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OPTIMAL CONTROL OF SEIHR MATHEMATICAL MODEL OF COVID-19

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Abstract. In this paper, we propose a SEIHR model of covid-19 in which we introduce the unreported infected people and persons who are hospitalized. We introduce a function that represents the effect on the compartments of different measures taken after the epidemic beginning. Then, we find the disease free equilibrium point and study its global stability. Also, we introduce two controls strategies in the model, namely: the vaccination and treatment strategies. After computing the pair of optimal functions of the controls, we perform some numerical simulations for illustrations.

1. Introduction

Caused by severe acute respiratory syndrome corona virus 2 (SARS-Cov 2), the COVID-19 is a disease of global concern according to the World Health Organization as of 30 January 2020. This pandemic first appeared in Wuhan in December 2019. Its virus spreads by many different routes like nose and mouth secretions, including tiny droplets of water from coughing, sneezing and speech. The disease becomes more contagious during the first three days after the onset of symptoms.
The first cases confirmed of COVID-19 in Burkina Faso date of 09 March 2020. It is begun by the capital Ouagadougou and spread in other towns like Bobo-Dioulasso and Hounde.
Basing on Guiro and al [6] work, we propose a more detailed model of COVID-19.

Mathematical modeling are much used to describe the dynamics of epidemic diseases [8, 9]. This domain is fast growing and has been playing capital roles in discovering relations between species and their interactions. The epidemiological models type are numerous [7]. According to the method of transmission(sexual contact, simple contact, etc) the nature of the disease (killer or curable disease), we can create a model basing on the satndard Susceptible-Infectious-Susceptible(SIS), Susceptible-Infectious-Recovered(SIR) and Susceptible-Exposed-Infectious-Recovered(SEIR)[4, 14]. Differential equations whether ordinary, partial or stochastic are fundamental tools in the modeling of infectious diseases. The main objective

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of our epidemiological model is to take into account unlisted patients (I_u) living in the population and patients in intensive care (H_D). Thus, we approximate the dynamics of infected people as well as possible.
Optimal control has long and been widely used as a control strategy in epidemic outbreaks [1, 3, 5, 11]. The principals idea by using the optimal control in epidemics are to show the real impact of the strategies by simulations and to search for among the possibles strategies, the most effective strategy that reduces the infection rate to a minimum level while optimizing the cost of deploying a therapy or vaccine that is used for controlling the disease progression [3]. In epidemic diseases, strategies can include therapies, vaccines, isolation, social distancing, etc [2, 19].

In this work, we consider a optimal control problem described by a system of differential equations with a function mi(t), which reflects the effect of the social distancing, the mask wearing and any other measure in the aim to break the spread of the disease. The equations of different states are described in a SEIHR-type model with four infected sub-compartments, two recovered sub-compartments and one compartment for deceased persons. By using Pontryangin's maximum principle [13], we compute the optimal control after showing its existence.

Our paper is organized as follows. We present in section 2, the model without control and some properties. The global stability of the disease free equilibrium point is studied in section 3. We analytically solved an optimal control problem in section 4. Then in section 5, with real data, we show the efficiency of the proposed optimal control through numerical simulations. We end by conclusion in section 6.

2. Model

The COVID-19 health crisis is too complex and the SEIR model too simple to describe it. Indeed, unidentified infected individuals require a separate compartment. We therefore add three compartments to represent unidentified infected individuals, identified infected individuals and those hospitalised in intensive care. Then we propose the following diagram :

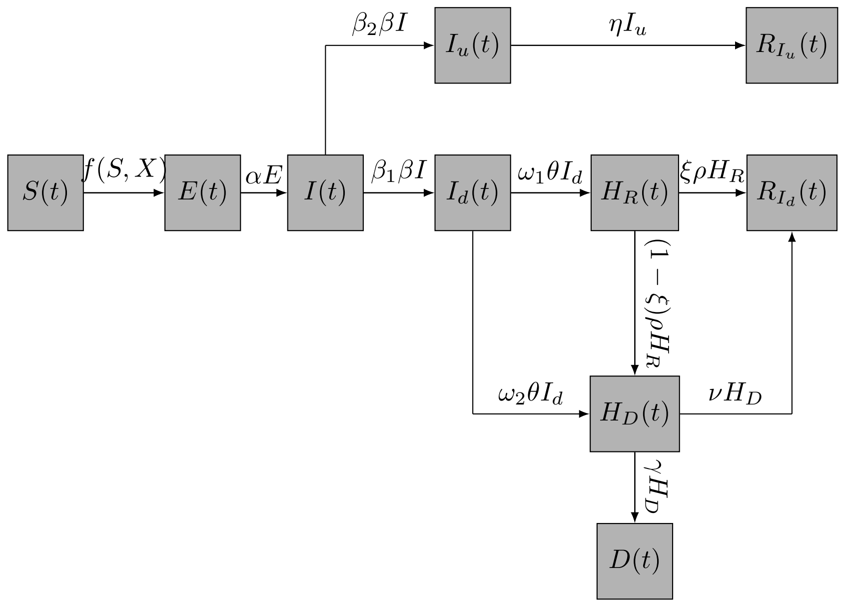


FIGURE 1. Transfer diagram

with

$$\begin{aligned} f(S,X) &= \frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_u}(t)\beta_{I_u} I_u(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\ &+ m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)) - \alpha E(t). \end{aligned}$$

Then, according to Figure 1 we obtain the following system of ten differential equations:

$$\left\{ \begin{aligned} \dot{S}(t) &= -\frac{S(t)}{N}\Big(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\ &+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) \\ &+ m_{H_D}(t)\beta_{H_D} H_D(t)\Big), \\ \dot{E}(t) &= \frac{S(t)}{N}\Big(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\ &+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) \\ &+ m_{H_D}(t)\beta_{H_D} H_D(t)\Big) - \alpha E(t), \\ \dot{I}(t) &= \alpha E(t) - \beta I(t), \\ \dot{I}_d(t) &= \beta_1 \beta I(t) - \theta I_d(t), \\ \dot{I}_u(t) &= \beta_2 \beta I(t) - \eta I_u(t), \\ \dot{H}_R(t) &= \omega_1 \theta I_d(t) - \rho H_R(t), \\ \dot{H}_D(t) &= \omega_2 \theta I_d(t) + (1-\xi)\rho H_R(t) - (\gamma+\nu) H_D(t), \\ \dot{R}_{I_d}(t) &= \xi \rho H_R(t) + \nu H_D(t), \\ \dot{R}_{I_u}(t) &= \eta I_u(t), \\ \dot{D}(t) &= \gamma H_D(t), \end{aligned} \right. \tag{1}$$

with the initial conditions
$S(t_0) = S_0 > 0, E(t_0) = E_0 > 0, I(t_0) = I_0 > 0, I_d(t_0) = I_{d_0} > 0,$
$I_u(t_0) = I_{u_0} > 0, H_R(t_0) = H_{R_0} > 0, H_D(t_0) = H_{d_0} > 0, R_{I_d}(t_0) = R_{I_{u_0}} > 0,$
$D(t_0) = D_0 > 0$ where :
•$S(t)$ (Susceptible) represents the persons who are not infected by the disease pathogen,
•$E(t)$ (Exposed) design the person is in the incubation period after being infected by the disease pathogen and has no visible clinical signs. The individual could infect other people but with a lower probability than people in the infectious compartments. After the incubation period, the person passes to one of the infectious states,
•$I(t)$ Infectious that will be detected design the person who starts developing clinical sign, these persons are symptomatic infectious,
•$I_d(t)$ Infectious that will be detected design the person can infect other people, starts developing clinical signs and will be detected and reported by authorities(when arriving to the compartments H_R or H_D). After this period, people in this compartment are taken in charge by sanitary authorities and we classify them as Hospitalized,
•$I_u(t)$ is the number of unreported symptomatic infectious individuals (i.e symptomatic infectious with mild symptoms) at time t,
•Hospitalized or in quarantine at home (but detected and reported by authorities) that will recover (denoted by H_R): The person is in hospital (or quarantine at home) and can still infect other people. At the end of this state, a person passes to the Recovered state,
•Hospitalized that are in intensive cares (denoted by H_D): The person is hospitalized and can still infect other people. At the end of this state, a person passes to the Dead state,
•Dead(denoted by D): The person has not survived the disease,
•Recovered(denoted by R): The person has survived the disease, is no longer infectious and has developed a natural immunity to the disease pathogen.
We assume that $\beta_1 + \beta_2 = 1$ and $\omega_1 + \omega_2 = 1$.
Here, $N = S + E + I + H_R + H_D + R_{I_d} + R_{I_u} + D$ is the total population number at time t. The parameters are defined in the table below

Parameter	Definition
$m_E, m_I, m_{I_u}, m_{I_d}, m_{H_R}, m_{H_D}$	The effect of measures (hands washing, public spaces closing, social distancing, mask carrying, ...) applying in the corresponding states
$\beta_E, \beta_I, \beta_{I_u}, \beta_{I_d}, \beta_{H_R}, \beta_{H_D}$	The disease contact rates ($days^{-1}$) of person in the corresponding states
α	The transition rate from (day^{-1}) state E to state I
β	The rate of infected people that is detected and unreported
β_1	Proportion of infected people that is detected
β_2	Proportion of infected people that is unreported
θ	Transition rate from state I_d
ω_1	Proportion of infected detected I_d people who are hospitalized and will recover H_R
ω_2	Proportion of infected detected I_d people who are hospitalized in intensive cares H_D
η	Transition rate from state I_u to R_{I_u}
ρ	Transition rate from state H_R
ξ	Proportion of H_R people among R_{I_u} people
ν	Transition rate from state H_D to R_{I_d}
γ	Transition rate from state H_D to D

Table 1. The definitions of parameters.

