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OSCILLATION THEORY FOR NONLINEAR ADVANCED DIFFERENTIAL EQUATIONS

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Abstract. The purpose of this paper is to investigate the first order nonlinear advanced differential equations with several arguments. Moreover, some new conditions for the solutions of oscillation of these equations are presented. Unlike other research in the literature, advanced arguments are not necessarily monotone. Finally, two examples are given to demonstrate the importance of the main results.

1. Introduction

The theory of oscillation of differential equations is a prominent research area in applied mathematics. Many scientists have focused their efforts in recent decades on developing more complicated numerical and analytical methods for solving mathematical models that arise in all sectors of science, technology, and engineering. Differential equations with advanced type appear by nature physical, biological, and chemical models and have applications in dynamical systems such as network mathematics and optimization, as well as engineering problems such as electrical power systems, energy, and materials. For instance, see from [1] to [16]] and the references cited therein. Advanced differential equations are differential equations in which derivate functions are based on both the present and future values. For broad information on oscillation theory, the reader is directed to monograph [7].
Consider the first order nonlinear advanced differential equation

$$z'(\xi)-\rho(\xi)\Phi(z\left(\varpi_1(\xi)\right),z\left(\varpi_2(\xi)\right),...,z\left(\varpi_n(\xi)\right))=0,\ \xi\geq\xi_0, \tag{1.1}$$

where the functions ρ, $\varpi_i\in C\left([\xi_0,\infty),\mathbb{R}^+\right)$ and $\varpi_i(\xi)$ are not necessarily monotone for $1\leq i\leq n$ such that

$$\varpi_i(\xi)\geq\xi\ \text{ for }\xi\geq\xi_0,\ \lim_{\xi\to\infty}\varpi_i(\xi)=\infty,\ 1\leq i\leq n \tag{1.2}$$

and $\Phi(z\left(\varpi_1(\xi)\right),z\left(\varpi_2(\xi)\right),...,z\left(\varpi_n(\xi)\right))$ is a continuous function on $\mathbb{R}^n$ such that

$$z\Phi(z\left(\varpi_1(\xi)\right),z\left(\varpi_2(\xi)\right),...,z\left(\varpi_n(\xi)\right))>0\text{ for }z\neq0. \tag{1.3}$$

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We refer to a continuously differentiable function defined on $[\varpi_i(T_0), \infty)$ for some $T_0 \geq \xi_0$ such that (1.1) holds for $\xi \geq T_0$, $1 \leq i \leq n$ by a solution of (1.1). A solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. If not, it is called *nonoscillatory*.

If $n = 1$ in (1.1), we have

$$z'(\xi) - \rho(\xi)\Phi(z(\varpi_1(\xi))) = 0, \quad \xi \geq \xi_0. \tag{1.4}$$

When $\Phi(z) = z$, we have the following equation which is the linear form of (1.4).

$$z'(\xi) - \rho(\xi)z(\varpi_1(\xi)) = 0, \quad \xi \geq \xi_0. \tag{1.5}$$

Many studies have been conducted to define oscillation criterion for all solutions of (1.5). You can see these results in [1],[3],[6],[11],[16].

The result given below was obtained by Fukagai and Kusano [6] in 1984 for following type of (1.4).

$$z'(\xi) + \rho(\xi)\Phi(z(\varpi_1(\xi))) = 0, \quad \xi \geq \xi_0. \tag{1.6}$$

Suppose that $\rho(\xi) \leq 0, \varpi_1(\xi) \geq \xi$ is nondecreasing for $\xi \geq \xi_0$ and $\limsup\limits_{|z|\to\infty} \frac{|z|}{|\Phi(z)|} = N_1 < \infty$. Furthermore, assume that

$$\Phi \in C(\mathbb{R}, \mathbb{R}), \ z\Phi(z) > 0 \text{ for } z \neq 0. \tag{1.7}$$

If

$$\liminf_{\xi\to\infty} \int_{\xi}^{\varpi_1(\xi)} [-\rho(s)]\, ds > \frac{N_1}{e}$$

then, all solutions of (1.6) oscillate.

In 2019, Öcalan et. al [14] obtained the criteria given below for the oscillatory solution of (1.4). Assume that (1.2), (1.7) hold and $\limsup\limits_{|z|\to\infty} \frac{z}{\Phi(z)} = N_2$. If $\varpi_1(\xi)$ is not necessarily monotone and

$$\liminf_{\xi\to\infty} \int_{\xi}^{\varpi_1(\xi)} \rho(s)ds > \frac{N_2}{e}, \ 0 \leq N_2 < \infty$$

or

$$\limsup_{\xi\to\infty} \int_{\xi}^{\sigma(\xi)} \rho(s)ds > N_2, \ 0 < N_2 < \infty$$

where $\sigma(\xi) := \inf\limits_{s\geq\xi} \{\varpi_1(s)\}, \xi \geq 0$, then all solutions of (1.4) oscillate.

