sided one dimensional Rosenblatt process mutually independent on $(\Omega,\mathscr{F},\mathbb{P})$. We consider a $\mathbb{X}_2$-valued stochastic process $R_Q^{\mathsf{H}}(t)$ given by the following series:

$$R_Q(t) = \sum_{n=1}^{\infty}\kappa_n(t)Q^{1/2}e_n, \quad t\geq 0.$$

Moreover, if Q is a non-negative self-adjoint trace class operator, then this series converges in the space $\mathbb{X}_2$, that is, it holds that $R_Q(t)\in L^2(\Omega,\mathbb{X}_2)$. Then, we say that the above $R_Q(t)$ is a $\mathbb{X}_2$-valued Q- Rosenblatt process with covariance operator Q. For exemple, if $\{\delta_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n=\delta_ne_n$, by assuming that Q is a nuclear operator in $\mathbb{X}_2$, then the stochastic process

$$R_Q(t) = \sum_{n=1}^{\infty}\kappa_n(t)Q^{1/2}e_n = \sum_{n=1}^{\infty}\sqrt{\delta_n}\kappa_n(t)e_n, \quad t\geq 0,$$

is well-defined as a $\mathbb{X}_2$-valued Q- Rosenblatt process.

Definition 2.1. *(Tudor [6]). Let $\varphi:[0,b]\to L_Q^0(\mathbb{X}_2,\mathbb{Y})$ such that $\sum_{n=1}^{\infty}\|N_{\mathsf{H}}^{*}(\varphi Q^{1/2}e_n)\|_{L^2([0,b];\mathbb{Y})}<\infty$. Then, its stochastic integral with respect to the Rosenblatt process $R_Q(t)$ is defined for $t\geq 0$ as follows:*

$$\begin{aligned}
\int_0^t\varphi(s)dR_Q(s): &= \sum_{n=1}^{\infty}\int_0^t\varphi(s)Q^{1/2}e_nd\kappa_n(s)\\
&= \sum_{n=1}^{\infty}\int_0^t\int_0^t(N_{\mathsf{H}}^{*}(\varphi Q^{1/2}e_n))(z_1,z_2)dB(z_1)dB(z_2). \qquad (4)
\end{aligned}$$

Lemma 2.1. *[30] For $\psi:[0,b]\to L_Q^0(\mathbb{X}_2,\mathbb{Y})$ such that $\sum_{n=1}^{\infty}\|\psi Q^{1/2}e_n\|_{L^{1/\mathsf{H}}([0,b];\mathbb{Y})}<\infty$ hold and for any $\alpha,\beta\in[0,b]$ with $\beta>\alpha$, we have*

$$\mathbb{E}\left\|\int_\alpha^\beta\psi(s)dR_Q(s)\right\|^2\leq c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\sum_{n=1}^{\infty}\int_\alpha^\beta\|\psi(s)Q^{1/2}e_n\|^2ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|\psi(t)Q^{1/2}e_n\| \text{ is uniformly convergent for } t \in [0,b], \quad \text{then}$$

$$\mathbb{E}\left\|\int_{\alpha}^{\beta} \psi(s)dR_Q(s)\right\|^2 \le c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\int_{\alpha}^{\beta} \|\psi(s)\|^2_{L_Q^0(\mathbb{X}_2,\mathbb{Y})}ds.$$

Lemma 2.2. *[16] For any $p \ge 1$ and for arbitrary $\mathscr{L}_2^1$-valued predictable process $\mathscr{X}(\cdot)$, we have that*

$$\sup_{r\in[0,t]} \mathbb{E}\left\|\int_0^r \mathscr{X}(\mu)dW(\mu)\right\|^{2p} \le (p(2p-1))^p \left(\int_0^t \left(\mathbb{E}\|\mathscr{X}(s)\|^{2p}_{\mathscr{L}_2^1}\right)^{1/p} ds\right)^p, \; t \in [0,b]. \tag{5}$$

In particular, for $p = 1$, we have $\sup_{r\in[0,t]} \mathbb{E}\left\|\int_0^r \mathscr{X}(\mu)dW(\mu)\right\|^2 \le \int_0^t \mathbb{E}\|\mathscr{X}(s)\|^2_{\mathscr{L}_2^1}ds.$

Now, we define the space $\mathscr{PC}(\mathbb{Y})$ formed by all $\mathscr{F}_t$-adapted, $\mathbb{Y}$-valued measurable stochastic processes $\{z(t) : t \in [0,b]\}$ such that z is continuous at $t \ne t_k$, $z(t_k^-) = z(t_k)$ and $z(t_k^+)$ exists for all $k = 1,\dots,m$ endowed with the norm

$$\|z\|_{\mathscr{PC}} = \left(\sup_{0\le t\le b} \mathbb{E}\|z(t)\|^2\right)^{1/2}.$$

Then, $(\mathscr{PC}(\mathbb{Y}), \|\cdot\|_{\mathscr{PC}})$ is Banach space.

The operator $C \in L_\infty(J, L(\mathscr{K},\mathbb{Y}))$, where $L_\infty(J, L(\mathscr{K},\mathbb{Y}))$ denotes the space of all operator valued functions, which are measurable and uniformly bounded equipped with the norm $\|\cdot\|$. The control function $u \in L^2_{\mathscr{F}}(J,\mathscr{K})$, where $L^2_{\mathscr{F}}(J,\mathscr{K})$ represents the space of all $\mathscr{K}$-valued stochastic processes, which are measurable and $\mathscr{F}_t$-adapted satisfying the condition $\mathbb{E}\int_0^b \|u(t)\|^2_{\mathscr{K}}dt < \infty$, and endowed with the following norm:

$$\|u\|^2_{L^2_{\mathscr{F}}} = \left(\mathbb{E}\int_0^b \|u(t)\|^2_{\mathscr{K}}dt\right)^{1/2}.$$

Let $\mathscr{U}$ be a nonempty closed bounded convex subset of $\mathscr{K}$. We define the admissible control set

$$\mathscr{U}_{ad} = \{u \in L^2_{\mathscr{F}}(J,\mathscr{K}); u(t) \in \mathscr{U} \text{ a.e. } t \in J\}.$$

Then, $Cu \in L^2(J,\mathbb{Y})$ for all $u \in \mathscr{U}_{ad}$.

Theorem 2.3. *(Bellman, 1943) [Gronwall-Beilman inequality]*
We begin by giving one of the simplest and most frequently used integral inequalities. Let $\rho(t)$ and $\varpi(t)$ be nonnegative continuous functions for $t > 0$, and let

$$\rho(t) \le \kappa + \int_0^t \varpi(s)\rho(s)ds \tag{6}$$

where $\kappa > 0$ is a constant. Then

$$\rho(t) \le \kappa \exp\left(\int_0^t \varpi(s)ds\right), \; t \ge 0 \tag{7}$$

Proof. Let $\kappa > 0$. Then (6) implies the inequality

$$\frac{\varpi(\tau)\rho(\tau)}{\kappa + \int_0^\tau \varpi(s)\rho(s)ds} \leq \varpi(\tau),\ \tau > 0$$

Integrating this from 0 to t yields

$$\log\left[\kappa + \int_0^t \varpi(s)\rho(s)ds\right] - \log\kappa \leq \int_0^t \varpi(s)ds$$

Together with (6) this implies (7). Let $\kappa = 0$. Then $\rho(t) = \varepsilon + \int_0^t \varpi(s)\rho(s)ds$ for any $t > 0$. Hence

$$\rho(t) \leq \varepsilon \exp\left(\int_0^t \varpi(s)ds\right)$$

and letting $\varepsilon \longrightarrow 0$ we find $\rho(t) \leq 0$.

We will prove some generalizations of the linear integral Gronwall's inequality. In the following we need Lemma2.4.

Lemma 2.4. *(Samoilenko and Perestyuk, 1977)*
Suppose that for $t \geq t_0$ the following inequality holds:

$$\rho(t) \leq \kappa + \int_{t_0}^t \varpi(s)\rho(s)ds + \sum_{t_0<t_k<t} \beta_k\rho(t_k)$$

where $\rho \in \mathscr{PC}(\mathbb{R}^+,\mathbb{R})$, $\varpi \in \mathscr{PC}(\mathbb{R}^+,\mathbb{R}^+)$, and $\beta_k \geq 0$, $k \in \mathbb{N}$, and a are constants. Then for $t \geq t_0$,

$$\rho(t) \leq \kappa \prod_{t_0<t_k<t} \left(1+\beta_k\right) \exp\left(\int_{t_0}^t \varpi(s)ds\right)$$

Proof. The proof of this theorem can be given by induction with respect to $k \in \mathbb{N}$, applying for $t \in (t_k, t_{k+1}]$ Theorem 2.3 to the Gronwall-Beilman inequality

$$\rho(t) \leq \kappa_k + \int_{t_K}^t \varpi(s)\rho(s)ds,\ \ t_k \leq t \leq t_{k+1}$$

where

$$\kappa_k = \rho(t_k^+) \leq \kappa + \int_{t_0}^{t_k} \varpi(s)\rho(s)ds + \sum_{i=1}^k \beta_i\rho(t_i) \leq \kappa\prod_{i=1}^k \left(1+\beta_i\right) \exp\left(\int_{t_0}^{t_k} \varpi(s)ds\right). \quad (8)$$

To be able to access existence of mild solutions for (1), we need to introduce partial integrodifferential equations and resolvent operators that will be used to develop the main results.

2.2. **Partial integrodifferential equations in Banach spaces.** Let $\mathscr{M}$ and $\mathscr{W}$ be two Banach spaces such that $\|w\|_{\mathscr{W}} = \|Aw\| + \|w\|$, $w \in \mathscr{W}$. A and $\mathscr{B}(t)$ are closed linear operators on $\mathscr{M}$. Let $\mathscr{C}(\mathbb{R}^+,\mathscr{W})$, $\mathscr{L}(\mathscr{W},\mathscr{M})$ stand for the space of all continuous functions from $\mathbb{R}^+$ into $\mathscr{W}$, the set of all bounded linear operators from $\mathscr{W}$ into $\mathscr{M}$, respectively. In what follows, we suppose the following assumptions:

(H1): A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on $\mathscr{M}$.