2.1. **Positivity and boundedness of the solutions.**

For showing the positivity of solutions, we state the following lemma.

Lemma 2.1. [18] *Suppose* $\Omega \subset \mathbb{R} \times \mathbb{C}^n$ *is open,* $f_i \in C(\Omega, \mathbb{R})$.
If $f_i|_{x_i(t)=0, X_t \in C^n_{+0}} \geqslant 0, X_t = (x_{1t}, x_{2t}, x_{3t}, ..., x_{nt})^T, i = 1, 2, 3, ..., n,$ *then* $C^n_{+0} = \{\phi = (\phi_1, ..., \phi_n) : \phi \in C([-\tau, 0], \mathbb{R}^n_{+0})\}$ *is the invariant domain of the following equations*

$$\frac{dx_i(t)}{dt} = f_i(t, X_t), t \geqslant \sigma, i = 1, 2, 3, ..., n$$

where $\mathbb{R}^n_{+0} = \{(x_1, ..., x_n), x_i \geqslant 0, i = 1, 2, 3, ..., n\}$.

Proposition 2.1. *The system* (1) *is invariant in* $\mathbb{R}^{10}_+$.

Proof. By re-writing the system (1), we get

$$\begin{cases} \dfrac{dX}{dt} = B(X(t)) \quad with \quad B(X(t)) = (B_1(X), B_2(X), ..., B_{10}(X))^T \\ X_0 = (S_0, E_0, I_0, H_{R_0}, H_{D_0}, R_{I_{d_0}}, R_{I_{u_0}}, D_0)^T \geqslant 0. \end{cases} \tag{2}$$

We have :

$$\begin{aligned}
\frac{dS}{dt}|_{S=0} &= 0 \geqslant 0, \\
\frac{dE}{dt}|_{E=0} &= \frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_u}(t)\beta_{I_u} I_u(t) + m_{I_d}(t)\beta_{I_d} I_d(t)) \\
&+ \frac{S(t)}{N}(m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)) \geqslant 0, \\
\frac{dI}{dt}|_{I=0} &= \alpha E(t) \geqslant 0, \\
\frac{dI_d}{dt}|_{I_d=0} &= \beta_1 \beta I(t) \geqslant 0, \\
\frac{dI_u}{dt}|_{I_u=0} &= \beta_2 \beta I(t) \geqslant 0, \\
\frac{dH_R}{dt}|_{H_R=0} &= \omega_1 \theta I_d(t) \geqslant 0, \\
\frac{dH_D}{dt}|_{H_D=0} &= \omega_2 \theta I_d(t) + (1-\xi)\rho H_R(t) \geqslant 0, \\
\frac{dR_{I_d}}{dt}|_{R_{I_d}=0} &= \xi \rho H_R(t) + \nu H_D(t) \geqslant 0, \\
\frac{dR_{I_u}}{dt}|_{R_{I_u}=0} &= \eta I_u(t) \geqslant 0, \\
\frac{dD}{dt}|_{D=0} &= \gamma H_D(t) \geqslant 0.
\end{aligned}$$

Then it follows that according to the lemma 2.1, $\mathbb{R}_+^{10}$ is a invariant set for model (1).

Proposition 2.2. *The system* (1) *is bounded in the region* $\Omega = \{X = (S, E, I, I_d, I_u, H_R, H_d, R_{I_d}, R_{I_u}, D) \in \mathbb{R}_+^{10} / S + E + I + I_d + I_u + H_R + H_d + R_{I_d} + R_{I_u} + D \leqslant c, c \in \mathbb{R}^+\}$.

Proof. We observe that

$$\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dH_R}{dt} + \frac{dH_D}{dt} + \frac{dR_{I_d}}{dt} + \frac{dR_{I_u}}{dt} + \frac{dD}{dt} = 0,$$

then $\dfrac{dN}{dt} = 0$. Consequently N is a constant. Hence the model (1) is bounded.

2.2. **Basic reproduction number $\mathcal{R}_0$.**

Proposition 2.3. *The basic reproduction number* R_0 *of model* (1) *is :*

$$\begin{aligned}
\mathcal{R}_0 &= \frac{1}{\alpha} m_E(t)\beta_E + \frac{1}{\beta} m_I(t)\beta_I + \frac{\beta_2}{\eta} m_{I_u}(t)\beta_{I_u} + \frac{\beta_1}{\theta} m_{I_d}(t)\beta_{I_d} + \frac{\omega_1 \beta_1}{\rho} m_{H_R}(t)\beta_{H_R} \\
&+ \frac{\beta_1}{\gamma + \nu}(\omega_2 + \omega_1(1-\xi)) m_{H_D}(t)\beta_{H_D}.
\end{aligned}$$

Proof. We use the next-generation matrix method [16] to calculate the reproduction number R_0 of model (1). Let $\mathcal{F}$ and $\mathcal{V}$, the transmission and flow matrix between the infectious compartments E, I, I_u, I_d, H_R and H_D :

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \mathcal{F}_4 \\ \mathcal{F}_5 \\ \mathcal{F}_6 \end{pmatrix} = \begin{pmatrix} \frac{S(t)}{N}\left(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_u}(t)\beta_{I_u} I_u(t) + m_{I_d}(t)\beta_{I_d} I_d(t)\right) \\ +\frac{S(t)}{N}\left(m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \\ \mathcal{V}_5 \\ \mathcal{V}_6 \end{pmatrix} = \begin{pmatrix} -\alpha E(t) \\ \alpha E(t) - \beta I(t) \\ \beta_1 \beta I(t) - \theta I_d(t) \\ \beta_2 \beta I(t) - \eta I_u(t) \\ \omega_1 \theta I_d(t) - \rho H_R(t) \\ \omega_2 \theta I_d(t) + (1-\xi)\rho H_R(t) - \gamma H_D(t) - \nu H_D(t) \end{pmatrix}.$$

On the disease free equilibrium $X^0 = (S^0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, we obtain:

$$\mathcal{DF} = \begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial E} & \frac{\partial \mathcal{F}_1}{\partial I} & \frac{\partial \mathcal{F}_1}{\partial I_u} & \frac{\partial \mathcal{F}_1}{\partial I_d} & \frac{\partial \mathcal{F}_1}{\partial H_R} & \frac{\partial \mathcal{F}_1}{\partial H_D} \\ \frac{\partial \mathcal{F}_2}{\partial E} & \frac{\partial \mathcal{F}_2}{\partial I} & \frac{\partial \mathcal{F}_2}{\partial I_u} & \frac{\partial \mathcal{F}_2}{\partial I_d} & \frac{\partial \mathcal{F}_2}{\partial H_R} & \frac{\partial \mathcal{F}_2}{\partial H_D} \\ \frac{\partial \mathcal{F}_3}{\partial E} & \frac{\partial \mathcal{F}_3}{\partial I} & \frac{\partial \mathcal{F}_3}{\partial I_u} & \frac{\partial \mathcal{F}_3}{\partial I_d} & \frac{\partial \mathcal{F}_3}{\partial H_R} & \frac{\partial \mathcal{F}_3}{\partial H_D} \\ \frac{\partial \mathcal{F}_4}{\partial E} & \frac{\partial \mathcal{F}_4}{\partial I} & \frac{\partial \mathcal{F}_4}{\partial I_u} & \frac{\partial \mathcal{F}_4}{\partial I_d} & \frac{\partial \mathcal{F}_4}{\partial H_R} & \frac{\partial \mathcal{F}_4}{\partial H_D} \\ \frac{\partial \mathcal{F}_5}{\partial E} & \frac{\partial \mathcal{F}_5}{\partial I} & \frac{\partial \mathcal{F}_5}{\partial I_u} & \frac{\partial \mathcal{F}_5}{\partial I_d} & \frac{\partial \mathcal{F}_5}{\partial H_R} & \frac{\partial \mathcal{F}_5}{\partial H_D} \\ \frac{\partial \mathcal{F}_6}{\partial E} & \frac{\partial \mathcal{F}_6}{\partial I} & \frac{\partial \mathcal{F}_6}{\partial I_u} & \frac{\partial \mathcal{F}_6}{\partial I_d} & \frac{\partial \mathcal{F}_6}{\partial H_R} & \frac{\partial \mathcal{F}_6}{\partial H_D} \end{pmatrix},$$