Fukagai and Kusano [6] also proved the following theorem in 1984 for the following form of (1.1).

$$z'(\xi) + \rho(\xi)\Phi(z(\varpi_1(\xi)), z(\varpi_2(\xi)), ..., z(\varpi_n(\xi))) = 0, \ \xi \geq \xi_0. \tag{1.8}$$

Theorem: Suppose that $\rho(\xi) \leq 0$, (1.2), (1.3) hold, $\varpi_i(\xi)$ are nondecreasing for $1 \leq i \leq n$ and

$$N_3 = \limsup_{|z|\to\infty} \frac{|z(\varpi_1(\xi))|^{\beta_1} |z(\varpi_2(\xi))|^{\beta_2} ... |z(\varpi_n(\xi))|^{\beta_n}}{|\Phi(z(\varpi_1(\xi)), z(\varpi_2(\xi)), ..., z(\varpi_n(\xi)))|} < \infty \tag{1.9}$$

where β_i are nonnegative constants with $\sum_{i=1}^{n}\beta_i = 1$. If there is a continuous non-decreasing function $\varpi_*(\xi)$ such that $\xi \leq \varpi_*(\xi) \leq \varpi_i(\xi)$ for $1 \leq i \leq n$, $\xi \geq a$ and

$$\liminf_{\xi\to\infty} \int_{\xi}^{\varpi_*(\xi)} [-\rho(s)]\, ds > \frac{N_3}{e}$$

then every solution of (1.8) is oscillatory.

The previous studies which are related to the oscillatory solution of (1.1) given in this section and the references are under the assumption that advanced arguments $\varpi_i(\xi)$ are monotone for $1 \leq i \leq n$. All well-known literature conclusions cannot be applied to the circumstance where the advanced arguments are not necessarily monotonous. Then there's the question of how to look into the oscillation of (1.1) when the arguments are not monotone.

To the best of our knowledge, this problem has yet to yield any results. The goal of this study is to find a solution to this problem. As a result, the purpose of this research is to radically evolve these conclusions based on the assumption that $\varpi_i(\xi)$ are not necessarily monotone arguments for $1 \leq i \leq n$. Because a huge number of oscillation conditions for higher-order and nonlinear differential equations can be simplified to oscillation conditions for these equations, the result achieved is significant (1.1).

The paper is arranged as noted below. Firstly, we give a chronological review of the advanced differential equations. Next, we establish some new criteria involving limsup and liminf for the all oscillatory solutions of (1.1). We present two examples to confirm the importance of the main result. Finally, we give the conclusion part.

2. Main Results

In this section, we offer new sufficient criteria for the oscillatory solutions of (1.1) when the advanced arguments $\varpi_i(\xi)$ are not necessarily monotone for $1 \leq i \leq n$. Let

$$\sigma_i(\xi) := \inf_{s\geq\xi} \{\varpi_i(s)\} \text{ and } \sigma(\xi) = \min_{1\leq i\leq n} \{\sigma_i(\xi)\}, \;\; \xi \geq 0. \tag{2.1}$$

Clearly, $\sigma(\xi) \leq \sigma_i(\xi) \leq \varpi_i(\xi)$ and $\sigma_i(\xi)$ are nondecreasing for all $\xi \geq 0$, $1 \leq i \leq n$. Also,

$$M = \limsup_{|z|\to\infty} \frac{z(\varpi(\xi))}{\Phi(z(\varpi_1(\xi)), z(\varpi_2(\xi)), ..., z(\varpi_n(\xi)))}, \;\; \varpi(\xi) = \min_{1\leq i\leq n} \{\varpi_i(\xi)\}. \tag{2.2}$$

The lemmas which are given below will be helpful to obtain our main theorems.

Lemma 2.1 ([13], Lemma 2.2). *Let (1.2) holds and*

$$\liminf_{\xi\to\infty} \int_{\xi}^{\varpi(\xi)} \rho(s)ds = L$$

then, we obtain

$$\liminf_{\xi\to\infty} \int_{\xi}^{\varpi(\xi)} \rho(s)ds = \liminf_{\xi\to\infty} \int_{\xi}^{\sigma(\xi)} \rho(s)ds = L \tag{2.3}$$

where $\varpi(\xi) = \min\limits_{1\le i\le n} \{\varpi_i(\xi)\}$.