(H2): For all $t \geq 0$, $\mathscr{B}(t)$ is a closed linear operator from $\mathscr{D}(A)$ to $\mathscr{M}$, and $\mathscr{B}(t) \in \mathscr{L}(\mathscr{W},\mathscr{M})$. For any $w \in \mathscr{W}$, the map $t \to \mathscr{B}(t)w$ is bounded, differentiable and the derivative $t \to \mathscr{B}(t)'w$ is bounded uniformly continuous on $\mathbb{R}^+$.

By Grimmer [21], under the assumptions **(H1)** and **(H2)**, the following Cauchy problem

$$\begin{cases} \xi'(t) = A\xi(t) + \int_0^t \mathscr{B}(t-s)\xi(s)ds \text{ for } t \geq 0 \\ \xi(0) = \xi_0 \in \mathscr{M}, \end{cases} \tag{9}$$

has an associated resolvent operator of bounded linear operator valued function $\mathscr{R}(t) \in \mathscr{L}(\mathscr{V})$, for $t \geq 0$.

Definition 2.2. *[21] A bounded linear operator valued function $\mathscr{R}(t) \in \mathscr{L}(\mathscr{M})$, for $t \geq 0$, is refered to be a resolvent operator associated with (9) if :*

(i): *$\mathscr{R}(0) = I$ and $\|\mathscr{R}(t)\|_{\mathscr{L}(\mathscr{M})} \leq \tilde{M}e^{\gamma t}$ for some constants $\tilde{M}$ and γ.*
(ii): *For all each $m \in \mathscr{M}$, $\mathscr{R}(t)m$ is strongly continuous for $t \geq 0$.*
(iii): *$\mathscr{R}(t) \in \mathscr{L}(\mathscr{W})$ for $t \geq 0$. For $m \in \mathscr{W}$, $\mathscr{R}(\cdot) \in \mathscr{C}^1([0,+\infty[,\mathscr{M}) \cap \mathscr{C}([0,+\infty[,\mathscr{W})$ and*

$$\begin{aligned} \mathscr{R}^{'}(t)m &= A\mathscr{R}(t)m + \int_0^t \mathscr{B}(t-s)\mathscr{R}(s)mds, \\ &= \mathscr{R}(t)Am + \int_0^t \mathscr{R}(t-s)\mathscr{B}(s)mds, \ t \geq 0. \end{aligned}$$

Now, we present some results on the existence of solutions for the following integrodifferential equation:

$$\begin{cases} \xi'(t) = A\xi(t) + \int_0^t \mathscr{B}(t-s)\xi(s)ds + \Xi(t) \text{ for } t \geq 0 \\ \xi(0) = \xi_0 \in \mathscr{M}, \end{cases} \tag{10}$$

where $\Xi : \mathbb{R}_+ \to \mathscr{M}$ is a continuous function.

Definition 2.3. *A continuous function $\xi : [0,\infty[\to \mathscr{M}$ is said to be a strict solution for equation (10) if*

(1) $\xi \in \mathscr{C}^1(\mathbb{R}_+,\mathscr{M}) \cap \mathscr{C}(\mathbb{R}_+,\mathscr{W})$,
(2) ξ satisfies equation (10) for $t \geq 0$.

Remark 2.1. *From this definition, we deduce that $\xi(t) \in \mathscr{D}(A)$, and the function $s \mapsto \mathscr{B}(t-s)\xi(s)$ is integrable, for all $t > 0$ and $s \geq 0$.*

Theorem 2.5. *[21] Suppose that hypotheses **(H1)** and **(H2)** hold. If ξ is a strict solution of (10), then the following variation of constants formula holds.*

$$\xi(t) = \mathscr{R}(t)\xi_0 + \int_0^t \mathscr{R}(t-s)\Xi(s)ds, \text{ for } t \geq 0. \tag{11}$$

Consequently, we can establish the following definition.

Definition 2.4. *[21] A function $\xi : \mathbb{R}_+ \to \mathscr{M}$ is called a mild solution of (10) for $\vartheta_0 \in \mathscr{M}$, if ξ sastisfies the variation of constants formula (11).*

Theorem 2.6. *[21] Let $\Xi \in \mathscr{C}^1([0,+\infty[;\mathscr{M})$ and ξ be defined by (11). If $\xi_0 \in \mathscr{D}(A)$, then ξ is a strict solution for equation (10).*

Lemma 2.7. *[23] Assume that assumptions **(H1)** and **(H2)** are satisfied. The resolvent operator $(\mathscr{R}(t))_{t\geq 0}$ is compact for $t>0$ if and only if the C_0-semigroup $(S(t))_{t\geq 0}$ is compact for $t>0$.*

Lemma 2.8. *[23] Suppose that hypotheses **(H1)** and **(H2)** hold. If the resolvent operator $(\mathscr{R}(t))_{t\geq 0}$ is compact for $t>0$ then it is norm continuous (or continuous in the uniform operator topology) for $t>0$.*

Lemma 2.9. *[23] Let (**H1**) and (**H2**) be satisfied. Then there is a constant $L=L(b)$ such that*

$$\| \mathscr{R}(t+\varepsilon)-\mathscr{R}(\varepsilon)\mathscr{R}(t) \|_{\mathscr{L}(\mathscr{M})}\leq L(\varepsilon), \quad 0<\varepsilon<t<b.$$

Let us define the bound which will be helpful throughout the calculations: $\sup_{t\in[0,b]} \|\mathscr{R}(t)\| = M$ for some positive constant M.

Now, we give the definition of mild solution for (1).

Definition 2.5. *An $\mathscr{F}_t$-adapted stochastic process $z: J\to \mathbb{Y}$ is said to be a mild solution of the stochastic system (1) if for every $t\in J$, $z(t)$ satisfies $z(0)=z_0$, $z(t)=I_k(t,z(t_k^-))$, $t\in (t_k,e_k]$, $k=1,2,\dots,m$ and*

$$\begin{aligned} z(t) &= \mathscr{R}(t)z_0+\int_0^t \mathscr{R}(t-s)C(s)u(s)ds \\ &\quad +\int_0^t \mathscr{R}(t-s)F(s,z(s))dW(s)+\int_0^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), \end{aligned}$$

for all $t\in[0,t_1]$, $k=0$ and

$$\begin{aligned} z(t) &= \mathscr{R}(t-e_k)I_k(e_k,z(t_k^-))+\int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds \\ &\quad +\int_{e_k}^t \mathscr{R}(t-s)F(s,z(s))dW(s)+\int_{e_k}^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s) \end{aligned} \tag{12}$$

for all $t\in(e_k,t_{k+1}]$, $k=1,2,\dots,m$.

3. Existence of mild solutions

In this section, we prove the existence of mild solutions for system (1). To prove our main results, we assume the following hypotheses:

(H3): $\mathscr{R}(t)$, $t\geq 0$ is compact.

(H4): The functions $I_k:(t_k,e_k]\times\mathbb{Y}\to\mathbb{Y}$, $k=1,2,\dots,m$, are continuous and there exist constants $\delta_{I_k},\gamma_k>0$, $k=1,2,\dots,m$ such that

$$\mathbb{E}\|I_k(t,z)\|^2\leq \delta_{I_k}(1+\mathbb{E}\|z\|^2), \text{ for all } z\in\mathbb{Y},$$

$$\mathbb{E}\|I_k(t,z_1)-I_k(t,z_2)\|^2\leq \gamma_k\mathbb{E}\|z_1-z_2\|^2, \text{ for all } z_1,z_2\in\mathbb{Y}.$$

(H5): The function $F: J\times\mathbb{Y}\to\mathscr{L}_2^1(\mathbb{X}_1,\mathbb{Y})$ satisfies the following conditions:

- **(i):** The function $F(t,\cdot):\mathbb{Y}\to\mathscr{L}_2^1(\mathbb{X}_1,\mathbb{Y})$ is continuous for a.e. $t\in J$ and for all $z\in\mathbb{Y}$, the function $t\to F(t,z)$ is strongly measurable.
- **(ii):** There exists a continuous function $\nu\in L^1(J,\mathbb{R}_+)$ such that

$$\mathbb{E}\|F(t,z)\|^2_{\mathscr{L}_2^1}\leq \nu(t)\mathbb{E}\|z\|^2$$

for all $(t,z)\in J\times\mathbb{Y}$.

(H6): The function $G : J \to \mathscr{L}_2^2(\mathbb{X}_2, \mathbb{Y})$ fulfills $\int_0^t \|G(s)\|^2_{\mathscr{L}_2^2} ds < \infty$ for every $t \in J$ and there exists a constant $\eta > 0$ such that $\|G(t)\|^2_{\mathscr{L}_2^2} \leq \eta$ uniformly in J.

(H7): The following inequalities hold:
(i): $\theta_1 = \max_{1\leq k\leq m} [\gamma_k, M^2\gamma_k] < 1$.
(ii): $\theta_2 = \max_{1\leq k\leq m} \{\delta_{I_k} + 4M^2\delta_{I_k} + 4M^2 \int_0^b \nu(s)ds\} < 1$.

Theorem 3.1. *Assume that the assumptions **(H1)-(H7)** hold. Then for each $u \in \mathscr{U}_{ad}$, the stochastic system* (1) *has at least one mild solution on* $[0,b]$.