$$F = \mathcal{DF},$$

$$F = \frac{S^0}{N} \begin{pmatrix} m_E(0)\beta_E & m_I(0)\beta_I & m_{I_u}(0)\beta_{I_u} & m_{I_d}(0)\beta_{I_d} & m_{H_R}(0)\beta_{H_R} & m_{H_D}(0)\beta_{H_D} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{D}\mathcal{V} = \begin{pmatrix} \frac{\partial \mathcal{V}_1}{\partial E} & \frac{\partial \mathcal{V}_1}{\partial I} & \frac{\partial \mathcal{V}_1}{\partial I_u} & \frac{\partial \mathcal{V}_1}{\partial I_d} & \frac{\partial \mathcal{V}_1}{\partial H_R} & \frac{\partial \mathcal{V}_1}{\partial H_D} \\ \frac{\partial \mathcal{V}_2}{\partial E} & \frac{\partial \mathcal{V}_2}{\partial I} & \frac{\partial \mathcal{V}_2}{\partial I_u} & \frac{\partial \mathcal{V}_2}{\partial I_d} & \frac{\partial \mathcal{V}_2}{\partial H_R} & \frac{\partial \mathcal{V}_2}{\partial H_D} \\ \frac{\partial \mathcal{V}_3}{\partial E} & \frac{\partial \mathcal{V}_3}{\partial I} & \frac{\partial \mathcal{V}_3}{\partial I_u} & \frac{\partial \mathcal{V}_3}{\partial I_d} & \frac{\partial \mathcal{V}_3}{\partial H_R} & \frac{\partial \mathcal{V}_3}{\partial H_D} \\ \frac{\partial \mathcal{V}_4}{\partial E} & \frac{\partial \mathcal{V}_4}{\partial I} & \frac{\partial \mathcal{V}_4}{\partial I_u} & \frac{\partial \mathcal{V}_4}{\partial I_d} & \frac{\partial \mathcal{V}_4}{\partial H_R} & \frac{\partial \mathcal{V}_4}{\partial H_D} \\ \frac{\partial \mathcal{V}_5}{\partial E} & \frac{\partial \mathcal{V}_5}{\partial I} & \frac{\partial \mathcal{V}_5}{\partial I_u} & \frac{\partial \mathcal{V}_5}{\partial I_d} & \frac{\partial \mathcal{V}_5}{\partial H_R} & \frac{\partial \mathcal{V}_5}{\partial H_D} \\ \frac{\partial \mathcal{V}_6}{\partial E} & \frac{\partial \mathcal{V}_6}{\partial I} & \frac{\partial \mathcal{V}_6}{\partial I_u} & \frac{\partial \mathcal{V}_6}{\partial I_d} & \frac{\partial \mathcal{V}_6}{\partial H_R} & \frac{\partial \mathcal{V}_6}{\partial H_D} \end{pmatrix},$$

and

$$V = \mathcal{D}\mathcal{V} = \begin{pmatrix} -\alpha & 0 & 0 & 0 & 0 & 0 \\ \alpha & -\beta & 0 & 0 & 0 & 0 \\ 0 & \beta_1\beta & -\theta & 0 & 0 & 0 \\ 0 & \beta_2\beta & 0 & -\eta & 0 & 0 \\ 0 & 0 & \omega_1\theta & 0 & -\rho & 0 \\ 0 & 0 & \omega_2\theta & 0 & (1-\xi)\rho & -\gamma-\nu \end{pmatrix}.$$

Then, we get

$$V^{-1} = \begin{pmatrix} \frac{-1}{\alpha} & 0 & 0 & -\frac{1}{\eta} & 0 & 0 \\ \frac{-1}{\beta} & \frac{-1}{\beta} & 0 & 0 & -\frac{1}{\rho} & 0 \\ -\frac{\beta_1}{\theta} & -\frac{\beta_1}{\theta} & -\frac{1}{\theta} & 0 & -\frac{1-\xi}{\gamma+\nu} & \frac{-1}{\gamma+\nu} \\ -\frac{\beta_2}{\eta} & -\frac{\beta_2}{\eta} & 0 & 0 & 0 & 0 \\ -\frac{\omega_1\beta_1}{\rho} & -\frac{\omega_1\beta_1}{\rho} & -\frac{\omega_1}{\rho} & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 \end{pmatrix}, \qquad (3)$$

$$a_1 = \frac{-\beta_1}{\gamma+\nu}(\omega_2+\omega_1(1-\xi)),$$

$$a_2 = \frac{-\beta_1}{\gamma+\nu}(\omega_2+\omega_1(1-\xi)),$$

$$a_3 = \frac{-1}{\gamma+\nu}(\omega_2+\omega_1(1-\xi)),$$

$$V_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V_2^{-1} = \begin{pmatrix} -\frac{1}{\eta} & 0 & 0 \\ 0 & -\frac{1}{\rho} & 0 \\ 0 & -\frac{1-\xi}{\gamma+\nu} & \frac{-1}{\gamma+\nu} \end{pmatrix}$$

and

$$-FV^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = -\frac{S^0}{N}\left(\begin{array}{c} \frac{-1}{\alpha}m_{E_0}\beta_E - \frac{1}{\beta}m_{I_0}\beta_I - \frac{\beta_2}{\eta}m_{I_{u_0}}\beta_{I_u} - \frac{\beta_1}{\theta}m_{I_{d_0}}\beta_{I_d} - \frac{\omega_1\beta_1}{\rho}m_{H_{R_0}}\beta_{H_R} \\ -\frac{\beta_1}{\gamma+\nu}(\omega_2+\omega_1(1-\xi))m_{H_{D_0}}\beta_{H_D} \end{array}\right) \in \mathcal{M}_{1,1}(\mathbb{R}),$$

$$B = (0) \in \mathcal{M}_{1,5}(\mathbb{R}), \quad C = (0) \in \mathcal{M}_{5,1}(\mathbb{R}), \quad D = (0) \in \mathcal{M}_{5,5}(\mathbb{R}).$$

The basic reproduction number $\mathcal{R}_0$ [16] is defined as the dominant eigeinvalue of the matrix $-FV^{-1}$. Therefore,

$$\begin{aligned} \mathcal{R}_0 &= \frac{S^0}{N}\Big(\frac{1}{\alpha}m_{E_0}\beta_E + \frac{1}{\beta}m_{I_0}\beta_I + \frac{\beta_2}{\eta}m_{I_{u_0}}\beta_{I_u} + \frac{\beta_1}{\theta}m_{I_{d_0}}\beta_{I_d} + \frac{\omega_1\beta_1}{\rho}m_{H_{R_0}}\beta_{H_R} \\ &\quad + \frac{\beta_1}{\gamma+\nu}(\omega_2+\omega_1(1-\xi))m_{H_{D_0}}(t)\beta_{H_D}\Big) \end{aligned}$$

and the effective reproduction number $\mathcal{R}_e(t)$ [6] is given by

$$\begin{aligned} \mathcal{R}_e(t) &= \frac{S(t)}{N}\Big(\frac{1}{\alpha}m_E(t)\beta_E + \frac{1}{\beta}m_I(t)\beta_I + \frac{\beta_2}{\eta}m_{I_u}(t)\beta_{I_u} + \frac{\beta_1}{\theta}m_{I_d}(t)\beta_{I_d} + \frac{\omega_1\beta_1}{\rho}m_{H_R}(t)\beta_{H_R} \\ &+ \frac{\beta_1}{\gamma+\nu}(\omega_2+\omega_1(1-\xi))m_{H_D}(t)\beta_{H_D}\Big). \end{aligned}$$

3. Global stability of X^0

Theorem 3.1. *The disease free equilibrium point X^0 of the system (1) is globally asymptotically stable when $\mathcal{R}_0 < 1$.*

Proof. Let us consider the infected classes E, I, I_d, I_u, H_R and H_D. By the equations corresponding to these states, we have at X_0, the following system :

$$\left\{\begin{array}{lcl}\dot{E}(t) & = & \frac{S^0}{N}\Big(m_{E_0}\beta_E E(t) + m_{I_0}\beta_I I(t) + m_{I_{d_0}}\beta_{I_d} I_d \\ & + & m_{I_{u_0}}\beta_{I_u} I_u(t) + m_{H_{R_0}}\beta_{H_R} H_R(t) \\ & + & m_{H_{D_0}}\beta_{H_D} H_D(t)\Big) - \alpha E(t), \\ \dot{I}(t) & = & \alpha E(t) - \beta I(t), \\ \dot{I}_d(t) & = & \beta_1\beta I(t) - \theta I_d(t), \\ \dot{I}_u(t) & = & \beta_2\beta I(t) - \eta I_u(t), \\ \dot{H}_R(t) & = & \omega_1\theta I_d(t) - \rho H_R(t), \\ \dot{H}_D(t) & = & \omega_2\theta I_d(t) + (1-\xi)\rho H_R(t) - (\gamma+\nu) H_D(t), \end{array}\right. \tag{4}$$

The matrix M associated to the linearized system (4) is :

$$M = \begin{pmatrix} \frac{S^0}{N} m_{E_0}\beta_E - \alpha & \frac{S^0}{N} m_{I_0}\beta_I & \frac{S^0}{N} m_{I_{u_0}}\beta_{I_u} & \frac{S^0}{N} m_{I_{d_0}}\beta_{I_d} & \frac{S^0}{N} m_{H_{R_0}}\beta_{H_R} & \frac{S^0}{N} m_{H_{D_0}}\beta_{H_D} \\ \alpha & -\beta & 0 & 0 & 0 & 0 \\ 0 & \beta_1\beta & -\theta & 0 & 0 & 0 \\ 0 & \beta_2\beta & 0 & -\eta & 0 & 0 \\ 0 & 0 & \omega_1\theta & 0 & -\rho & 0 \\ 0 & 0 & \omega_2\theta & 0 & (1-\xi)\rho & -\gamma-\nu \end{pmatrix},$$

and the linearized system (4) can be rewritten

$$\dot{Y} \leqslant MY, \tag{5}$$

where $Y = (E, I, I_d, I_u, H_R, H_D)^t$. Let $M = A + B$, with :

$$A = \begin{pmatrix} \frac{S^0}{N} m_{E_0}\beta_E & \frac{S^0}{N} m_{I_0}\beta_I & \frac{S^0}{N} m_{I_{u_0}}\beta_{I_u} & \frac{S^0}{N} m_{I_{d_0}}\beta_{I_d} & \frac{S^0}{N} m_{H_{R_0}}\beta_{H_R} & \frac{S^0}{N} m_{H_{D_0}}\beta_{H_D} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} -\alpha & 0 & 0 & 0 & 0 & 0 \\ \alpha & -\beta & 0 & 0 & 0 & 0 \\ 0 & \beta_1\beta & -\theta & 0 & 0 & 0 \\ 0 & \beta_2\beta & 0 & -\eta & 0 & 0 \\ 0 & 0 & \omega_1\theta & 0 & -\rho & 0 \\ 0 & 0 & \omega_2\theta & 0 & (1-\xi)\rho & -\gamma-\nu \end{pmatrix}.$$

We remark that B is invertible matrix and $B^{-1} = V^{-1}$ (3). We can see that $A \geqslant 0$ and $-B^{-1} \geqslant 0$.