Lemma 2.2 ([8], Lemma 2)**.** *Let* $z(\xi)$ *be an eventually positive solution of (1.1). If*

$$\limsup_{\xi\to\infty} \int_{\xi}^{\sigma(\xi)} \rho(s)ds > 0 \tag{2.4}$$

where $\sigma(\xi)$ *is defined by (2.1), then,* $\lim\limits_{\xi\to\infty} z(\xi) = \infty$.

Also, let $z(\xi)$ *be an eventually negative solution of (1.1). If (2.4) satisfies, then,* $\lim\limits_{\xi\to\infty} z(\xi) = -\infty$.

Theorem 2.3. *Suppose that (1.2) and (1.3) are satisfied. If*

$$\liminf_{\xi\to\infty} \int_{\xi}^{\varpi(\xi)} \rho(s)ds > \frac{M}{e},\ 0 \le M < \infty \tag{2.5}$$

where $\sigma(\xi)$ *is defined by (2.1) and* $\varpi(\xi) = \min\limits_{1\le i\le n} \{\varpi_i(\xi)\}$, *then every solution of (1.1) is oscillatory.*

Proof. Assume, for the sake of contradiction, that there is an eventually positive solution $z(\xi)$ of (1.1). If there is an eventually negative solution of (1.1), then the proof of theorem can be done in the same way as shown below. So, there is $\xi_1 > \xi_0$ such that $z(\xi),\ z(\varpi_i(\xi)),\ z(\sigma_i(\xi)) > 0$, for all $\xi \ge \xi_1$, $1 \le i \le n$. Therefore, from (1.1) we get

$$z'(\xi) = \rho(\xi)\Phi(z(\varpi_1(\xi)), z(\varpi_2(\xi)), ..., z(\varpi_n(\xi))) \ge 0$$

for all $\xi \ge \xi_1$, which shows that $z(\xi)$ is an eventually nondecreasing. (2.5) and Lemma 2.2 imply that $\lim\limits_{\xi\to\infty} z(\xi) = \infty$.

Case I: Let $M > 0$. Then, by (2.2), we can take $\xi_2 \ge \xi_1$ so large that

$$\Phi(z(\varpi_1(\xi)), z(\varpi_2(\xi)), ..., z(\varpi_n(\xi))) \ge \frac{1}{2M} z(\varpi(\xi)) \tag{2.6}$$

for $\xi \ge \xi_2$. Since $\sigma(\xi) \le \varpi(\xi)$ and $z(\xi)$ is nondecreasing by (2.6), we have from (1.1)

$$z'(\xi) - \frac{1}{2M}\rho(\xi)z(\varpi(\xi)) \ge 0$$

or

$$z'(\xi) - \frac{1}{2M}\rho(\xi)z(\sigma(\xi)) \ge 0. \tag{2.7}$$

Also, from (2.5) and Lemma 2.1, there is a constant $\lambda > 0$ such that

$$\int_{\xi}^{\sigma(\xi)} \rho(s)ds \ge \lambda > \frac{M}{e},\ \xi \ge \xi_3 \ge \xi_2. \tag{2.8}$$

Moreover, from (2.5) there is a real number $\xi^* \in (\xi, \sigma(\xi))$ for all $\xi \ge \xi_3$ such that

$$\int_{\xi}^{\xi^*} \rho(s)ds > \frac{M}{2e} \text{ and } \int_{\xi^*}^{\sigma(\xi)} \rho(s)ds > \frac{M}{2e}. \tag{2.9}$$

Integrating (2.7) from ξ to ξ^* by using $z(\xi)$ and $\sigma(\xi)$ are nondecreasing and (2.9), we have

$$z(\xi^*) - z(\xi) - \frac{1}{2M}\int_{\xi}^{\xi^*} \rho(s)z(\sigma(s))ds \geq 0,$$

$$z(\xi^*) - z(\xi) - \frac{1}{2M}z(\sigma(\xi))\frac{M}{2e} > 0$$

or

$$z(\xi^*) > \frac{1}{4e}z(\sigma(\xi)). \tag{2.10}$$

Integrating (2.7) from ξ^* to $\sigma(\xi)$ to by using the same facts as above, we have

$$z(\sigma(\xi)) - z(\xi^*) - \frac{1}{2M}\int_{\xi^*}^{\sigma(\xi)} \rho(s)z(\sigma(s))ds \geq 0,$$