Proof. For a constant $\omega > 0$, we define

$$\mathscr{L}_\omega = \{z \in \mathscr{PC}(\mathbb{Y}) : \|z\|^2_{\mathscr{PC}} \leq \omega\}.$$

Clearly, $\mathscr{L}_\omega$ is convex bounded and closed subset of $\mathscr{PC}(\mathbb{Y})$. Define the operator Γ : $\mathscr{L}_\omega \to \mathscr{PC}(\mathbb{Y})$ by

$$(\Gamma z)(t) = \begin{cases} \mathscr{R}(t)z_0 + \int_0^t \mathscr{R}(t-s)C(s)u(s)ds \\ + \int_0^t \mathscr{R}(t-s)F(s,z(s))dW(s) + \int_0^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t \in [0,t_1],\ k=0, \\ I_k(t, z(t_k^-)), & t \in (t_k, e_k],\ k \geq 1, \\ \mathscr{R}(t-e_k)I_k(e_k, z(t_k^-)) + \int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds \\ + \int_{e_k}^t \mathscr{R}(t-s)F(s,z(s))dW(s) + \int_{e_k}^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t \in (e_k, t_{k+1}],\ k \geq 1. \end{cases}$$

Now, we decompose Γ as $\Gamma_1 + \Gamma_2$, where

$$(\Gamma_1 z)(t) = \begin{cases} \mathscr{R}(t)z_0, & t \in [0,t_1],\ k=0, \\ I_k(t, z(t_k^-)), & t \in (t_k, e_k],\ k \geq 1, \\ \mathscr{R}(t-e_k)I_k(e_k, z(t_k^-)), & t \in (e_k, t_{k+1}],\ k \geq 1, \end{cases}$$

and

$$(\Gamma_2 z)(t) = \begin{cases} \int_0^t \mathscr{R}(t-s)C(s)u(s)ds + \int_0^t \mathscr{R}(t-s)F(s,z(s))dW(s) \\ + \int_0^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t \in [0,t_1],\ k=0, \\ 0, & t \in (t_k, e_k],\ k \geq 1, \\ \int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds + \int_{e_k}^t \mathscr{R}(t-s)F(s,z(s))dW(s) \\ + \int_{e_k}^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t \in (e_k, t_{k+1}],\ k \geq 1. \end{cases}$$

Now we proceed as follows:

Step 1. There exists $\omega > 0$ such that $\Gamma(\mathscr{L}_\omega) \subset \mathscr{L}_\omega$.
Suppose on the contrary that this is not true. We can choose $z^\omega \in \mathscr{L}_\omega$ and $t \in J$ such that $\mathbb{E}\|\Gamma(z^\omega)(t)\|^2 > \omega$. Using Hölder's inequality, **(H3)**, **(H5)**, **(H6)**, Lemma 2.1 and Lemma 2.2, we have for $t \in [0,t_1]$,

$$\begin{aligned}
\omega &< \mathbb{E}\|\Gamma(z^\omega)(t)\|^2 \\
&\leq 4\mathbb{E}\|\mathscr{R}(t)z_0\|^2 + 4\mathbb{E}\left\|\int_0^t \mathscr{R}(t-s)C(s)u(s)ds\right\|^2 \\
&\quad +4\mathbb{E}\left\|\int_0^t \mathscr{R}(t-s)F(s,z^\omega(s))dW(s)\right\|^2 + 4\mathbb{E}\left\|\int_0^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2 \\
&\leq 4M^2\mathbb{E}\|z_0\|^2 + 4\mathbb{E}\left[\int_0^t \|\mathscr{R}(t-s)\|\|C(s)u(s)\|ds\right]^2 \\
&\quad +4M^2\int_0^t \mathbb{E}\|F(s,z^\omega(s))\|^2_{\mathscr{L}_2^1}ds + 8c_{\mathsf{H}}M^2t_1^{2\mathsf{H}-1}\int_0^t \|G(s)\|^2_{\mathscr{L}_2^2}ds \\
&\leq 4M^2\mathbb{E}\|z_0\|^2 + 4M^2t_1\|C\|_\infty\|u\|_{L^2_{\mathscr{F}}} \\
&\quad +4M^2\int_0^t \nu(s)\mathbb{E}\|z^\omega\|^2ds + 8c_{\mathsf{H}}M^2t_1^{2\mathsf{H}-1}\int_0^t \eta ds \\
&\leq 4M^2\mathbb{E}\|z_0\|^2 + 4M^2t_1\|C\|_\infty\|u\|_{L^2_{\mathscr{F}}} + 4M^2\omega\int_0^t \nu(s)ds + 8c_{\mathsf{H}}M^2t_1^{2\mathsf{H}}\eta.
\end{aligned}$$

For $t \in]t_k,e_k],\ k = 1,2,\dots,m,$, we obtain

$$\begin{aligned}
\omega < \mathbb{E}\|\Gamma(z^\omega)(t)\|^2 = \mathbb{E}\|I_k(t,z^\omega(t_k^-))\|^2 &\leq \delta_{I_k}\left(1+\mathbb{E}\|z^\omega\|^2\right) \\
&\leq \delta_{I_k}(1+\omega).
\end{aligned}$$

Similarly, for $t \in (e_k,t_{k+1}],\ k = 1,2,\dots,m$, we obtain

$$\begin{aligned}
\omega < \mathbb{E}\|\Gamma(z^\omega)(t)\|^2 &\leq 4\mathbb{E}\|\mathscr{R}(t-e_k)I_k(e_k,z^\omega(t_k^-))\|^2 + 4\mathbb{E}\left\|\int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds\right\|^2 \\
&\quad +4\mathbb{E}\left\|\int_{e_k}^t \mathscr{R}(t-s)F(s,z^\omega(s))dW\right\|^2 + 4\mathbb{E}\left\|\int_{e_k}^t \mathscr{R}(t-s)G(s)dR_Q^H(s)\right\|^2 \\
&\leq 4M^2\mathbb{E}\|I_k(e_k,z^\omega(t_k^-))\|^2 + 4\mathbb{E}\left[\int_{e_k}^t \|\mathscr{R}(t-s)\|\|C(s)u(s)\|ds\right]^2 \\
&\quad +4M^2\int_{e_k}^t \mathbb{E}\|F(s,z^\omega(s))\|^2_{\mathscr{L}_2^1}ds + 8c_{\mathsf{H}}M^2t_1^{2\mathsf{H}-1}\int_{e_k}^t \|G(s)\|^2_{\mathscr{L}_2^2}ds \\
&\leq 4M^2\delta_{I_k}(1+\mathbb{E}\|(z^\omega\|^2) + 4M^2t_{k+1}\|C\|_\infty\|u\|_{L^2_{\mathscr{F}}} \\
&\quad +4M^2\int_{e_k}^t \nu(s)\mathbb{E}\|z^\omega\|^2ds + 8c_{\mathsf{H}}M^2t_{k+1}^{2\mathsf{H}-1}\int_{e_k}^t \eta ds \\
&\leq 4M^2\delta_{I_k}(1+\omega) + 4M^2t_{k+1}\|C\|_\infty\|u\|_{L^2_{\mathscr{F}}} \\
&\quad +4M^2\omega\int_{e_k}^t \nu(s)ds + 8c_{\mathsf{H}}M^2t_{k+1}^{2\mathsf{H}}\eta.
\end{aligned}$$

For any $t \in [0,b]$, we obtain

$$\omega < \mathbb{E}\|\Gamma(z^{\omega})(t)\|^2 \leq D + \delta_{I_k}\omega + 4M^2\delta_{I_k}\omega + 4M^2\omega \int_0^b \nu(s)ds, \tag{13}$$

where

$$D = \max_{1\leq k\leq m}\left\{4M^2\mathbb{E}\|z_0\|^2\delta_{I_k} + 4M^2\delta_{I_k} + 4M^2 b + \|C\|_\infty^2\|u\|^2_{L^2_{\mathscr{F}}} + 8c_{\mathsf{H}}M^2 b^{2\mathsf{H}}\eta\right\},$$

is independent on ω. Dividing both sides of (13) by ω and taking $\omega \to \infty$, we obtain

$$1 < \delta_{I_k} + 4M^2\delta_{I_k} + 4M^2\int_0^b \nu(s)ds,$$

which contradicts to **(H7)**. Hence, for some $\omega > 0$, $\Gamma(\mathscr{L}_\omega) \subset \mathscr{L}_\omega$.
Step 2. Γ_1 is a contraction map on $\mathscr{L}_\omega$.
Let $z_1, z_2 \in \mathscr{L}_\omega$. If $t \in [0,t_1]$. Then we have

$$\mathbb{E}\|(\Gamma_1 z_1)(t) - (\Gamma_1 z_2)(t)\|^2 = 0. \tag{14}$$

If $t \in (t_k, e_k],\ k = 1,2,\ldots,m$, then by **(H4)**, we have

$$\begin{aligned}\mathbb{E}\|(\Gamma_1 z_1)(t) - (\Gamma_1 z_2)(t)\|^2 &= \mathbb{E}\|I_k(t, z_1(t_j^-)) - I_k(t, z_2(t_j^-))\|^2\\ &\leq \gamma_k\|z_1 - z_2\|^2_{\mathscr{P}\mathscr{C}}.\end{aligned} \tag{15}$$

Likewise, if $t \in (e_k, t_{k+1}],\ k = 1,2,\ldots,m$, then we have

$$\begin{aligned}\mathbb{E}\|(\Gamma_1 z_1)(t) - (\Gamma_1 z_2)(t)\|^2 &= \mathbb{E}\|\mathscr{R}(t-e_k)[I_k(t, z_1(t_j^-)) - I_k(t, z_2(t_j^-))]\|^2\\ &\leq M^2\gamma_k\|z_1 - z_2\|^2_{\mathscr{P}\mathscr{C}}.\end{aligned} \tag{16}$$

From (14) to (16), we obtain

$$\|\Gamma_1 z_1 - \Gamma_1 z_2\|^2_{\mathscr{P}\mathscr{C}} \leq \theta_1\|z_1 - z_2\|^2_{\mathscr{P}\mathscr{C}},$$

where $\theta_1 = \max_{1\leq k\leq m}\{\gamma_k, M^2\gamma_k\}$. By **(H7)**, we see that $\theta_1 < 1$. Hence, Γ_1 is a contraction map.
Step 3. Γ_2 is continuous on $\mathscr{L}_\omega$.
Let $(z^n)_{n\geq 1}$ be a sequence such that $z^n \underset{n\to\infty}{\overset{\mathscr{L}_\omega}{\to}} \hat{z}$. By hypothesis **(H5)**, we obtain

$$F(s, z^n(s)) \underset{n\to\infty}{\to} F(s, \hat{z}),$$

for any $s \in [0,t]$ and since

$$\begin{aligned}\mathbb{E}\|F(s, z^n(s)) - F(s, \hat{z})\|^2_{\mathscr{L}_2^1} &\leq \nu(s)\mathbb{E}\|z^n\|^2 + \nu(s)\mathbb{E}\|\hat{z}\|^2\\ &\leq 2\omega\nu(s).\end{aligned}$$

For any $t \in (e_k, t_{k+1}],\ k = 0,1,\ldots,m$, we obtain

$$\begin{aligned}\mathbb{E}\|(\Gamma_2 z^n)(t) - (\Gamma_2\hat{z})(t)\|^2 &\leq \mathbb{E}\left\|\int_{e_k}^t \mathscr{R}(t-s)[F(s, z^n(s)) - F(s, \hat{z}(s))]dW(s)\right\|^2 ds\\ &\leq M^2\int_{e_k}^t \mathbb{E}\|F(s, z^n(s)) - F(s, \hat{z}(s))\|^2_{\mathscr{L}_2^1} ds.\end{aligned}$$

The Lebesgue dominated convergence theorem allows us to derive

$$\|\Gamma_2 z^n - \Gamma_2\hat{z}\|^2_{\mathscr{P}\mathscr{C}} \to 0 \text{ as } n \to \infty.$$

Consequently, Γ_2 is continuous on $\mathscr{L}_\omega$.