Thus, $\mathcal{R}_0 = \rho(-AB^{-1}) < 1$ and from Varga theorem [17] the matrix M is asymptotically stable. The eigenvalue of matrix M has negative real part, by a standard comparison theorem [10] when $t \to +\infty$, $E \to 0, I \to 0, I_d \to 0, I_u \to 0, H_R \to 0$

and $H_D \to 0$ for system (4) and substituting $E = 0, I = 0, I_d = 0, I_u = 0, H_R = 0$ in (1), we get $S \to S^0$, $R_{I_d} \to 0$, $R_{I_u} \to 0$, $D \to 0$ as well as $t \to +\infty$.
Thus, $(S, E, I, I_d, I_u, H_R, H_D, R_{I_d}, R_{I_u}, D) \to (S^0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ as $t \to \infty$ for system (1), when $\mathcal{R}_0 < 1$.

3.1. **Some numerical simulations with real data**[6]**.**

FIGURE 2. $S(t)$

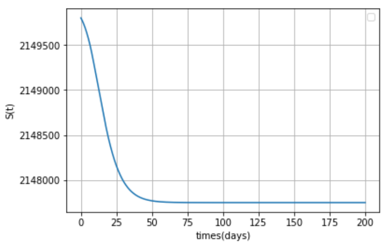


FIGURE 3. E(t)

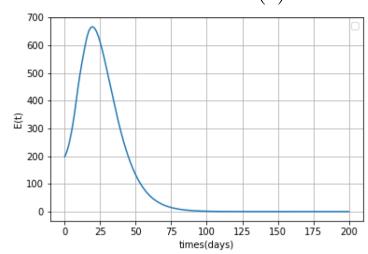


FIGURE 4. I(t)

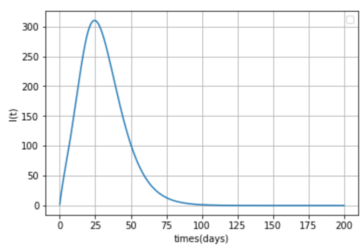


FIGURE 5. $I_d(t)$

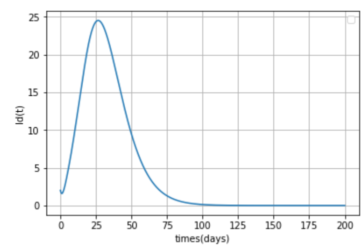


FIGURE 6. $I_u(t)$

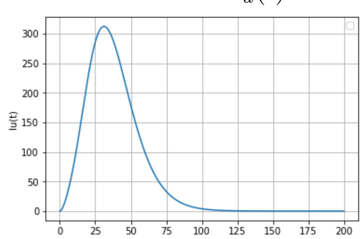


FIGURE 7. $H_R(t)$

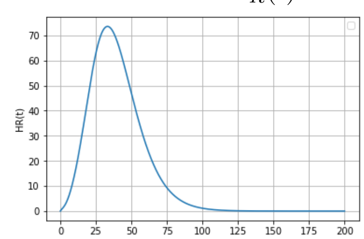
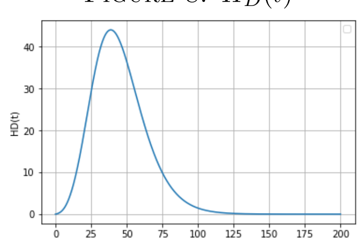


FIGURE 8. $H_D(t)$



These different numerical simulations effectively show us a stability around the free equilibrium point X^0.

4. Statement of the optimal control problem

In this section, we compute the optimal function of the control $(u(t), v(t))$ to determine the best measures in terms of vaccination and treatment to maximize the population of cured individuals and minimize both the population of infected and susceptible individuals. This optimal couple minimizes at the same time the cost of implementing the vaccination and treatment strategies. So we consider the following optimal control problem :

$$J(u^*, v^*) = \min\{J(u, v) : (u, v) \in U\} \tag{6}$$

where

$$J(u, v) = \int_0^{t_f} \left[I(t) - R_{I_d}(t) + \frac{A_1}{2} u^2(t) + \frac{A_2}{2} v^2(t) \right] dt, \tag{7}$$

subject to the equation

$$\left\{\begin{array}{lcl}
\dot{S}(t) & = & -(1-u(t))\dfrac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
& + & m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) \\
& + & m_{H_D}(t)\beta_{H_D} H_D(t)), \\
\dot{E}(t) & = & (1-u(t))\dfrac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
& + & m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) \\
& + & m_{H_D}(t)\beta_{H_D} H_D(t)) - \alpha E(t), \\
\dot{I}(t) & = & \alpha E(t) - \beta I(t), \\
\dot{I}_d(t) & = & \beta_1 \beta I(t) - \theta I_d(t), \\
\dot{I}_u(t) & = & \beta_2 \beta I(t) - \eta I_u(t), \\
\dot{H}_R(t) & = & \omega_1 \theta I_d(t) - (\rho + v(t)) H_R(t), \\
\dot{H}_D(t) & = & \omega_2 \theta I_d(t) + (1-\xi)\rho H_R(t) - (\gamma + \nu + v(t)) H_D(t), \\
\dot{R}_{I_d}(t) & = & \xi \rho H_R(t) + \nu H_D(t) + v(t)(H_R(t) + H_D(t)), \\
\dot{R}_{I_u}(t) & = & \eta I_u(t), \\
\dot{D}(t) & = & \gamma H_D(t).
\end{array}\right. \tag{8}$$

The two functions $u(t)$ and $v(t)$ represent vaccination and treatment. These controls function are assumed to be elements of U,

$$U = \{(u, v) : 0 \leqslant u, v \leqslant 1, t \in [0, t_f], t_f \in \mathbb{R}^+, u, v \text{ are Lebesgue measurable}\}.$$

The two constants $A_1 \geqslant 0, A_2 \geqslant 0$ are weighted cost with the use of the controls u and v respectively.

Theorem 4.1. ***(Existence of optimal control)***
Consider the optimal control problem (6) *subject to* (7)*. Then there exists an optimal control* (u^*, v^*) *in* U *and a corresponding solution*

$X^*(t) = (S^*, E^*, I^*, I_d^*, I_u^*, H_R^*, H_D^*, R_{I_d}^*, R_{I_u}^*, D^*)$ *that minimize* $J(u,v)$ *over set of admissible controls* U.

Proof. To prove the existence of optimal control, we use the lemma of Fellippo-Cesari [12]. Then, we have to show the following points :

(1) The set of controls and the corresponding solutions is no empty.
(2) The set of admissible controls U is convex and closed in $L^2(0,T)$.
(3) The vectors field of state system is borned by linear control fonction.
(4) The integrante of objective fonction

$$f^0(X(t),u(t),v(t)) = I(t) - R_{I_d}(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t)$$

is convex. The hessien matrix of f^0 on U is :

$$H = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

We have $spec(M) = \{A_1, A_2\} \subset \mathbb{R}_+^*$, then f^0 is strictly convex.

(5) It exists constants $k_1, k_2 > 0$ et $\rho > 1$ such as the integrante f^0 of objective function verifiy $f^0(X(t),u(t),v(t)) \geqslant k_1|(u,v)|^\rho - k_2$. We have :

$$\begin{aligned} f^0(X(t),u(t),v(t)) &= I(t) - R_{I_d}(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) \\ &\geqslant \frac{1}{2}\min(A_1,A_2)(u(t)^2 + v(t)^2) + I(t) - R_{I_d}(t) \\ &\geqslant \frac{1}{2}\min(A_1,A_2)||(u,v)||_2^2 - R_{I_d}(t) \end{aligned}$$

$N(t) = S(t) + E(t) + I(t) + R(t)$ is borned, then $R(t)$ is borned too. Thus, it exists $\tau_1, \tau_2 \in \mathbb{R}_+$ such as $\tau_1 < R(t) = R_{I_d}(t) + R_{I_u}(t) < \tau_2, \forall\ t \in \mathbb{R}_+$.
Let $k_1 = \frac{1}{2}\min(A_1,A_2)$ and $k_2 = \tau_2$. We get :

$$f^0(X(t),u(t),v(t)) \geqslant k_1||(u,v)||_2^2 - k_2.$$

Proposition 4.1. ***(Hamiltonian characterization of minimization problem)***
The minimization problem (6) *induce to a problem of minimization of Hamiltonian* H *defined by :*

$$H(X(t),p(t),p^0,u(t),v(t)) = I(t) - R_{I_d}(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) + \sum_{i=1}^{10}\lambda_i f_i, \quad (9)$$

where f_i *is the right side of the differential equation of* i^{th} *state variable,* $p(\cdot)$ *is absolutely continuous application defined to* $[0,t_f] \longrightarrow \mathbb{R}^n \setminus \{0\}$*,* p^0 *a positive or null real and*
$X(t) = (S, E, I, I_d, I_u, H_R, H_D, R_{I_d}, R_{I_u}, D)$.