$$z(\sigma(\xi)) - z(\xi^*) - \frac{1}{2M}z(\sigma(\xi^*))\frac{M}{2e} > 0$$

or

$$z(\sigma(\xi)) > \frac{1}{4e}z(\sigma(\xi^*)). \tag{2.11}$$

Combining (2.10) and (2.11), we get

$$z(\xi^*) > \frac{1}{4e}z(\sigma(\xi)) > \frac{1}{(4e)^2}z(\sigma(\xi^*))$$

or

$$\frac{z(\sigma(\xi^*))}{z(\xi^*)} < (4e)^2, \ \xi \geq \xi_4. \tag{2.12}$$

Let

$$u = \liminf_{\xi\to\infty}\frac{z(\sigma(\xi))}{z(\xi)} \geq 1 \tag{2.13}$$

and due to $1 \leq u < (4e)^2$, u is finite.
Dividing (1.1) with $z(\xi)$ and integrating from ξ to $\sigma(\xi)$, we obtain

$$\int_{\xi}^{\sigma(\xi)} \frac{z'(s)}{z(s)}ds - \int_{\xi}^{\sigma(\xi)} \rho(s)\frac{\Phi(z(\varpi_1(s)), z(\varpi_2(s)), ..., z(\varpi_n(s)))}{z(s)}ds = 0$$

or

$$\ln\frac{z(\sigma(\xi))}{z(\xi)} - \int_{\xi}^{\sigma(\xi)} \rho(s)\frac{\Phi(z(\varpi_1(s)), z(\varpi_2(s)), ..., z(\varpi_n(s)))}{z(\varpi(s))}\frac{z(\varpi(s))}{z(s)}ds = 0.$$

Also, using the fact that $z(\xi)$ is nondecreasing and $\sigma(\xi) \leq \varpi(\xi)$, we get

$$\ln\frac{z(\sigma(\xi))}{z(\xi)} - \int_{\xi}^{\sigma(\xi)} \rho(s)\frac{\Phi(z(\varpi_1(s)), z(\varpi_2(s)), ..., z(\varpi_n(s)))}{z(\varpi(s))}\frac{z(\sigma(s))}{z(s)}ds \geq 0$$

and there is a μ such that $\xi \leq \mu \leq \sigma(\xi)$. So, we get

$$\ln \frac{z(\sigma(\xi))}{z(\xi)} \geq \frac{\Phi(z\left(\varpi_1(\mu)\right), z\left(\varpi_2(\mu)\right), ..., z\left(\varpi_n(\mu)\right))}{z(\varpi(\mu))} \frac{z(\sigma(\mu))}{z(\mu)} \int_{\xi}^{\sigma(\xi)} \rho(s)ds. \tag{2.14}$$

Finally, if we take lower limit on both side of (2.14), then we obtain $\ln u > \frac{u}{e}$. Since $\ln x \leq \frac{x}{e}$ for all $x > 0$, it is impossible.

Case II: Let $M = 0$. It is clear that $\frac{z(\varpi(\xi))}{\Phi(z(\varpi_1(\xi)), z(\varpi_2(\xi)), ..., z(\varpi_n(\xi)))} > 0$ and

$$\lim_{|z| \to \infty} \frac{z(\varpi(\xi))}{\Phi(z\left(\varpi_1(\xi)\right), z\left(\varpi_2(\xi)\right), ..., z\left(\varpi_n(\xi)\right))} = 0. \tag{2.15}$$

By (2.15), we obtain

$$\frac{z(\varpi(\xi))}{\Phi(z\left(\varpi_1(\xi)\right), z\left(\varpi_2(\xi)\right), ..., z\left(\varpi_n(\xi)\right))} < \varepsilon \tag{2.16}$$

or

$$\frac{\Phi(z\left(\varpi_1(\xi)\right), z\left(\varpi_2(\xi)\right), ..., z\left(\varpi_n(\xi)\right))}{z(\varpi(\xi))} > \frac{1}{\varepsilon} \tag{2.17}$$

where ε is an arbitrary real number. Because of this $\sigma(\xi) \leq \varpi(\xi)$ and $z(\xi)$ is nondecreasing and using (2.17), we have from (1.1)

$$z'(\xi) - \frac{1}{\varepsilon}\rho(\xi)z\left(\varpi(\xi)\right) > 0$$

or

$$z'(\xi) - \frac{1}{\varepsilon}\rho(\xi)z\left(\sigma(\xi)\right) > 0. \tag{2.18}$$