Step 4. $\{\Gamma_2 z : z \in \mathscr{L}_\omega\}$ is equicontinuous.
Let $z \in \mathscr{L}_\omega$ and $\tau_1, \tau_2 \in (e_k, t_{k+1}]$. Then, if $e_k < \tau_1 < \tau_2 \leq t_{k+1},\ k = 0,1,2,\ldots,m$, we obtain

$$\begin{aligned}
&\mathbb{E}\|(\Gamma_2 z)(\tau_2) - (\Gamma_2 z)(\tau_1)\|^2 \\
&\leq 6\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} \mathscr{R}(\tau_2 - s)C(s)u(s)ds \right\|^2 + 6\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} \mathscr{R}(\tau_2 - s)F(s,z(s))dW(s) \right\|^2 \\
&+ 6\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} \mathscr{R}(\tau_2 - s)G(s)dR_Q^{\mathsf{H}}(s) \right\|^2 + 6\mathbb{E}\left\| \int_{e_k}^{\tau_1} [\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)]C(s)u(s)ds \right\|^2 \\
&+ 6\mathbb{E}\left\| \int_{e_k}^{\tau_1} [\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)]F(s,z(s))dW(s) \right\|^2 \\
&+ 6\mathbb{E}\left\| \int_{e_k}^{\tau_1} [\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)]G(s)dR_Q^{\mathsf{H}}(s) \right\|^2 \\
&:= C_1 + C_2 + C_3 + C_4 + C_5 + C_6.
\end{aligned} \tag{17}$$

By Hölder's inequality, we have,

$$C_1 \leq 6M^2(\tau_2 - \tau_1)\|C\|_\infty^2 \mathbb{E}\int_{\tau_1}^{\tau_2} \|u(s)\|_{\mathscr{K}}^2 ds \to 0 \text{ as } \tau_2 \to \tau_1. \tag{18}$$

Using Lemma 2.2 and **(H5)**, we have

$$\begin{aligned}
C_2 &\leq 6\int_{\tau_1}^{\tau_2} \|\mathscr{R}(\tau_2 - s)\|^2 \mathbb{E}\|F(s,z(s))\|_{\mathscr{L}_2^1}^2 ds \\
&\leq 6M^2 \int_{\tau_1}^{\tau_2} \nu(s)\mathbb{E}\|z\|^2 ds \\
&\leq 6M^2\omega \int_{\tau_1}^{\tau_2} \nu(s)ds \to 0 \text{ as } \tau_2 \to \tau_1.
\end{aligned} \tag{19}$$

For C_3, by Lemma 2.1 and **(H6)**, we have

$$\begin{aligned}
C_3 &\leq 12c_{\mathsf{H}} M^2(\tau_2 - \tau_1)^{2\mathsf{H}-1} \int_{\tau_1}^{\tau_2} \|G(s)\|_{\mathscr{L}_2^2}^2 ds \\
&\leq 12c_{\mathsf{H}} M^2(\tau_2 - \tau_1)^{2\mathsf{H}}\eta \to 0 \text{ as } \tau_2 \to \tau_1.
\end{aligned} \tag{20}$$

For the term C_4, we have by Hölder's inequality

$$C_4 \leq 6(\tau_2 - \tau_1)\|C\|_\infty^2 \mathbb{E}\int_{e_k}^{\tau_1} \|\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)\|^2 \|u(s)\|_{\mathscr{K}}^2 ds. \tag{21}$$

Since $\mathscr{R}(t)$ is continuous in the uniform operator topology, then $C_4 \to 0$ as $\tau_2 \to \tau_1$.
On the other hand, in view of Lemma 2.2 and **(H5)**, we have

$$\begin{aligned}
C_5 &\leq 6\int_{e_k}^{\tau_1} \|\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)\|^2 \mathbb{E}\|F(s,z(s))\|_{\mathscr{L}_2^1}^2 ds \\
&\leq 6\int_{e_k}^{\tau_1} \|\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)\|^2 \nu(s)\mathbb{E}\|z\|^2 ds \\
&\leq 6\omega \int_{e_k}^{\tau_1} \|\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)\|^2 \nu(s)ds.
\end{aligned} \tag{22}$$

Due to the continuity in the uniform operator topology of $\mathscr{R}(t)$, we obtain that $C_5 \to 0$ as $\tau_2 \to \tau_1$.

Using Lemma 2.1, **(H6)** and the continuity in the uniform operator topology of $\mathscr{R}(t)$, we obtain

$$\begin{aligned} C_6 &\leq 12c_{\mathsf{H}}(\tau_1-e_k)^{2\mathsf{H}-1}\int_{e_k}^{\tau_1}\|\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)\|^2\|G(s)\|^2_{\mathscr{L}_2^2}ds \\ &\leq 12c_{\mathsf{H}}(\tau_1-e_k)^{2\mathsf{H}-1}\eta\int_{e_k}^{\tau_1}\|\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)\|^2 \to 0 \text{ as } \tau_2\to\tau_1. \quad (23)\end{aligned}$$

Hence, the right-hand of (17) converges to 0 as $\tau_2\to\tau_1$. Thus, $\{\Gamma_2 z : z\in\mathscr{L}_\omega\}$ is equicontinuous. Also, it is clear that $\{\Gamma_2 z : z\in\mathscr{L}_\omega\}$ is bounded.

Step 5. $\mathscr{V}(t)=\{(\Gamma_2 z)(t) : z\in\mathscr{L}_\omega\}$ is relatively compact in $\mathbb{Y}$.

Let $0<s\leq t\leq t_1$ be fixed and let h be a real number satisfying $0<h<t$. For $z\in\mathscr{L}_\omega$, we define the operators

$$\begin{aligned}(\Gamma_2^h z)(t) = \mathscr{R}(t)z_0 + \int_0^{t-h}\mathscr{R}(t-s)C(s)u(s)ds + \int_0^{t-h}\mathscr{R}(t-s)F(s,z(s))dW(s) \\ +\int_0^{t-h}\mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)\end{aligned}$$

and

$$\begin{aligned}\left(\Gamma_2^{*h}z\right)(t) = \mathscr{R}(t)z_0 + \mathscr{R}(h)\int_0^{t-h}\mathscr{R}(t-s-h)C(s)u(s)ds + \mathscr{R}(h)\int_0^{t-h}\mathscr{R}(t-s-h)F(s,z(s))dW(s) \\ +\mathscr{R}(h)\int_0^{t-h}\mathscr{R}(t-s-h)G(s)dR_Q^{\mathsf{H}}(s).\end{aligned}$$

By Lemma 2.9 and using the compactness of $\left(\mathscr{R}(h)\right)_{h>0}$, we deduce that the set $\tilde{\mathscr{V}}(t)=\{(\Gamma_2^h z)(t) : z\in\mathscr{L}_\omega\}$ is precompact in $\mathbb{Y}$ for every h, $0<h<t$. Moreover, by Lemma 2.9 and Hölder's inequality, for every $z\in\mathscr{L}_\omega$, we have:

$$\begin{aligned}&\mathbb{E}\|(\Gamma_2^h z)(t)-\left(\Gamma_2^{*h}z\right)(t)\|^2 \\ &\leq 3\mathbb{E}\left\|\mathscr{R}(h)\int_0^{t-h}\mathscr{R}(t-s-h)C(s)u(s)ds-\int_0^{t-h}\mathscr{R}(t-s)C(s)u(s)ds\right\|^2 \\ &\quad+3\mathbb{E}\left\|\mathscr{R}(h)\int_0^{t-h}\mathscr{R}(t-s-h)F(s,z(s))dW(s)-\int_0^{t-h}\mathscr{R}(t-s)F(s,z(s))dW(s)\right\|^2 \\ &\quad+3\mathbb{E}\left\|\mathscr{R}(h)\int_0^{t-h}\mathscr{R}(t-s-h)G(s)dR_Q^{\mathsf{H}}(s)-\int_0^{t-h}\mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2 \\ &\leq 3\mathbb{E}\left\|\int_0^{t-h}\left[\mathscr{R}(h)\mathscr{R}(t-s-h)-\mathscr{R}(t-s)\right]C(s)u(s)ds\right\|^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 3\mathbb{E}\left\| \int_0^{t-h} \left[\mathscr{R}(h)\mathscr{R}(t-s-h) - \mathscr{R}(t-s)\right] F(s,z(s))dW(s) \right\|^2 \\
&\quad + 3\mathbb{E}\left\| \int_0^{t-h} \left[\mathscr{R}(h)\mathscr{R}(t-s-h) - \mathscr{R}(t-s)\right] G(s)dR_Q^{\mathsf{H}}(s) \right\|^2 \\
&\le 3\mathbb{E}\int_0^{t-h} \left\|\mathscr{R}(h)\mathscr{R}(t-s-h) - \mathscr{R}(t-s)\right\|^2 \left\|C(s)u(s)\right\|^2 ds \\
&\quad + 3\mathbb{E}\int_0^{t-h} \left\|\mathscr{R}(h)\mathscr{R}(t-s-h) - \mathscr{R}(t-s)\right\|^2 \left\|F(s,z(s))\right\|^2 dW(s) \\
&\quad + 3\mathbb{E}\int_0^{t-h} \left\|\mathscr{R}(h)\mathscr{R}(t-s-h) - \mathscr{R}(t-s)\right\|^2 \left\|G(s)\right\|^2 dR_Q^{\mathsf{H}}(s) \\
&\le 3L(h)^2\mathbb{E}\int_0^{t-h} \left\|C(s)u(s)\right\|^2 ds + 3L(h)^2\mathbb{E}\int_0^{t-h} \left\|F(s,z(s))\right\|^2 dW(s) \\
&\quad + 3L(h)^2\mathbb{E}\int_0^{t-h} \left\|G(s)\right\|^2 dR_Q^{\mathsf{H}}(s) \\
&\le 3L(h)^2 h\|C\|_\infty^2 \mathbb{E}\int_0^{t-h} \|u(s)\|_{\mathscr{K}}^2 ds + 3L(h)^2\omega\int_0^{t-h} \nu(s)ds + 3L(h)^2 c_{\mathsf{H}} b^{2\mathsf{H}-1}\mathbb{E}\int_0^{t-h} \left\|G(s)\right\|_{\mathscr{L}_2^0}^2 ds \\
&\le 3L(h)^2\left[h\|C\|_\infty^2 \mathbb{E}\int_0^{t-h} \|u(s)\|_{\mathscr{K}}^2 ds + \omega\int_0^{t-h} \nu(s)ds + c_{\mathsf{H}} b^{2\mathsf{H}-1}\mathbb{E}\int_0^{t-h} \left\|G(s)\right\|_{\mathscr{L}_2^0}^2 ds\right] \\
&\quad \underset{h\longrightarrow 0}{\longrightarrow} 0.
\end{aligned}
$$