Proof. Let
$p^0 = 1$,
$p(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), ..., \lambda_{10}(t))$,

$$\begin{aligned} f^0(X(t),u(t),v(t)) &= I(t) - R_{I_d}(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t), \\ f(X(t),u(t),v(t)) &= (f_1(X(t),u(t),v(t)), f_2(X(t),u(t),v(t)), ..., f_{10}(X(t),u(t),v(t))) \end{aligned}$$

where

$$X(t) = (S, E, I, I_d, I_u, H_R, H_D, R_{I_d}, R_{I_u}, D)$$

and

$$\begin{aligned}
f_1(X(t), u(t), v(t)) &= -(1-u(t))\frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)), \\
f_2(X(t), u(t), v(t)) &= (1-u(t))\frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)) - \alpha E(t), \\
f_3(X(t), u(t), v(t)) &= \alpha E(t) - \beta I(t), \\
f_4(X(t), u(t), v(t)) &= \beta_1 \beta I(t) - \theta I_d(t), \\
f_5(X(t), u(t), v(t)) &= \beta_2 \beta I(t) - \eta I_u(t), \\
f_6(X(t), u(t), v(t)) &= \omega_1 \theta I_d(t) - (\rho + v(t)) H_R(t), \\
f_7(X(t), u(t), v(t)) &= \omega_2 \theta I_d(t) + (1-\xi)\rho H_R(t) - (\gamma + \nu + v(t)) H_D(t), \\
f_8(X(t), u(t), v(t)) &= \xi \rho H_R(t) + \nu H_D(t) + v(t)(H_R(t) + H_D(t)), \\
f_9(X(t), u(t), v(t)) &= \eta I_u(t), \\
f_{10}(X(t), u(t), v(t)) &= \gamma H_D(t).
\end{aligned}$$

Then, the Hamiltonian of optimal problem is defined by

$$\begin{aligned}
H(t, X, p, p^0, u, v) &= \langle p, f(X(t), u(t), v(t)) \rangle + p^0 f^0(X(t), u(t), v(t)) \\
&= \langle (\lambda_1, \lambda_2, ..., \lambda_{10}), (f_1, f_2, ..., f_{10}) \rangle + p^0 f^0 \\
&= \sum_{i=1}^{10} \lambda_i f_i + I(t) - R_{I_d}(t) + \frac{A_1}{2} u^2(t) + \frac{A_2}{2} v^2(t) \\
H(t, x, p, p^0, u, v) &= I(t) - R_{I_d}(t) + \frac{A_1}{2} u^2(t) + \frac{A_2}{2} v^2(t) + \sum_{i=1}^{10} \lambda_i f_i.
\end{aligned}$$

Proposition 4.2. ***(Existence of adjoint vector*** $p(\cdot)$***)***

The application $p(\cdot)$

$$\begin{aligned}
p(\cdot): \quad [0, t_f] &\longrightarrow \mathbb{R}^{10} \\
t &\longmapsto (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t), \lambda_6(t), \lambda_6(t), \lambda_7(t), \lambda_8(t), \lambda_9(t), \lambda_{10}(t))
\end{aligned}$$

and verify

$$\begin{cases} \dot{\lambda_1} &= (1-u(t))\dfrac{\lambda_1-\lambda_2}{N}(m_E(t)\beta_E E(t)+m_I(t)\beta_I I(t)+m_{I_d}(t)\beta_{I_d}I_d(t) \\ &+ m_{I_u}(t)\beta_{I_u}I_u(t)+m_{H_R}(t)\beta_{H_R}H_R(t)+m_{H_D}(t)\beta_{H_D}H_D(t)), \\ \dot{\lambda_2} &= (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_E(t)\beta_E+(\lambda_2-\lambda_3)\alpha, \\ \dot{\lambda_3} &= (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_I(t)\beta_I \\ &+ (\lambda_3-\lambda_4\beta_1-\lambda_5\beta_2)\beta-1, \\ \dot{\lambda_4} &= (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{I_d}(t)\beta_{I_d} \\ &+ (\lambda_4-\lambda_6\omega_1-\lambda_7\omega_2)\theta, \\ \dot{\lambda_5} &= (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{I_u}(t)\beta_{I_u}+(\lambda_5-\lambda_9)\eta, \\ \dot{\lambda_6} &= (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{H_R}(t)\beta_{H_R} \\ &+ (\lambda_6-\lambda_7)\rho+(\lambda_7-\lambda_8)\xi\rho+(\lambda_6-\lambda_8)v(t) \\ \dot{\lambda_7} &= (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{H_D}(t)\beta_{H_D} \\ &+ (\lambda_7-\lambda_8)(\nu+v(t))+(\lambda_7-\lambda_{10})\gamma, \\ \dot{\lambda_8} &= 1, \quad \dot{\lambda_9}=\dot{\lambda_{10}}=0, \\ \lambda_i(t_f) &= 0 \quad \forall\ i\ \in\{1,2,...,10\}. \end{cases} \tag{10}$$

Proof. According to the theorem (4.1) the couple of controls (u^*,v^*) associated to the solution X^* minimize $J(u,v)$ sur U. According to the maximum principle of Pontryagin, it exists a absolutely continuous application

$$\begin{array}{rccl} p(\cdot): & [0,t_f] & \longrightarrow & \mathbb{R}^{10} \\ & t & \longmapsto & (\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t)),,\lambda_5(t),...,,\lambda_{10}(t)) \end{array}$$

such as for almost all $t\in[0,t_f]$

$$\dot{p}(t)=-\frac{\partial H}{\partial X} \quad \text{and} \quad p(t_f)=0.$$

Then,

$$\dot{p}(t) = -\frac{\partial H}{\partial X} \Longrightarrow \begin{cases} \dot{\lambda_1} = -\dfrac{\partial H}{\partial S}, \\ \dot{\lambda_2} = -\dfrac{\partial H}{\partial E}, \\ \dot{\lambda_3} = -\dfrac{\partial H}{\partial I}, \\ \dot{\lambda_4} = -\dfrac{\partial H}{\partial I_d}, \\ \dot{\lambda_5} = -\dfrac{\partial H}{\partial I_u}, \\ \dot{\lambda_6} = -\dfrac{\partial H}{\partial H_R}, \\ \dot{\lambda_7} = -\dfrac{\partial H}{\partial H_D}, \\ \dot{\lambda_8} = -\dfrac{\partial H}{\partial R_{I_d}}, \\ \dot{\lambda_9} = -\dfrac{\partial H}{\partial R_{I_u}}, \\ \dot{\lambda_{10}} = -\dfrac{\partial H}{\partial D}. \end{cases} \tag{11}$$

Then, we have:
$$\begin{aligned}
\dot{\lambda_1} &= -\frac{\partial H}{\partial S}(t, X, p, p^0, u, v) \\
&= -\frac{\partial}{\partial S}(I(t) - R_{I_d}(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) + \sum_{i=1}^{10} \lambda_i f_i(X, u, v)), \\
&= -\sum_{i=1}^{10} \lambda_i \frac{\partial f_i}{\partial S}(X, u, v), \\
&= -\lambda_1 \frac{\partial f_1}{\partial S}(X, u, v) - \lambda_2 \frac{\partial f_2}{\partial S}(X, u, v), \\
&= (1 - u(t))\frac{\lambda_1}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)) \\
&- (1 - u(t))\frac{\lambda_2}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)) \\
\dot{\lambda_1} &= (1 - u(t))\frac{\lambda_1 - \lambda_2}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)).
\end{aligned}$$

By the same method, we have :

$$
\begin{aligned}
\dot{\lambda_2} &= (\lambda_1-\lambda_2)(1-u(t))\frac{S(t)}{N}m_E(t)\beta_E+(\lambda_2-\lambda_3)\alpha,\\
\dot{\lambda_3} &= (\lambda_1-\lambda_2)(1-u(t))\frac{S(t)}{N}m_I(t)\beta_I\\
&\quad +(\lambda_3-\lambda_4\beta_1-\lambda_5\beta_2)\beta-1,\\
\dot{\lambda_4} &= (\lambda_1-\lambda_2)(1-u(t))\frac{S(t)}{N}m_{I_d}(t)\beta_{I_d}\\
&+ (\lambda_4-\lambda_6\omega_1-\lambda_7\omega_2)\theta,\\
\dot{\lambda_5} &= (\lambda_1-\lambda_2)(1-u(t))\frac{S(t)}{N}m_{I_u}(t)\beta_{I_u}+(\lambda_5-\lambda_9)\eta\\
\dot{\lambda_6} &= (\lambda_1-\lambda_2)(1-u(t))\frac{S(t)}{N}m_{H_R}(t)\beta_{H_R}\\
&+ (\lambda_6-\lambda_7(1-\xi)-\lambda_8)\rho+(\lambda_6-\lambda_8)v(t),\\
\dot{\lambda_7} &= (\lambda_1-\lambda_2)(1-u(t))\frac{S(t)}{N}m_{H_D}(t)\beta_{H_D}\\
&+ (\lambda_7-\lambda_8)(\nu+v(t))+(\lambda_7-\lambda_{10})\gamma,\\
\dot{\lambda_8} &= 1,\\
\dot{\lambda_9} &= \dot{\lambda_{10}}=0.
\end{aligned}
$$

