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ON EIGENVALUE, SINGULAR VALUE AND NORMS OF A REAL SKEW-SYMMETRIC MATRIX

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Abstract. Many problems in applied mathematics are solved by computing the eigenvalues, singular values, spectral and Euclidean norms of the skew-symmetric matrices. In this article, we first introduce the eigenvalues and singular values of the matrix $A = [x_i - x_j]_{i,j=1}^n$. Then we obtain the spectral norm and Euclidean norms of A. Finally, some numerical examples are taken to illustrate the correctness of the concluded results.

1. Introduction and Preliminaries

There has been a lot of interest in the the symmetric, skew-symmetric, Cauchy, Hankel, Toeplitz and Circulant matrices and some profound results were established [1], [3], [4], [7].

Toeplitz matrices has attracted the continuous attention of the scholars in the filed of applied mathematics [1], [3], [6], [8].

In this paper we obtain some new results on the eigenvalues, singular values, the spectral and Euclidean norms of the n -by- n real skew-symmetric matrix for any $n \geq 2$

$$A = [x_i - x_j]_{i,j=1}^n = \begin{bmatrix} 0 & x_1 - x_2 & x_1 - x_3 & \cdots & x_1 - x_n \\ x_2 - x_1 & 0 & x_2 - x_3 & \cdots & x_2 - x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n - x_1 & x_n - x_2 & x_n - x_3 & \cdots & 0 \end{bmatrix}. \quad (1)$$

When $x_i = i, i = 1, 2, ..., n$, for the elements of the matrix A, the matrix A becomes the n -by- n real skew-symmetric Toeplitz matrix.

We recall some basic definitions and properties of the spectral and Euclidean norms of matrices. For a comprehensive exposition of the subject we refer the reader to [2] and [5].

Let $\mathbb{C}^{n\times n}$ denote the space of $n \times n$ complex matrices. The conjugate $\bar{A}$ of $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is the matrix such that $\bar{A_{ij}} = \bar{a}_{ij}, \quad 1 \leq i, j \leq n$. The transpose of A is the $n \times n$ matrix $A^\top$ such that $A_{ij}^\top = a_{ji}, \quad 1 \leq i, j \leq n$. The conjugate transpose of A is the $n \times n$ matrix A^* such that $A^* = \overline{(A^\top)} = (\bar{A})^\top$. When A is a

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real matrix ($A \in \mathbb{R}^{n\times n}$), $A^* = A^\top$. A matrix A is Hermitian if $A^* = A$. If A is a real matrix ($A \in \mathbb{R}^{n\times n}$), we say that A is symmetric if $A^\top = A$. The trace of A is the sum of its diagonal elements $tr(A) = a_{11} + \cdots + a_{nn}$.

Hermitian matrices $AA^* \in \mathbb{C}^{n\times n}$ and $A^*A \in \mathbb{C}^{n\times n}$ have the same eigenvalues. The singular values of A are uniquely determined by the eigenvalues of A^*A (equivalently, by the eigenvalues of AA^*). Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ have eigenvalues $\lambda_1, \dots, \lambda_n$ ordered so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and singular values $\sigma_1, \dots, \sigma_n$ ordered so that $\sigma_1 \geq \cdots \geq \sigma_n$. Then, it is well known that

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = trA^*A = \sum_{i=1}^{n} \sigma_i^2.$$

A matrix $A \in \mathbb{C}^{n\times n}$ is called normal if $A^*A = AA^*$. If $A^* = -A$, we have $A^*A = AA^* = -A^2$. Hence matrices for which $A^* = -A$, called skew-Hermitian, are normal. $\sigma_i = |\lambda_i|$ for all $i = 1, \dots, n$ if and only if A is normal.

The Euclidean norm of A is given by

$$\|A\|_E = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{i}^{n} \sigma_i^2(A)} = \sqrt{tr\,(AA^*)}. \tag{2}$$

The specral radius $\rho(A)$ of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as:

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

The spectral norm $\|\cdot\|_2$ is defined on $\mathbb{C}^{n\times n}$ by

$$\|A\|_2 = \sigma_1(A).$$

The spectral norm of normal matrices are equal to the maximum eigenvalue in absolute value which is the spectral radius $\rho(A)$:

$$\rho(A) = \|A\|_2.$$

We now outline the organization of the paper and the results obtained there. Section 2 presents some new results on the eigenvalues, singular values, spectral and Euclidean norms of a real skew-symmetric matrix $A = [x_i - x_j]_{i,j=1}^n$. Section 3 presents two numerical examples illustrated Theorem 1 and Thereom 2, respectively. Section 4 presents the conclusions of this study.

2. Main Results

With the help of preliminaries given in Section 1, we are now ready to prove the following theorem characterizing the eigenvalues and singular values of the matrix considered in (1).

Theorem 2.1. *Except for $n-2$ zeroes, the eigenvalues and singular values of A in (1), respectively, are*

$$\lambda_1 = -i\sqrt{\beta n - \alpha^2}, \quad \lambda_2 = i\sqrt{\beta n - \alpha^2}$$

and

$$\sigma_1 = \sigma_2 = \sqrt{\beta n - \alpha^2} \tag{3}$$

where $i^2 = -1$, $\alpha = \sum_{k=1}^{n} x_k$ and $\beta = \sum_{k=1}^{n} x_k^2$.