So the set $\tilde{\mathscr{V}}(t) = \{(\Gamma_2^h z)(t) : z \in \mathscr{L}_\omega\}$ is precompact in $\mathbb{Y}$ by using the total boundedness. Using this idea again, we obtain

$$
\begin{aligned}
&\mathbb{E}\|(\Gamma_2 z)(t) - (\Gamma_2^h z)(t)\|^2 \\
&\le 3\mathbb{E}\left\| \int_{e_k}^{t} \mathscr{R}(t-s)C(s)u(s)ds - \int_{e_k}^{t-h} \mathscr{R}(t-s)C(s)u(s)ds \right\|^2 \\
&\quad + 3\mathbb{E}\left\| \int_0^{t} \mathscr{R}(t-s)F(s,z(s))dW(s) - \int_0^{t-h} \mathscr{R}(t-s)F(s,z(s))dW(s) \right\|^2 \\
&\quad + 3\mathbb{E}\left\| \int_0^{t} \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s) - \int_0^{t-h} \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s) \right\|^2 \\
&\le 3M^2\left[h\|C\|_\infty^2 \mathbb{E}\int_{t-h}^{t} \|u(s)\|_{\mathscr{K}}^2 ds + \omega\int_{t-h}^{t} \nu(s)ds + 2c_{\mathsf{H}}\eta h^{2\mathsf{H}}\right] \to 0 \text{ as } h \to 0.
\end{aligned}
$$

Similarly, for any $t \in (e_k, t_{k+1}]$ with $k = 1, \cdots, N$. Let $e_k < t \le s \le t_{k+1}$ be fixed and let h be a real number satisfying $0 < h < t$. For $z \in \mathscr{L}_\omega$, we define the operators

$$
\begin{aligned}
(\tilde{\Gamma}_2^h z)(t) = &\int_{e_k}^{t-h} \mathscr{R}(t-s)C(s)u(s)ds + \int_{e_k}^{t-h} \mathscr{R}(t-s)F(s,z(s))dW(s) \\
&+ \int_{e_k}^{t-h} \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\tilde{\Gamma}_2^{*h} z\right)(t) = &\mathscr{R}(h)\int_{e_k}^{t-h} \mathscr{R}(t-s-h)C(s)u(s)ds + \mathscr{R}(h)\int_{e_k}^{t-h} \mathscr{R}(t-s-h)F(s,z(s))dW(s) \\
&+\mathscr{R}(h)\int_{e_k}^{t-h} \mathscr{R}(t-s-h)G(s)dR_Q^{\mathsf{H}}(s).
\end{aligned}
$$

If we using the Lemma 2.9 and compactness of $\big(\mathscr{R}(h)\big)_{h>0}$, we deduce that the set $\tilde{\mathscr{V}}'(t)$ is precompact in $\mathbb{Y}$ for every h, $0<h<t$. Moreover, by Lemma 2.9 and Hölder's inequality, for every $z\in\mathscr{L}_\omega$ we have:

$$\begin{aligned}
&\mathbb{E}\|(\tilde{\Gamma}_2^h z)(t)-\big(\tilde{\Gamma}_2^{*h}z\big)(t)\|^2\\
&\le 3\mathbb{E}\left\|\mathscr{R}(h)\int_{e_k}^{t-h}\mathscr{R}(t-s-h)C(s)u(s)ds-\int_{e_k}^{t-h}\mathscr{R}(t-s)C(s)u(s)ds\right\|^2\\
&\quad+3\mathbb{E}\left\|\mathscr{R}(h)\int_{e_k}^{t-h}\mathscr{R}(t-s-h)F(s,z(s))dW(s)-\int_{e_k}^{t-h}\mathscr{R}(t-s)F(s,z(s))dW(s)\right\|^2\\
&\quad+3\mathbb{E}\left\|\mathscr{R}(h)\int_{e_k}^{t-h}\mathscr{R}(t-s-h)G(s)dR_Q^{\mathsf{H}}(s)-\int_{e_k}^{t-h}\mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2\\
&\le 3\big(L(h)\big)^2\left[h\|C\|_\infty^2\mathbb{E}\int_0^{t-h}\|u(s)\|_{\mathscr{K}}^2ds+\omega\int_{e_k}^{t-h}\nu(s)ds+c_{\mathsf{H}}b^{2\mathsf{H}-1}\mathbb{E}\int_{e_k}^{t-h}\big\|G(s)\big\|_{\mathscr{L}_2^0}^2ds\right]\\
&\qquad\underset{\varepsilon\longrightarrow 0}{\longrightarrow}0.
\end{aligned}$$

So the set $\tilde{\mathscr{V}}'(t)=\{(\tilde{\Gamma}_2^h z)(t):z\in\mathscr{L}_\omega\}$ is precompact in $\mathbb{Y}$ by using the total boundedness. Using this idea again, we obtain

$$\begin{aligned}
&\mathbb{E}\|(\Gamma_2 z)(t)-(\tilde{\Gamma}_2^h z)(t)\|^2\\
&\le 3\mathbb{E}\left\|\int_{t-h}^{t}\mathscr{R}(t-s)C(s)u(s)ds\right\|^2+3\mathbb{E}\left\|\int_{t-h}^{t}\mathscr{R}(t-s)F(s,z(s))dW(s)\right\|^2\\
&\quad+3\mathbb{E}\left\|\int_{t-h}^{t}\mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2\\
&\le 3M^2\left[h\|C\|_\infty^2\mathbb{E}\int_{t-h}^{t}\|u(s)\|_{\mathscr{K}}^2ds+\omega\int_{t-h}^{t}\nu(s)ds+2c_{\mathsf{H}}\eta h^{2\mathsf{H}}\right]\to 0 \text{ as } h\to 0.
\end{aligned}$$

Therefore, as $h\longrightarrow 0$, there are precompact sets arbitrarily close to the set $\mathscr{V}(t)=\{(\Gamma_2 z)(t):z\in\mathscr{L}_\omega\}$ is relatively compact in $\mathbb{Y}$. It is easy to see that the set $\mathscr{L}_\omega$ is uniformly bounded. By using the steps 3 to 5 and the Arzela-Ascoli theorem, we get that the operator Γ_2 is completely continuous. Consequently, by Krasnoselskii's fixed point Theorem (see [29]), we see that the operator Γ has at least one fixed point on $\mathscr{L}_\omega$, which is a mild solution of stochastic system (1).

4. Existence of a stochastic optimal control

This section is devoted to the exploration of the existence of optimal state-control pairs of the Lagrange problem corresponding to the stochastic system(1).

Let $\mathscr{U}_z(u)$ be the set of all solutions of the stochastic system (1) and z^u denote the mild solution of the stochastic system (1) with respect to $u\in\mathscr{U}_{ad}$. We consider the following Lagrange problem $(\mathscr{L}\mathscr{P})$

$$(\mathscr{L}\mathscr{P})\begin{cases}\text{Find an optimal pair } (z^0,u^0)\in\mathscr{P}\mathscr{C}\times\mathscr{U}_{ad} \text{ such that}\\ \mathscr{J}(z^0,u^0)\le\mathscr{J}(z^u,u),\ \text{for all } u\in\mathscr{U}_{ad},\end{cases}$$

where

$$\mathscr{J}(z^u,u)=\mathbb{E}\int_0^b\mathscr{T}(t,z^u(t),u(t))dt.$$

The following hypotheses are imposed in order to discuss the Lagrange problem $(\mathscr{L}\mathscr{P})$.

(H8): The Borel measurable function $\mathscr{T} : J \times \mathbb{Y} \times \mathscr{K} \to \mathbb{R} \cup \{\infty\}$ satisfies

- **(i):** For almost all $t \in J$, $\mathscr{T}(t,z,\cdot)$ is convex function on $\mathscr{K}$ for each $z \in \mathbb{Y}$.
- **(ii):** For almost all $t \in J$, $\mathscr{T}(t,\cdot,\cdot)$ is sequentially lower semi-continuous on $\mathbb{Y} \times \mathscr{K}$.
- **(iii):** There exist constants $\tau_1 \geq 0, \tau_2 > 0$ and Υ is nonnegative function in $L^1(J,\mathbb{R})$ such that

$$\mathscr{T}(t,z,u) \geq \Upsilon(t) + \tau_1\|z\| + \tau_2\|u\|^2_{\mathscr{K}}.$$

Denote

$$\Lambda = \max_{1\leq k\leq m}\left[(n_1+n_{10})\exp(n_2),\ \delta_{I_k}/(1-\delta_{I_k}),\ (n_{2k}+n_{1k})(1+d_\varepsilon)^k\exp(n_2)\right],$$

where $n_1 = 4M^2\mathbb{E}\|z_0\|^2$, $n_{1k} = 4M^2 t_{k+1}\|C\|^2_\infty\|u\|^2_{L^2_{\mathscr{F}}} + 8c_{\mathsf{H}}M^2 t_{k+1}^{2\mathsf{H}}\eta$, $n_2 = 4M^2\|\nu\|_{L^1}$, $n_{2k} = 4M^2\delta_{I_k}$,
$d_\varepsilon = \sup_{1\leq k\leq m}\{4M^2\delta_{I_k}\}$, and $\delta_{I_k} < 1$.