The condition of transversality at final time t_f is $p(t_f)=0$. Then,

$$
p(t_f)=0 \Longrightarrow \begin{cases}
\lambda_1(t_f)=0,\\
\lambda_2(t_f)=0,\\
\lambda_3(t_f)=0,\\
\lambda_4(t_f)=0,\\
\lambda_5(t_f)=0,\\
\lambda_6(t_f)=0,\\
\lambda_7(t_f)=0,\\
\lambda_8(t_f)=0,\\
\lambda_9(t_f)=0,\\
\lambda_{10}(t_f)=0.
\end{cases} \tag{12}
$$

Finally, the characteristics of the vector
$p(\cdot): t \longmapsto (\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t),\lambda_5(t),\lambda_6(t),\lambda_7(t),\lambda_8(t),\lambda_9(t),\lambda_{10}(t))$ are

$$
\left\{
\begin{array}{lcl}
\dot{\lambda_1} & = & (1-u(t))\dfrac{\lambda_1-\lambda_2}{N}(m_E(t)\beta_E E(t)+m_I(t)\beta_I I(t)+m_{I_d}(t)\beta_{I_d}I_d(t) \\
 & + & m_{I_u}(t)\beta_{I_u}I_u(t)+m_{H_R}(t)\beta_{H_R}H_R(t)+m_{H_D}(t)\beta_{H_D}H_D(t)), \\
\dot{\lambda_2} & = & (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_E(t)\beta_E+(\lambda_2-\lambda_3)\alpha, \\
\dot{\lambda_3} & = & (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_I(t)\beta_I+(\lambda_3-\lambda_4\beta_1-\lambda_5\beta_2)\beta-1, \\
\dot{\lambda_4} & = & (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{I_d}(t)\beta_{I_d}+(\lambda_4-\lambda_6\omega_1-\lambda_7\omega_2)\theta, \\
\dot{\lambda_5} & = & (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{I_u}(t)\beta_{I_u}+(\lambda_5-\lambda_9)\eta, \\
\dot{\lambda_6} & = & (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{H_R}(t)\beta_{H_R} \\
 & + & (\lambda_6-\lambda_7)\rho+(\lambda_7-\lambda_8)\xi\rho+(\lambda_6-\lambda_8)v(t), \\
\dot{\lambda_7} & = & (\lambda_1-\lambda_2)(1-u(t))\dfrac{S(t)}{N}m_{H_D}(t)\beta_{H_D} \\
 & + & (\lambda_7-\lambda_8)(\nu+v(t))+(\lambda_7-\lambda_{10})\gamma, \\
\dot{\lambda_8} & = & 1, \\
\dot{\lambda_9} & = & \dot{\lambda_{10}}=0, \\
\lambda_i(t_f) & = & 0 \quad \forall\ i\ \in\{1,2,...,10\}.
\end{array}
\right. \tag{13}
$$

Theorem 4.2. ***(Characterization of optimal control)***
The optimal control (u^*,v^*) is defined by :

$$u^*=\min\left(1,\max\left(0,\frac{(\lambda_2-\lambda_1)S(t)M_1(X^*(t))}{A_1N}\right)\right)$$

and

$$v^*=\min\left(1,\max\left(0,\frac{M_2(X^*(t))}{A_2}\right)\right)$$

where

$$
\begin{array}{rcl}
M_1(X^*,t) & = & m_E(t)\beta_E E^*(t)+m_I(t)\beta_I I^*(t)+m_{I_d}(t)\beta_{I_d}I_d^*(t)+m_{I_u}(t)\beta_{I_u}I_u^*(t) \\
 & + & m_{H_R}(t)\beta_{H_R}H_R^*(t)+m_{H_D}(t)\beta_{H_D}H_D^*(t)
\end{array}
$$

and

$$M_2(X^*,t) \quad = \quad (\lambda_8-\lambda_6)H_R^*(t)+(\lambda_8-\lambda_7)H_D^*(t).$$

Proof. To prove the characterizations of optimal contol, we define the Lagrangien associated to the problem. It corresponds to Hamiltonian increased by coefficients of penality.

$$L(t,X,u,v,p)=H(t,X,p,p^0,u,v)+w_{11}u+w_{12}(1-u)+w_{21}v+w_{22}(1-v),$$

where $w_{ij}(t)\geqslant 0$ are penalisation coefficients that verify

$$w_{11}u(t)=w_{12}(1-u(t))=0 \quad \text{for the control} \quad u^*$$

and

$$w_{21}v(t)=w_{22}(1-v(t))=0 \quad \text{for the control} \quad v^*.$$

The optimal control (u^*, v^*) obtained is the resultant of application of equations of contrainte

$$\begin{cases} \dfrac{\partial L}{\partial u} = 0 \quad in \quad u^*, \\ \dfrac{\partial L}{\partial v} = 0 \quad in \quad v^*. \end{cases}$$

That imply,

$$\begin{cases} \dfrac{\partial H}{\partial u} - w_{11} + w_{12} = 0 \quad in \quad u^*, \\ \dfrac{\partial H}{\partial v} - w_{21} + w_{22} = 0 \quad in \quad v^*. \end{cases}$$

The partial derivative of H in relation to u is given by

$$\begin{aligned}
\frac{\partial H}{\partial u} &= \frac{\partial}{\partial u}\left(I(t) - R_{I_d}(t) + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) + \sum_{i=1}^{10}\lambda_i f_i(X,u,v)\right), \\
&= A_1 u(t) + \lambda_1 \frac{\partial f_1}{\partial u} + \lambda_2 \frac{\partial f_2}{\partial u}, \\
&= A_1 u(t) + \lambda_1 \frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)) \\
&- \lambda_2 \frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)), \\
\frac{\partial H}{\partial u} &= A_1 u(t) + (\lambda_1 - \lambda_2)\frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\
&+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)).
\end{aligned}$$

The partial derivative of H in relation to v is given by

$$\begin{aligned}
\frac{\partial H}{\partial v} &= \frac{\partial}{\partial v}\left(I(t) - R_{I_d} + \frac{A_1}{2}u^2(t) + \frac{A_2}{2}v^2(t) + \sum_{i=1}^{10}\lambda_i f_i(X,u,v)\right), \\
&= A_2 v(t) + \lambda_6 \frac{\partial f_6}{\partial v} + \lambda_7 \frac{\partial f_7}{\partial v} + \lambda_8 \frac{\partial f_8}{\partial v}, \\
&= A_2 v(t) + \lambda_6 \frac{\partial}{\partial v}(\omega_1 \theta I_d(t) - (\rho + v(t))H_R(t)) \\
&+ \lambda_7 \frac{\partial}{\partial v}(\omega_2 \theta I_d(t) + (1-\xi)\rho H_R(t) - (\gamma + \nu + v(t))H_D(t)) \\
&+ \lambda_8 \frac{\partial}{\partial v}(\xi \rho H_R(t) + \nu H_D(t) + v(t)(H_R(t) + H_D(t))), \\
&= A_2 v(t) - \lambda_6 H_R(t) - \lambda_7 H_D(t) + \lambda_8 (H_R(t) + H_D(t)), \\
\frac{\partial H}{\partial v} &= A_2 v(t) + (\lambda_8 - \lambda_6)H_R(t) + (\lambda_8 - \lambda_7)H_D(t).
\end{aligned}$$

We obtain

$$\begin{cases} A_1 u(t) + (\lambda_1 - \lambda_2)\dfrac{S(t)}{N}M_1(X(t)) - w_{11} + w_{12} = 0 \quad for \quad u = u^*, \\ A_2 v(t) + M_2(X(t)) - w_{21} + w_{22} = 0 \quad for \quad v = v^*, \end{cases}$$

where

$$\begin{aligned} M_1(X(t)) &= m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) + m_{I_u}(t)\beta_{I_u} I_u(t) \\ &+ m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t) \end{aligned}$$

and

$$M_2(X(t)) = (\lambda_8 - \lambda_6)H_R(t) + (\lambda_8 - \lambda_7)H_D(t).$$

At u^* and v^*, we have: $\bullet A_1 u^* + (\lambda_1 - \lambda_2)\dfrac{S^*(t)}{N} M_1(X^*(t)) - w_{11} + w_{12} = 0$
$u^* = \dfrac{1}{A_1}\left((\lambda_2 - \lambda_1)\dfrac{S^*(t)}{N} M_1(X^*(t)) + w_{11} - w_{12}\right)$, $\bullet A_2 v^* + M_2(X^*(t)) - w_{21} + w_{22} = 0$
$v^* = \dfrac{1}{A_2}(-M_2(X^*(t)) + w_{21} - w_{22})$. Let be the set $\{t : 0 < u^* < 1\}$. We have : $w_{11}u^* = w_{12}(1 - u^*) \Rightarrow w_{11} = w_{12} = 0$, therefore

$$u^* = \frac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t))}{A_1 N}.$$

Let be the set $\{t : u^* = 0\}$. We have $w_{12}(1 - u^*) = 0 \Rightarrow w_{12} = 0$, therefore

$$0 = u^* = \frac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t)) + w_{11}}{A_1 N}.$$

Since $w_{11} \geqslant 0$ then $\dfrac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t))}{A_1 N} \leqslant u^* = 0$.
Thus, on the set $\{t : 0 \leqslant u^* < 1\}$, u^* is defined like the following :

$$\max\left(0, \frac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t))}{A_1 N}\right).$$

Let be the set $\{t : u^* = 1\}$. We have $w_{11} \times 1 = w_{12} \times 0 = 0 \Rightarrow w_{11} = 0$ then $1 = u^* = \dfrac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t)) - w_{12}}{A_1 N}$.
Since $w_{12} \geqslant 0$ then $\dfrac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t))}{A_1 N} \leqslant u^* = 1$.
On the set $\{t : 0 \leqslant u^* \leqslant 1\}$, u^* is defined by :

$$u^* = \min\left(1, \max\left(0, \frac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t))}{A_1 N}\right)\right).$$