Proof. We write the matrix A given in (1) as follows:

$$A = \begin{bmatrix} 0 & x_1 - x_2 & x_1 - x_3 & \cdots & x_1 - x_n \\ x_2 - x_1 & 0 & x_2 - x_3 & \cdots & x_2 - x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n - x_1 & x_n - x_2 & x_n - x_3 & \cdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

We know that the eigenvalues of A are 0 or purely imaginary. Thus if λ is a purely imaginary eigenvalue of A, then its conjugate $\bar{\lambda} = -\lambda$ is also an eigenvalue of A since A is a real matrix. Thus, nonzero eigenvalues come in pairs $\lambda, -\lambda$. Let

$$v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \text{ and } e = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

Except for $n-2$ zeroes, the eigenvalues of A are the same as those of

$$B = \begin{bmatrix} e^T v & -e^T e \\ v^T v & -v^T e \end{bmatrix}.$$

We, respectively, get the real numbers $e^T v$, $-e^T e$, $v^T v$ and $-v^T e$ as follows:

$$e^T v = \sum_{k=1}^{n} x_k,$$
$$-e^T e = -\sum_{k=1}^{n} 1 = -n,$$
$$v^T v = \sum_{k=1}^{n} x_k^2,$$
$$-v^T e = -\sum_{k=1}^{n} x_k.$$

Let

$$\alpha = \sum_{k=1}^{n} x_k \text{ and } \beta = \sum_{k=1}^{n} x_k^2. \tag{4}$$

So, the B matrix will be in the form of

$$B = \begin{bmatrix} \alpha & -n \\ \beta & -\alpha \end{bmatrix}.$$

The eigenvalues of the B matrix are

$$\lambda_1 = -\sqrt{\alpha^2 - \beta n}, \quad \lambda_2 = \sqrt{\alpha^2 - \beta n}.$$

From Cauchy–Schwarz inequality, we have $\alpha^2 - \beta n \leq 0$. Equality occurs if and only if $x_1 = x_2 = ... = x_n$. Therefore, we get $\beta n - \alpha^2 \geq 0$. Thus, the eigenvalues of the B matrix are

$$\lambda_1 = -i\sqrt{\beta n - \alpha^2}, \quad \lambda_2 = i\sqrt{\beta n - \alpha^2},$$

where i is complex unity. Since A is a skew-symmetric matrix, A is a normal matrix. Thus, since A is a normal matrix, $\sigma_i = |\lambda_i|$ for all $i = 1, \ldots, n$. The proof of the theorem is completed with this result. □

By summarizing the above discussions and using Theorem 2.1, we can obtain the following result.

Theorem 2.2. *The spectral norm and Euclidean norms of A in (1), respectively, are*

$$\|A\|_2 = \sqrt{\beta n - \alpha^2}$$

and

$$\|A\|_E = \sqrt{2\left(\beta n - \alpha^2\right)}$$

with $\alpha = \sum_{k=1}^{n} x_k$ and $\beta = \sum_{k=1}^{n} x_k^2$.

Proof. The singular values of A are the (positive) square roots of the eigenvalues of the matrix $A^T A = -A^2$. Since the spectral norm of A is defined by the largest singular value of A we obtain from Theorem 1 ,

$$\|A\|_2 = \sqrt{\beta n - \alpha^2}.$$

The Euclidean norm of A is from (2) and (3)

$$\begin{aligned}\|A\|_E &= \sqrt{\sum_{i}^{n} \sigma_i^2(A)} \\ &= \sqrt{2\left(\beta n - \alpha^2\right)}.\end{aligned}$$

Consequently, the proof of the theorem is completed. □

3. Numerical examples

The following example illustrates Theorem 1.

Example 3.1. *We consider the 3 -by-3 real skew-symmetric Toeplitz matrix*

$$C = [i-j]_{i,j=1}^{n} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}. \tag{5}$$

The values of α and β in (4) for the C matrix are calculated as follows:

$$\alpha = \sum_{k=1}^{3} x_k = 6, \tag{6}$$

$$\beta = \sum_{k=1}^{3} x_k^2 = 14, \tag{7}$$

where $x_1 = 1, x_2 = 2, x_3 = 3$. Except for $n-2 = 3-2 = 1$ zero, the eigenvalues λ_1, λ_2 and singular values σ_1, σ_2 of C from Theorem 2.1, respectively, are

$$\lambda_1 = -i\sqrt{\beta n - \alpha^2} = -i\sqrt{6}, \quad \lambda_2 = i\sqrt{\beta n - \alpha^2} = i\sqrt{6},$$

and

$$\sigma_1 = \sigma_2 = \sqrt{\beta n - \alpha^2} = \sqrt{6}.$$

Using a computer program, we get the eigenvalues and singular values of the C matrix as

$$\lambda_1 = -i\sqrt{6}, \quad \lambda_2 = i\sqrt{6}, \quad \lambda_3 = 0$$

and

$$\sigma_1 = \sigma_2 = \sqrt{6}, \quad \sigma_3 = 0,$$

respectively,

The following example illustrates Theorem 2.2.

Example 3.2. *We consider the 3 -by-3 real skew-symmetric Toeplitz matrix in 5. Using a computer program, we find the spectral and Euclidean norms of C as* $\sqrt{6}$ *and* $\sqrt{12}$*, respectively. For* $x_1 = 1, x_2 = 2, x_3 = 3$ *and* $n = 3$*, we verify Theorem (2.2) by (6) and (7).*

4. Conclusion

We have obtained the eigenvalues, singular values, the spectral and Euclidean norms of the matrix $A = [x_i - x_j]_{i,j=1}^{n}$ in (1). Then, we have show that the results obtained in Theorem 1 and Thereom 2 to calculate the eigenvalues, singular values, the spectral and Euclidean norms of A are simply applicable.

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