Lemma 4.1. *Assume that* ***(H1)-(H7)*** *hold. Then, for given $u \in \mathscr{U}_{ad}$, there exits a constant $\Lambda > 0$ such that $\|z\|^2_{\mathscr{PC}} \leq \Lambda$ for any mild solution z of the stochastic system* (1).

Proof. Let z be a mild solution of stochastic system (1) with respect to $u \in \mathscr{U}_{ad}$ on $[0,b]$. We split the proof into several cases.
Case 1. For any $t \in [0,t_1]$, we have

$$\begin{aligned}
\mathbb{E}\|z(t)\|^2 &\leq 4\mathbb{E}\|\mathscr{R}(t)z_0\|^2 + 4\mathbb{E}\left\|\int_0^t \mathscr{R}(t-s)C(s)u(s)ds\right\|^2\\
&\quad +4\mathbb{E}\left\|\int_0^t \mathscr{R}(t-s)F(s,z(s))dW(s)\right\|^2 + 4\mathbb{E}\left\|\int_0^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2\\
&\leq 4M^2\mathbb{E}\|z_0\|^2 + 4M^2t_1\|C\|^2_\infty\|u\|^2_{L^2_{\mathscr{F}}}\\
&\quad +4M^2\int_0^t \mathbb{E}\|F(s,z(s))\|^2_{\mathscr{L}_2^1}ds + 8M^2c_{\mathsf{H}}t_1^{2\mathsf{H}-1}\int_0^t\|G(s)\|^2_{\mathscr{L}_2^2}ds\\
&\leq 4M^2\mathbb{E}\|z_0\|^2 + 4M^2t_1\|C\|^2_\infty\|u\|^2_{L^2_{\mathscr{F}}}\\
&\quad +8M^2c_{\mathsf{H}}t_1^{2\mathsf{H}}\eta + 4M^2\int_0^t \nu(s)\mathbb{E}\|z(s)\|^2 ds.
\end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned}
\mathbb{E}\|z(t)\|^2 &\leq \max_{1\leq k\leq m}\left(4M^2\mathbb{E}\|z_0\|^2 + 4M^2t_1\|C\|^2_\infty\|u\|^2_{L^2_{\mathscr{F}}} + 8M^2c_{\mathsf{H}}t_1^{2\mathsf{H}}\eta\right)\\
&\quad \times\exp[4M^2\|\nu\|_{L^1}] = \Lambda.
\end{aligned}$$

Case 2. For any $t \in (t_k, e_k],\ k = 1,2,\ldots,m$, we have

$$\mathbb{E}\|z(t)\|^2 = \mathbb{E}\|I_k(t,z(t_k^-))\|^2 \leq \delta_{I_k}(1+\mathbb{E}\|z(t_k^-)\|^2),$$

which implies that

$$\|z\|^2_{\mathscr{PC}} \leq \frac{\delta_{I_k}}{1-\delta_{I_k}} \leq \Lambda.$$

Case 3. For any $t \in (e_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, using Lemma 2.1, Lemma 2.2, **(H4)**, **(H5)**, **(H6)** and Hölder's inequality, we obtain

$$\begin{aligned}
\mathbb{E}\|z(t)\|^2 &\leq 4\mathbb{E}\|\mathscr{R}(t-e_k)I_k(e_k, z(t_k^-))\|^2 + 4\mathbb{E}\left\|\int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds\right\|^2 \\
&\quad +4\mathbb{E}\left\|\int_{e_k}^t \mathscr{R}(t-s)F(s,z(s))dW(s)\right\|^2 \\
&\quad +4\mathbb{E}\left\|\int_{e_k}^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2 \\
&\leq 4M^2\mathbb{E}\|I_k(e_k, z(t_k^-))\|^2 + 4\mathbb{E}\left[\int_{e_k}^t \|\mathscr{R}(t-s)C(s)u(s)\|ds\right]^2 \\
&\quad +4M^2\int_{e_k}^t \mathbb{E}\|F(s,z(s))\|_{\mathscr{L}_2^1}^2 ds \\
&\quad +8M^2\mathsf{H}t_{k+1}^{2\mathsf{H}-1}\int_{e_k}^t \|G(s)\|_{\mathscr{L}_2^2}^2 ds \\
&\leq 4\delta_{I_k}(1+\mathbb{E}\|z(t_k^-)\|^2) + 4M^2 t_{k+1}\|C\|_\infty^2\|u\|_{L_{\mathscr{F}}^2}^2 \\
&\quad +8M^2\mathsf{H}t_{k+1}^{2\mathsf{H}}\eta + 4M^2\int_{e_k}^t \nu(s)\mathbb{E}\|z(s)\|^2 ds. \\
&\quad +4\sum_{i=1}^k 4M^2\delta_{I_i}\mathbb{E}\|z(t_i^-)\|^2.
\end{aligned}$$

Applying impulsive Gronwall's inequality given by Theorem 2.4, we obtain

$$\begin{aligned}
\mathbb{E}\|z(t)\|^2 &\leq \left(4M^2\delta_{I_k} + 4M^2 t_{k+1}\|C\|_\infty^2\|u\|_{L_{\mathscr{F}}^2}^2 + 8M^2 c_{\mathsf{H}} t_{k+1}^{2\mathsf{H}}\eta\right) \\
&\quad \times(1+d_\varepsilon)^k \exp[4M^2\|\nu\|_{L^1}] \leq \Lambda,
\end{aligned}$$

where $d_\varepsilon = \sup_{1\leq i\leq m}(4M^2\delta_{I_i})$.
From the above, we obtain

$$\|z\|_{\mathscr{PC}}^2 \leq \Lambda.$$

We have the following result:

Lemma 4.2. *The operator $\Pi : L_{\mathscr{F}}^2(J, \mathscr{K}) \to \mathscr{PC}(\mathbb{Y})$ given by*

$$(\Pi u)(\cdot) = \begin{cases} \int_0^t \mathscr{R}(t-s)C(s)u(s)ds & t \in [0, t_1], \\ 0, & t \in (t_k, e_k],\ k \geq 1, \\ \int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds, & t \in (e_k, t_{k+1}],\ k \geq 1, \end{cases}$$

is completely continuous.

Proof. Suppose $\{u^n(\cdot)\} \subseteq L^p([0,t_1], \mathbb{Y})$ is bounded, *i.e., there exists $\delta > 0$ such that*

$$\|u\|_{L^p([0,t_1],\mathbb{Y})}^p \leq \delta \qquad \forall\, n \in \mathbb{N}.$$

We define $\Theta_n(t) = \big(\Pi u^n\big)(t)$, $t \in [0,b]$ i.e.

$$\Theta_n(t) = \int_0^t \mathscr{R}(t-s)C(s)u^n(s)ds.$$

By virtue of Holder inequality one can verify that for ant fixed $t \in [0,b]$, $\|\Theta_n(t)\|$ *is bounded. Similar to the proof of Theorem 3.1 it is easy to verify that* $\Theta_n(t)$ *is relatively compact in* $\mathbb{Y}$ *and is also equicontinuous. Due to Arzela-Ascoli theorem again,* $\big(\Pi u\big)(t)$ *is compact in* $\mathbb{Y}$.

Obviously, Πu *is linear and continuous. Hence,* Πu *is a completely continuous operator.*

Theorem 4.3. *Assume that hypotheses **(H3)-(H8)** are satisfied. Then the problem* $(\mathscr{L}\mathscr{P})$ *admits at least one optimal control pair.*

Proof. Whithout loss of generality, we suppose that $\mathscr{J}(u) = \inf_{z^u \in \mathscr{U}_z(u)} \mathscr{J}(z^u,u) < +\infty$ for $u \in \mathscr{U}_{ad}$. Otherwise, there is nothing to prove. By **(H8)**, we have

$$\mathscr{J}(z^u,u) \geq \int_0^b \Upsilon(t)dt + \tau_1 \int_0^b \|z^u(t)\|dt + \int_0^b \tau_2 \|u(t)\|^2_{\mathscr{K}}dt \geq -b > -\infty,$$

where b is a nonnegative constant. Therefore $\mathscr{J}(u) \geq -b \geq -\infty$. For better readability, we break the proof into a sequence of steps.

Step 1. By using definition of infimum, there exists a minimizing sequence $\{z^u_n\} \subseteq \mathscr{U}_z(u)$ such that

$$\mathscr{J}(z^u_n,u) \to \mathscr{J}(u) \text{ as } n \to \infty.$$

Let $\big(z^u_n\big)_n$ denote the sequence of mild solutions of the stochastic system (1) corresponding to the control u

$$(z^u_n)(t) = \begin{cases} \mathscr{R}(t)z_0 + \displaystyle\int_0^t \mathscr{R}(t-s)C(s)u(s)ds \\ + \displaystyle\int_0^t \mathscr{R}(t-s)F(s,z^u_n(s))dW(s) + \int_0^t \mathscr{R}(t-s)G(s)dR^{\mathsf{H}}_Q(s), & t \in [0,t_1],\ k=0, \\ I_k(t,z^u_n(t_k^-)), & t \in (t_k,e_k],\ k \geq 1, \\ \mathscr{R}(t-e_k)I_k(e_k,z^u_n(t_k^-)) + \displaystyle\int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds \\ + \displaystyle\int_{e_k}^t \mathscr{R}(t-s)F(s,z^u_n(s))dW(s) + \int_{e_k}^t \mathscr{R}(t-s)G(s)dR^{\mathsf{H}}_Q(s), & t \in (e_k,t_{k+1}],\ k \geq 1. \end{cases}$$