By the same method, we get the expression of v^* :

$$v^* = \min\left(1, \max\left(0, \frac{M_2(X^*(t))}{A_2}\right)\right).$$

Finally on the set U, the optimal control (u^*, v^*) is given by :

$$u^* = \min\left(1, \max\left(0, \frac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t))}{A_1 N}\right)\right)$$

and

$$v^* = \min\left(1, \max\left(0, \frac{M_2(X^*(t))}{A_2}\right)\right)$$

with

$$\begin{aligned} M_1(X^*(t)) &= \frac{S^*(t)}{N}(m_E(t)\beta_E E^*(t) + m_I(t)\beta_I I^*(t) + m_{I_d}(t)\beta_{I_d} I_d^*(t) + m_{I_u}(t)\beta_{I_u} I_u^*(t) \\ &+ m_{H_R}(t)\beta_{H_R} H_R^*(t) + m_{H_D}(t)\beta_{H_D} H_D^*(t)) \\ M_2(X^*(t)) &= (\lambda_8 - \lambda_6)H_R^*(t) + (\lambda_8 - \lambda_7)H_D^*(t). \end{aligned}$$

5. Numerical simulations

In this section, we present the optimality system and the results of simulations obtained by Python 3.7.(see the **Annex** (6)). The system is given by :

$$\left\{ \begin{aligned} \dot{S}(t) &= -(1-u(t))\frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\ &+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) \\ &+ m_{H_D}(t)\beta_{H_D} H_D(t)), \\ \dot{E}(t) &= (1-u(t))\frac{S(t)}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\ &+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) \\ &+ m_{H_D}(t)\beta_{H_D} H_D(t)) - \alpha E(t), \\ \dot{I}(t) &= \alpha E(t) - \beta I(t), \\ \dot{I}_d(t) &= \beta_1 \beta I(t) - \theta I_d(t), \\ \dot{I}_u(t) &= \beta_2 \beta I(t) - \eta I_u(t), \\ \dot{H}_R(t) &= \omega_1 \theta I_d(t) - (\rho + v(t))H_R(t), \\ \dot{H}_D(t) &= \omega_2 \theta I_d(t) + (1-\xi)\rho H_R(t) - (\gamma + \nu + v(t))H_D(t), \\ \dot{R}_{I_d}(t) &= \xi \rho H_R(t) + \nu H_D(t) + v(t)(H_R(t) + H_D(t)), \\ \dot{R}_{I_u}(t) &= \eta I_u(t), \\ \dot{D}(t) &= \gamma H_D(t), \\ \dot{\lambda_1} &= (1-u(t))\frac{\lambda_1 - \lambda_2}{N}(m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_d}(t)\beta_{I_d} I_d(t) \\ &+ m_{I_u}(t)\beta_{I_u} I_u(t) + m_{H_R}(t)\beta_{H_R} H_R(t) + m_{H_D}(t)\beta_{H_D} H_D(t)), \\ \dot{\lambda_2} &= (\lambda_1 - \lambda_2)(1-u(t))\frac{S(t)}{N} m_E(t)\beta_E + (\lambda_2 - \lambda_3)\alpha, \\ \dot{\lambda_3} &= (\lambda_1 - \lambda_2)(1-u(t))\frac{S(t)}{N} m_I(t)\beta_I + (\lambda_3 - \lambda_4\beta_1 - \lambda_5\beta_2)\beta - 1, \\ \dot{\lambda_4} &= (\lambda_1 - \lambda_2)(1-u(t))\frac{S(t)}{N} m_{I_d}(t)\beta_{I_d} + (\lambda_4 - \lambda_6\omega_1 - \lambda_7\omega_2)\theta, \\ \dot{\lambda_5} &= (\lambda_1 - \lambda_2)(1-u(t))\frac{S(t)}{N} m_{I_u}(t)\beta_{I_u} + (\lambda_5 - \lambda_9)\eta, \\ \dot{\lambda_6} &= (\lambda_1 - \lambda_2)(1-u(t))\frac{S(t)}{N} m_{H_R}(t)\beta_{H_R} \\ &+ (\lambda_6 - \lambda_7)\rho + (\lambda_7 - \lambda_8)\xi\rho + (\lambda_6 - \lambda_8)v(t), \\ \dot{\lambda_7} &= (\lambda_1 - \lambda_2)(1-u(t))\frac{S(t)}{N} m_{H_D}(t)\beta_{H_D} \\ &+ (\lambda_7 - \lambda_8)(\nu + v(t)) + (\lambda_7 - \lambda_{10})\gamma, \\ \dot{\lambda_8} &= 1, \\ \dot{\lambda_9} &= \dot{\lambda}_{10} = 0, \\ \lambda_i(t_f) &= 0 \quad \forall\ i\ \in \{1,2,...,10\}, \\ u^* &= \min\left(1, \max\left(0, \frac{(\lambda_2 - \lambda_1)S^*(t)M_1(X^*(t))}{A_1 N}\right)\right), \\ v^* &= \min\left(1, \max\left(0, \frac{M_2(X^*(t))}{A_2}\right)\right), \end{aligned} \right. \tag{14}$$

with

$$\begin{aligned} M_1(X^*(t)) &= \frac{S^*(t)}{N}(m_E(t)\beta_E E^*(t) + m_I(t)\beta_I I^*(t) + m_{I_d}(t)\beta_{I_d} I_d^*(t) + m_{I_u}(t)\beta_{I_u} I_u^*(t) \\ &+ m_{H_R}(t)\beta_{H_R} H_R^*(t) + m_{H_D}(t)\beta_{H_D} H_D^*(t)) \\ M_2(X^*(t)) &= (\lambda_8 - \lambda_6)H_R^*(t) + (\lambda_8 - \lambda_7)H_D^*(t). \end{aligned}$$

The different values we use for simulations are estimated and sum up in the next table:

Parameters	Values	Source
$m_E, m_I, m_{I_u}, m_{I_d}, m_{H_R}, m_{H_D}$	$\gamma(t)$	[6]
β_E	0.2	[6]
β_I	0.2850	[6]
β_{I_u}	0.1222	[6]
β_{I_d}	0.3373	[6]
β_{H_R}	0.126	[6]
β_{H_D}	0.126	[6]
α	0.1	[6]
β	0.2	[6]
β_1	2/10	[6]
β_2	8/10	[6]
θ	0.5	[6]
ω_1	95/100	[6]
ω_2	5/100	[6]
η	0.143	[6]
ρ	0.143	[6]
ξ	0.33	[6]
ν	0.05	[6]
γ	1/9	[6]

TABLE 2. The values of parameters.

where $\gamma(t)$ [6] is the contact rate defined by

$$\gamma(t) = \begin{cases} \gamma_0, \quad 0 \leqslant t \leqslant 2, \\ \\ \gamma_0 \exp(-\mu(t-2)), \quad t \geqslant 2. \end{cases} \tag{15}$$

The fonction $\gamma(t)$ describes the contact between the compartments of susceptibles and the infectious compartments. The contact curve can be seen in figure 9.
With $S_0 = 21499800, E_0 = 198, I_0 = 2, I_{d_0} = 2, I_{u_0} = 0, H_{R_0} = 0, H_{d_0} = 0,$ $R_{I_{u_0}} = 0, D_0 = 0$, we obtain :

FIGURE 9. $\gamma(t)$

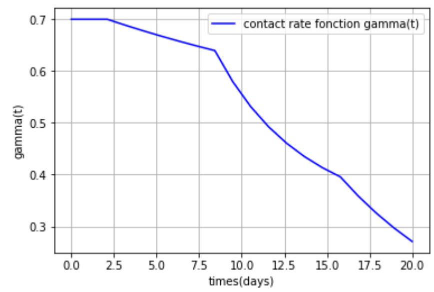


FIGURE 10. S(t)

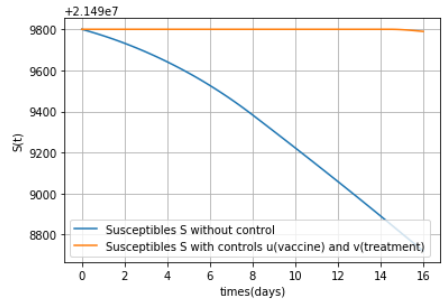


FIGURE 11. E(t)

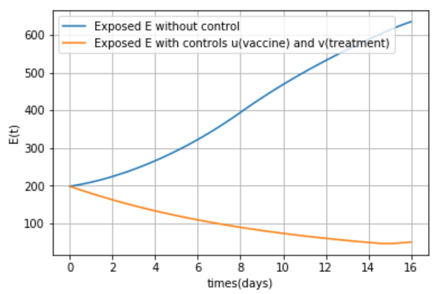


FIGURE 12. I(t)