Step 2. We show that there exists some $\tilde{z}^u \in \mathscr{U}_z(u)$ such that $\mathscr{J}(\tilde{z}^u,u) = \inf_{z^u \in \mathscr{U}_z(u)} \mathscr{J}(z^u,u) = \mathscr{J}(u)$. Therefore, we have to show that for each $u \in \mathscr{U}_{ad}$, the set $\{z^u_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathscr{P}\mathscr{C}(\mathbb{Y})$. Lemma 4.1 shows that the set $\{z^u_n\}$ is uniformely bounded. Now, we prove that $\{z^u_n\}$ is equicontinuous on J. To achieve our aim, we consider the following three cases:

Case 1. For $0 < \tau_1 < \tau_2 \leq t_1$, we have

$$\begin{aligned}
\mathbb{E}\|z(\tau_2)-z(\tau_1)\|^2 &\leq 7\|\mathscr{R}(\tau_2)-\mathscr{R}(\tau_1)\|^2\mathbb{E}\|z_0\|^2+7\mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\mathscr{R}(\tau_2-s)C(s)u(s)ds\right\|^2\\
&+ 7\mathbb{E}\left\|\int_0^{\tau_1}[\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)]C(s)u(s)ds\right\|^2\\
&+ 7\mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\mathscr{R}(\tau_2-s)F(s,z(s))dW(s)\right\|^2\\
&+ 7\mathbb{E}\left\|\int_0^{\tau_1}[\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)]F(s,z(s))dW(s)\right\|^2\\
&+ 7\mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\mathscr{R}(\tau_2-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2\\
&+ 7\mathbb{E}\left\|\int_0^{\tau_1}[\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)]G(s)dR_Q^{\mathsf{H}}(s)\right\|^2\\
&= 7[\|\mathscr{R}(\tau_2)-\mathscr{R}(\tau_1)\|^2\mathbb{E}\|z_0\|^2+N_1+N_2+N_3],
\end{aligned}$$

where

$$\begin{aligned}
N_1 &= \mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\mathscr{R}(\tau_2-s)C(s)u(s)ds\right\|^2\\
&+ \mathbb{E}\left\|\int_0^{\tau_1}[\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)]C(s)u(s)ds\right\|^2\\
N_2 &= \mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\mathscr{R}(\tau_2-s)F(s,z(s))dW(s)\right\|^2\\
&+ \mathbb{E}\left\|\int_0^{\tau_1}[\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)]F(s,z(s))dW(s)\right\|^2\\
N_3 &= \mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\mathscr{R}(\tau_2-s)G(s)dR_Q^{\mathsf{H}}(s)\right\|^2\\
&+ \mathbb{E}\left\|\int_0^{\tau_1}[\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)]G(s)dR_Q^{\mathsf{H}}(s)\right\|^2.
\end{aligned}$$

For N_1, we have

$$\begin{aligned}
N_1 &\leq M^2(\tau_2-\tau_1)\|C\|_\infty^2\mathbb{E}\int_{\tau_1}^{\tau_2}\|u(s)\|_{\mathscr{K}}^2ds\\
&+ \tau_1M^2\|C\|_\infty^2\mathbb{E}\int_0^{\tau_1}\|\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)\|^2\|u(s)\|_{\mathscr{K}}^2ds.
\end{aligned}$$

For N_2, by Lemma 2.2 and **(H4)**, we have

$$N_2 \leq M^2\Lambda\int_{\tau_1}^{\tau_2}\nu(s)ds+\Lambda\int_0^{\tau_1}\|\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)\|^2\nu(s)ds.$$

Using Lemma 2.1, **(H6)** and Hölder's inequality, we obtain

$$N_3 \leq 2c_{\mathsf{H}}M^2\eta(\tau_2-\tau_1)^{2\mathsf{H}}+2c_{\mathsf{H}}\tau_1^{2\mathsf{H}-1}\eta\int_0^{\tau_1}\|\mathscr{R}(\tau_2-s)-\mathscr{R}(\tau_1-s)\|^2ds.$$

Case 2. For $t_k < \tau_1 < \tau_2 \leq e_k,\ k = 1,2,\ldots,m$, we have

$$\mathbb{E}\|z(\tau_2) - z(\tau_1)\|^2 = \mathbb{E}\|I_k(\tau_2, z(t_k^-)) - I_k(\tau_1, z(t_k^-))\|^2$$

Case 3. For $e_k < \tau_1 < \tau_2 \leq t_{k+1},\ k = 1,2,\ldots,m$, we have

$$\begin{aligned}
\mathbb{E}\|z(\tau_2) - z(\tau_1)\|^2 &\leq 7\mathbb{E}\|\mathscr{R}(\tau_2 - e_k) - \mathscr{R}(\tau_1 - e_k)\|^2 \|I_k(e_k, z(t_k^-))\|^2 \\
&+ 7\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} \mathscr{R}(\tau_2 - s)C(s)u(s)ds \right\|^2 \\
&+ 7\mathbb{E}\left\| \int_{e_k}^{\tau_1} [\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)]C(s)u(s)ds \right\|^2 \\
&+ 7\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} \mathscr{R}(\tau_2 - s)F(s, z(s))dW(s) \right\|^2 \\
&+ 7\mathbb{E}\left\| \int_{e_k}^{\tau_1} [\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)]F(s, z(s))dW(s) \right\|^2 \\
&+ 7\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} \mathscr{R}(\tau_2 - s)G(s)dR_Q^{\mathsf{H}}(s) \right\|^2 \\
&+ 7\mathbb{E}\left\| \int_{e_k}^{\tau_1} [\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)]G(s)dR_Q^{\mathsf{H}}(s) \right\|^2 \\
&= 7[\|\mathscr{R}(\tau_2) - \mathscr{R}(\tau_1)\|^2 \mathbb{E}\|z_0\|^2 + N_1^{'} + N_2^{'} + N_3^{'}],
\end{aligned}$$

where

$$\begin{aligned}
N_1^{'} &\leq M^2(\tau_2 - \tau_1)\|C\|_\infty^2 \mathbb{E}\int_{\tau_1}^{\tau_2} \|u(s)\|_{\mathscr{K}}^2 ds \\
&+ \tau_1 M^2 \|C\|_\infty^2 \mathbb{E}\int_{e_k}^{\tau_1} \|\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)\|^2 \|u(s)\|_{\mathscr{K}}^2 ds, \\
N_2^{'} &\leq M^2 \Lambda \int_{\tau_1}^{\tau_2} \nu(s)ds + \Lambda \int_{e_k}^{\tau_1} \|\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)\|^2 \nu(s)ds, \\
N_3^{'} &\leq 2c_{\mathsf{H}} M^2 \eta (\tau_2 - \tau_1)^{2\mathsf{H}} + 2c_{\mathsf{H}} \tau_1^{2\mathsf{H}-1} \eta \int_0^{\tau_1} \|\mathscr{R}(\tau_2 - s) - \mathscr{R}(\tau_1 - s)\|^2 ds.
\end{aligned}$$

By using the hypothesis **(H3)** and the continuity of the functions I_k we see that the right-hand side of $N_j^{'},\ j = 1,2,3$ tends to zero as $\tau_2 \to \tau_1$. Thus, $\{z_n^u\}$ is equicontinuous on J. As in the proof of steps 4 and 5 in Theorem 3.1, we can see that $\{z_n^u\}$ is relatively compact on $\mathscr{PC}(\mathbb{Y})$. Consequently, there exists $\tilde{z}^u \in \mathscr{PC}(\mathbb{Y})$ such that $z_n^u \to \tilde{z}^u$ in $\mathscr{PC}(\mathbb{Y})$ for each $u \in \mathscr{U}_{ad}$.

$$(\tilde{z}^u)(t) = \begin{cases}
\mathscr{R}(t)z_0 + \int_0^t \mathscr{R}(t-s)C(s)u(s)ds \\
+ \int_0^t \mathscr{R}(t-s)F(s, \tilde{z}^u(s))dW(s) + \int_0^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t \in [0, t_1],\ k = 0, \\
I_k(t, \tilde{z}^u(t_k^-)), & t \in (t_k, e_k],\ k \geq 1, \\
\mathscr{R}(t - e_k)I_k(e_k, \tilde{z}^u(t_k^-)) + \int_{e_k}^t \mathscr{R}(t-s)C(s)u(s)ds \\
+ \int_{e_k}^t \mathscr{R}(t-s)F(s, \tilde{z}^u(s))dW(s) + \int_{e_k}^t \mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t \in (e_k, t_{k+1}],\ k \geq 1,
\end{cases}$$

which gives that $\tilde{z}^u \in \mathscr{U}_z(u)$. Now, we claim that $\mathscr{J}(\tilde{z}^u, u) = \inf_{\tilde{z}^u \in \mathscr{U}_z(u)} \mathscr{J}(\tilde{z}^u, u) = \mathscr{J}(u)$. Since $\mathscr{PC}(\mathbb{Y}) \subset L^1(J, \mathbb{Y})$, by using the Balder's theorem [9] and hypothesis **(H8)**, we obtain

$$\mathscr{J}(u)=\lim_{n\to\infty}\mathbb{E}\int_0^b \mathscr{T}(t,z_n^u(t),u(t)dt \geq \mathbb{E}\int_0^b \mathscr{T}(t,\tilde{z}^u(t),u(t)dt=\mathscr{J}(\tilde{z}^u,u)\geq \mathscr{J}(u),$$

which implies that $\mathscr{J}(u)$ attains its minimum at $\tilde{z}^u \in \mathscr{PC}(\mathbb{Y})$ for each $u\in\mathscr{U}_{ad}$.