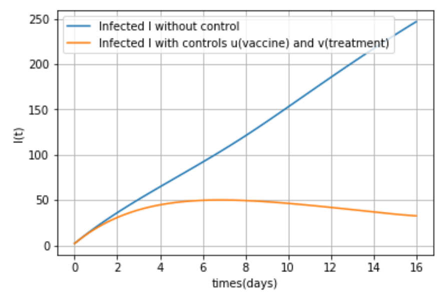


FIGURE 13. $I_d(t)$

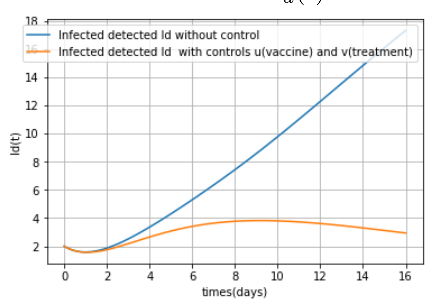


FIGURE 14. $I_u(t)$

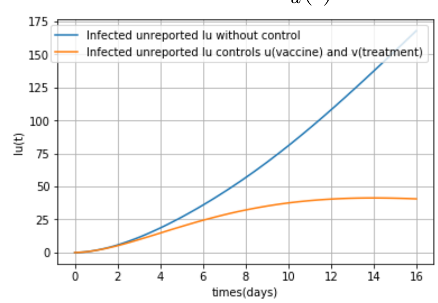


FIGURE 15. $H_R(t)$

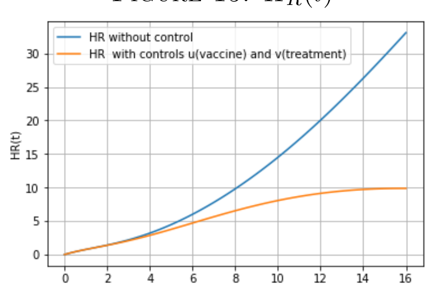


FIGURE 16. $H_D(t)$

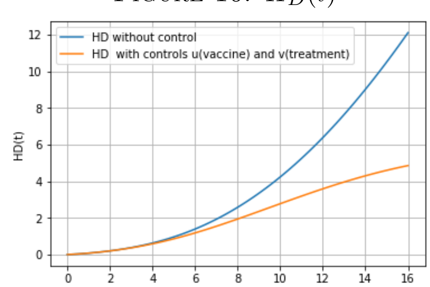


FIGURE 17. $R_{I_d}(t)$

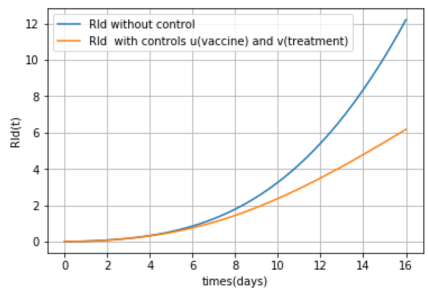


FIGURE 18. $R_{I_u}(t)$

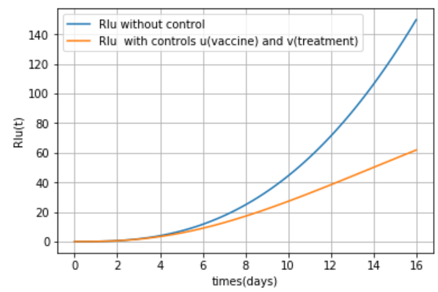


FIGURE 19. $R_{I_d}(t) + R_{I_u}(t)$

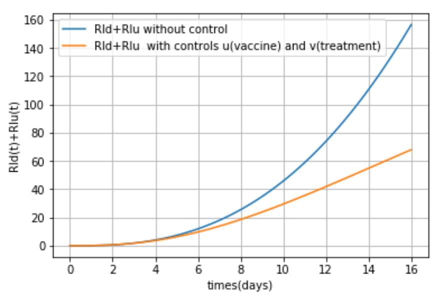
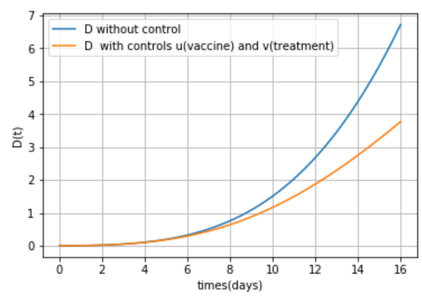


FIGURE 20. D(t)



COMMENTS.

Fig.9 represents the contact rate. Before the beginning of public policies, we assume $\gamma(t) = \gamma_0 = 0.7$ [6]. According to the measures taken by the authorities, the value of μ increases thus reducing the contact rate. Figures $10 - 20$ show the evolution of individuals in the differents compartments with $\mu = 0.05$, $\mu = 0.1$, $\mu = 0.25$.

Fig.10 represents the different dynamics of susceptible population. The orange color represent the population when there are vaccination and treatment ($u \neq 0$ and $v \neq 0$). The blue color represent the population when there aren't vaccination and treatment ($u = 0$ and $v = 0$). This show that, without vaccination and treatment people leave susceptible compartment for exposed compartment. They are therefore more exposed for covid19.

Fig.11 show in orange color the exposed population, when there are vaccination and treatment ($u \neq 0$ and $v \neq 0$) and in blue color the exposed population when there aren't vaccination and treatment ($u = 0$ and $v = 0$). This show that, without vaccination and treatment people, people are more exposed to covid-19 disease.

Fig.12, 13 and 14 represent the different dynamics of infected population. The orange color represent the population when there are vaccination and treatment ($u \neq 0$ and $v \neq 0$). The blue color represent the population when there aren't vaccination and treatment ($u = 0$ and $v = 0$). This show that, without vaccination and treatment, individuals will be more numerous to be infected by the disease.

The curves Fig.15, 16 respectively show us the dynamics of people hospitalized in simple and intensive care. Those in orange represent the dynamics when there are vaccination and treatment ($u \neq 0$ and $v \neq 0$), and those in blue when there are not vaccination and treatment ($u = 0$ and $v = 0$). It is clear that the hospitalizations decrease drastically when there are vaccination and treatment.

Fig.17, 18 and 19 depict the dynamics of people recovering from the disease. We get the orange color curves when there are vaccination and treatment ($u \neq 0$ and $v \neq 0$), and the blue color curves when there aren't vaccination and treatment ($u \neq 0$ and $v \neq 0$). Globally, we observe that there are more people cured when the optimal controls are applied.

Fig.20 represent the different dynamics of death compartment. The orange color represent the population when there are vaccination and treatment ($u \neq 0$ and $v \neq 0$). The blue color represent the population when there aren't vaccination and treatment ($u = 0$ and $v = 0$). This show that, without vaccination and treatment, death rate will increase.

Consider the following tables in which we have summarize the number of person in the different states, before and after applying the controls with a total population $N = 2150000$.

States to the 15^{th} day	Without control	With controls
S	2148809	2149777
E	612	50
I	231	38
I_d	16	3
I_u	152	46
H_R	29	3
H_D	10	0
R_{I_d}	10	21
R_{I_u}	126	62
D	5	0

States to the 30^{th} day	Without control	With controls
S	2147987	2149724
E	504	38
I	288	22
I_d	24	2
I_u	311	30
H_R	72	6
H_D	37	4
R_{I_d}	66	27
R_{I_u}	665	142
D	46	5

Table 3. Table of different states before and after controls.

Comments.

When there is no control, on the 15th day, 16 infected persons are detected compared to 3 when controls are applied. The impact of the controls is still visible on day 30. Indeed, on the 30th day without controls, the number of infected persons detected is 288 against 22 with controls.

6. Conclusion

We have considered an optimal control problem for a SEIHR model with individuals infected reported and unreported. The objective being to describe as best as possible the reality of COVID-19, we took into account the individuals in intensive care and those in simple care.
By applying the Pontryangin's maximum principle, we have proposed an optimal control pair. Then, with estimated data, we exhibited the efficiency of the optimal functions that we determined. These results reveal the importance of the optimal control theory in the fight against COVID-19. Indeed the results of the numerical simulations obtained with control are clearly better than those without control. The optimal control (u^*, v^*) drastically reduces the number of patients.

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Annex

Numerical method with python 3.7

In first, we implement (1), the model without control by using the function odeint of PYTHON. We obtain for example

$$\begin{aligned} def\ F0(Y,t) &:& \\ f &=& [f_1,\\ && f_2,\\ && f_3,\\ && f_4,\\ && f_5,\\ && f_6,\\ && f_7,\\ && f_8,\\ && f_9,\\ && f_{10}]\\ return && f \\ sol &=& odeint(F0, Y0, T), \end{aligned}$$

where Y0 is the initial condition and T the time.
Secondly, we implement the model (14) by using the method of shoot [15]. Let

$$\begin{cases} y = (S, E, I, I_d, I_u, H_R, H_D, R_{I_d}, R_{I_u}, D, \lambda_1, ..., \lambda_{10}) \\ y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, ..., y_{19}, y_{20}). \end{cases} \tag{16}$$

By re-writing the model (14), we get the two point boundary value problem :

$$\begin{cases} \dot{y}(t) = F(t, y(t)), \\ y(0) = (S_0, E_0, I_0, I_{d_0}, ..., R_{I_{d_0}}, R_{I_{u_0}}, D_0, \lambda_1(0), ..., \lambda_{10}(0)) = y^0, \\ y(T) = (S(T), E(T), I(T), I_d(T), ..., R_{I_u}(T), D(T), 0, ..., 0) = y^T. \end{cases} \tag{17}$$

The solution of (17) depends on T and y^0, and be written $y(T, y^0)$.
At final time T,

$$y(T, y^0) = y(T), \tag{18}$$

and this means

$$y(T, y^0) - y(T) = 0. \tag{19}$$

By posing $G(y^0) = y(T, y^0) - y(T)$, the problem becomes :
Find y_0 such that

$$G(y^0) = 0. \tag{20}$$

Solving the system of differential equations (17) is the same as finding a zero of the firing function $G(y^0)$. This is possible with the **fsolve** function in Python.

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