Step 3. Next, we show that there exists $\hat{u}\in\mathscr{U}_{ad}$ such that

$$\mathscr{J}(\hat{u})\leq \mathscr{J}(u) \text{ for all } u\in\mathscr{U}_{ad}.$$

We assume that $\inf_{u\in\mathscr{U}_{ad}}\mathscr{J}(u)<+\infty$. Otherwise there is nothing to do. Using the assumptions **(H8)** again, we have $\inf_{u\in\mathscr{U}_{ad}}\mathscr{J}(u)>-\infty$. By definition of infimum, there exists a minimizing sequence $\{u_n\}\subseteq\mathscr{U}_{ad}$ such that

$$\mathscr{J}(u_n)\to\inf_{u\in\mathscr{U}_{ad}}\mathscr{J}(u).$$

Since $\{u_n\}\subseteq\mathscr{U}_{ad}$, $\{u_n\}$ is bounded in the space $L^2_{\mathscr{F}}(J,\mathscr{K})$, there exists a subsequence $\{u_{n_l}\}_{l\geq 1}$ of $\{u_n\}$ which converges weakly to $\hat{u}\in L^2_{\mathscr{F}}(J,\mathscr{K})$ as $l\to\infty$. Since $\mathscr{U}_{ad}$ is convex and closed, then Marzur Theorem implies that $\hat{u}\in\mathscr{U}_{ad}$. Let $\{\tilde{z}^{u_n}\}$ denote the sequence of mild solutions of stochastic system (1) corresponding to the sequence control $\{u_n\}$

$$\tilde{z}^{u_n}(t)=\begin{cases}\mathscr{R}(t)z_0+\int_0^t\mathscr{R}(t-s)C(s)u(s)ds \\ +\int_0^t\mathscr{R}(t-s)F(s,\tilde{z}^{u_n}(s))dW(s)+\int_0^t\mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t\in[0,t_1],\ k=0,\\ I_k(t,\tilde{z}^{u_n}(t_k^-)), & t\in(t_k,e_k],\ k\geq 1,\\ \mathscr{R}(t-e_k)I_k(e_k,\tilde{z}^{u_n}(t_k^-))+\int_{e_k}^t\mathscr{R}(t-s)C(s)u(s)ds \\ +\int_{e_k}^t\mathscr{R}(t-s)F(s,\tilde{z}^{u_n}(s))dW(s)+\int_{e_k}^t\mathscr{R}(t-s)G(s)dR_Q^{\mathsf{H}}(s), & t\in(e_k,t_{k+1}],\ k\geq 1.\end{cases}$$

In the same way to the proof of the above **step 2**, we can demonstrate that $\{\tilde{z}^{u_n}\}$ is relatively compact on $\mathscr{PC}(\mathbb{Y})$. Thus, there exists $\tilde{z}^{\hat{u}}\in\mathscr{PC}(\mathbb{Y})$ such that $\tilde{z}^{u_n}\to\tilde{z}^{\hat{u}}$ for $\hat{u}\in\mathscr{U}_{ad}$. As $n\to\infty$, we have

$$\begin{aligned}&I_k(t,\tilde{z}^{u_n}(t_k^-))\to I_k(t,\tilde{z}^{\hat{u}}(t_k^-)),\ k=1,2,\ldots,m,\\ &\int_{e_k}^t\mathscr{R}(t-s)C(s)u_n(s)ds\to\int_{e_k}^t\mathscr{R}(t-s)C(s)\hat{u}(s)ds,\ k=0,1,2,\ldots,m,\\ &\int_{e_k}^t\mathscr{R}(t-s)F(s,\tilde{z}^{u_n}(s))dW(s)\to\int_{e_k}^t\mathscr{R}(t-s)F(s,\tilde{z}^{\hat{u}}(s))dW(s),\ k=0,1,2,\ldots,m.\end{aligned}$$

Hence, $\tilde{z}^{\hat{u}}$ denotes the solution of stochastic system (1) corresponding to $\hat{u}$. Since $\mathscr{PC}(\mathbb{Y})\subset L^1(J,\mathbb{Y})$, by Balder's theorem [9] and hypothesis **(H8)**, we obtain

$$\begin{aligned}\inf_{u\in\mathscr{U}_{ad}}\mathscr{J}(u) &= \lim_{n\to\infty}\mathbb{E}\int_0^b\mathscr{T}(t,\tilde{z}^{u_n}(t),u_n(t)dt\\ &\geq \mathbb{E}\int_0^b\mathscr{T}(t,\tilde{z}^{\hat{u}}(t),\hat{u}(t))dt=\mathscr{J}(\tilde{z}^{\hat{u}},\hat{u})\geq\inf_{u\in\mathscr{U}_{ad}}\mathscr{J}(u).\end{aligned}$$

Consequently, $\mathscr{J}(\tilde{z}^{\hat{u}},\hat{u})=\mathscr{J}(\hat{u})=\inf_{z^{\hat{u}}\in\mathscr{U}_z(\hat{u})}\mathscr{J}(\tilde{z}^{\hat{u}},\hat{u})$. Additionally, $\mathscr{J}(\hat{u})=\inf_{u\in\mathscr{U}_{ad}}\mathscr{J}(u)$, ie, $\mathscr{J}$ attains its minimum at $\hat{u}\in\mathscr{U}_{ad}$.

5. Example

Let us consider the following impulsive stochastic control system:

$$\begin{cases} dz(t,\zeta) = \Big[\frac{\partial^2}{\partial \zeta^2} z(t,\zeta) + u(t,\zeta) + \int_0^t b(t-s)\frac{\partial^2}{\partial \zeta^2} z(s,\zeta)ds\Big]dt \\ \qquad\qquad + \frac{\sqrt{t}e^{-t}z(t,\zeta)}{2(1+|z(t,\zeta)|)}dW(t) + e^{-t}dR_Q^{\mathsf{H}}(t), \quad t \in (0,0.30] \cup (0.70,\ 1], u \in \mathscr{U}_{ad}, \zeta \in [0,\pi], \\ z(t,\zeta) = \frac{1}{5}(\sin t) z(0.30^-,\zeta), \quad t \in (0,0.30], \zeta \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, \\ z(0,\zeta) = z_0(\zeta), \ \ \zeta \in [0,\pi], \end{cases} \tag{24}$$

with the following cost function

$$\mathscr{J}(z,u) = \int_0^1 \int_0^\pi \|z^u(t,\zeta)\|^2 d\zeta dt + \int_0^1 \int_0^\pi \|u(t,\zeta)\|^2 d\zeta dt,$$

where $R_Q^{\mathsf{H}} = \{R_Q^{\mathsf{H}}(t) \ : \ t \geq 0\}$ is a Rosenblatt process with $\mathsf{H} \in (1/2,1)$, $W(t)$ is a Wiener process, $b : \mathbb{R}^+ \to \mathbb{R}$ is bounded and $\mathscr{C}^1$ function such that b' is bounded and uniformly continuous. $0 = e_0 < t_0 < t_1 < e_1 < t_2 = b$, with $t_1 = 0.30$, $e_1 = 0.70$ and $t_2 = 1$. Let $Q = 1, \mathbb{X}_1 = \mathbb{X}_2 = \mathbb{R}, \delta_1 = 1, \ \delta_n = 0, n > 1$. Let $\mathbb{Y} = \mathscr{K} = L^2[0,\pi], \mathbb{X}_1 = \mathbb{X}_2 = \mathbb{R}$. Define the operator $A : \mathscr{D}(A) \subset \mathbb{Y} \to \mathbb{Y}$ by $A\omega = \frac{\partial^2 \omega}{\partial y^2}$ with

$$\mathscr{D}(A) = \left\{ y \in \mathbb{Y} : y, y' \text{are absolutely continuous}, \ \ y\text{"} \in \mathbb{Y}, \ y(0) = y(\pi) = 0 \right\}.$$

It is well know that A generates a C_0-semi-group $(S(t))_{t\geq 0}$ on $\mathbb{Y}$, which implies that **(H1)** is satisfied. Let $\mathscr{B} : \mathscr{D}(A) \subset \mathbb{Y} \to \mathbb{Y}$ be the operator defined by

$$\mathscr{B}(t)(z) = b(t)Az \ \text{ for } t \geq 0 \ \text{ and } z \in \mathscr{D}(A).$$

Now, we define the control set

$$\mathscr{U}_{ad} = \Big\{ u(\cdot,\zeta) : [0,1] \to \mathscr{K} \text{ is } \mathscr{F}_t\text{-adapted and mesurable stochastic process and } \|u\|_{L^2_{\mathscr{F}}} \leq \rho, \rho > 0 \Big\}.$$

Let $z(t)(\zeta) = z(t,\zeta)$ and the functions F, G and I_1 are defined as

$$\begin{aligned} F(t,z)(\zeta) &= \frac{\sqrt{t}e^{-t}z(t,\zeta)}{2(1+|z(t,\zeta)|)}, \\ G(t) &= e^{-t}, \\ I_1(t,z(t_1^-))(\zeta) &= \frac{1}{5}(\sin t) z(0.30^-,\zeta). \end{aligned}$$

For all $u \in L^2([0,1]\times[0,\pi])$, we define an operator C as follows: $Cu(t,\zeta) = u(t,x)$. Using these definitions, we can represent the system (24) in the abstract form of (1) for $m = 1$. We assume that $b = 1, M = 1, c_{\mathsf{H}} = 1, \mathsf{H} = 0.75$. We obtain $\int_0^1 \nu(s)ds = \frac{1}{4}\int_0^1 se^{-2s}ds \approx 0.0371, \delta_{I_1} = \gamma_1 = \frac{1}{25}$.Thus, we have $\delta_{I_1} \approx 0.04 < 1$, $\theta_1 = \max\Big[\gamma_1, \gamma_1\Big] \approx \max\Big[0.04, 0.04\Big] = 0.04 < 1$ and $\theta_2 = \Big[\delta_{I_1} + 4\delta_{I_1} + 4\int_0^b \nu(s)ds\Big] \approx \big[0.04 + 0.16 + 0.1484\big] = 0.3484 < 1.$

Hence all conditions of Theorem 3.1 and Theorem 4.3 are satisfied Thus, the problem $(\mathscr{L}\mathscr{P})$ corresponding to stochastic system (24) admits at least one optimal state-control pair.

6. Conclusion

The existence of some impulsive stochastic integro-differential equations driven by the Rosenblatt process with non-instantaneous impulses, as well as optimal control findings for these equations, have been examined. By employing the fixed point technique, specifically Krasnoselskii's fixed point theorem and the theory of resolvent operators, we were able to demonstrate the existence of mild solutions to the suggested system. Furthermore, we demonstrated the optimal control outcomes by employing the minimizing sequence concept, which was then used to derive the optimization criteria. Finally, an example has been used to validate the theoretical conclusions reached.

There is one direct issue which require further study. In a coming paper we will investigate the approximate controllability of impulsive stochastic integro-differential equations driven by Rosenblatt process with non-instantaneous impulses.

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