Integrating last inequality from ξ to $\sigma(\xi)$, we get

$$z(\sigma(\xi)) - z(\xi) - \frac{1}{\varepsilon} \int_{\xi}^{\sigma(\xi)} \rho(s)z\left(\sigma(s)\right)ds > 0$$

or

$$z(\sigma(\xi)) \left[1 - \frac{1}{\varepsilon} \int_{\xi}^{\sigma(\xi)} \rho(s)ds\right] > 0.$$

Then, from (2.8), we obtain

$$1 > \frac{\lambda}{\varepsilon}$$

or

$$\varepsilon > \lambda \tag{2.19}$$

but this contradicts with (2.15), hence this completes the proof. □

Theorem 2.4. *Suppose that (1.2) and (1.3) hold. If*

$$\limsup_{\xi \to \infty} \int_{\xi}^{\sigma(\xi)} \rho(s)ds > M, \ 0 < M < \infty, \tag{2.20}$$

where $\sigma(\xi)$ is defined by (2.1), then every solution of (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that there is an eventually positive solution $z(\xi)$ of (1.1). If there is an eventually negative solution of (1.1), then the proof can be done in the same way as shown below. So, there is a $\xi_1 \geq \xi_0$ such that $z(\xi),\ z(\varpi_i(\xi)),\ z(\sigma_i(\xi)) > 0$, for all $\xi \geq \xi_1$, $1 \leq i \leq n$. From Theorem 2.3, $z(\xi)$ is an eventually nondecreasing, also from (2.20) and Lemma 2.2, $\lim_{\xi\to\infty} z(\xi) = \infty$. We have the following statement for $\theta > 1$ by (2.2),

$$\Phi(z(\varpi_1(\xi)), z(\varpi_2(\xi)), ..., z(\varpi_n(\xi))) \geq \frac{1}{\theta M} z(\varpi(\xi)). \tag{2.21}$$

From, (2.20), there is a constant $\Gamma > 0$ such that

$$\limsup_{\xi\to\infty} \int_{\xi}^{\sigma(\xi)} \rho(s)ds = \Gamma > M. \tag{2.22}$$

Since $\Gamma > M$, we get $M < \frac{\Gamma+M}{2} < \Gamma$. Also, by (2.21) and using the fact that $\varpi(\xi) \geq \sigma(\xi)$ and $z(\xi)$ is nondecreasing from (1.1), we obtain

$$z'(\xi) - \frac{1}{\theta M}\rho(\xi) z(\varpi(\xi)) \geq 0$$

or

$$z'(\xi) - \frac{1}{\theta M}\rho(\xi) z(\sigma(\xi)) \geq 0. \tag{2.23}$$

Integrating (2.23) from ξ to $\sigma(\xi)$, we have

$$z(\sigma(\xi)) - z(\xi) - \frac{1}{\theta M} \int_{\xi}^{\sigma(\xi)} \rho(s) z(\sigma(s)) ds \geq 0$$

$$z(\sigma(\xi)) \left[1 - \frac{1}{\theta M} \int_{\xi}^{\sigma(\xi)} \rho(s)ds\right] \geq 0$$

and hence

$$\int_{\xi}^{\sigma(\xi)} \rho(s)ds < \theta M$$

for sufficiently large t. Thus,

$$\limsup_{\xi\to\infty} \int_{\xi}^{\sigma(\xi)} \rho(s)ds \leq \theta M.$$

Because of $\theta > 1$ and $\frac{\Gamma+M}{2M} > 1$, this term can be choosen instead of θ. If the term $\theta = \frac{\Gamma+M}{2M} > 1$ is replaced in the last inequality, we have

$$\limsup_{\xi\to\infty} \int_{\xi}^{\sigma(\xi)} \rho(s)ds = \Gamma \leq \frac{\Gamma+M}{2},$$

which contradicts with $\Gamma > \frac{\Gamma+M}{2}$and this completes the proof. $\square$

Example 2.5. *Consider the first order nonlinear advanced differential equation.*

$$z'(\xi) - \frac{3}{e} z(\varpi_1(\xi)) \ln\left(z(\varpi_1(\xi)) z(\varpi_2(\xi)) + 3\right) = 0, \ \xi \geq 0, \tag{2.24}$$

where

$$\varpi_1(\xi) = \begin{cases} 4\xi - 6a - 2, & \xi \in [2a+1, \ 2a+2] \\ -2\xi + 6a + 10, & \xi \in [2a+2, \ 2a+3] \end{cases}, \quad a \in \mathbb{N}_0$$

$$\varpi_2(\xi) = \varpi_1(\xi) + 2$$

and

$$\sigma_1(\xi) := \inf_{s \geq \xi} \{\varpi_1(s)\} = \begin{cases} 4\xi - 6a - 2, & \xi \in [2a+1, \ 2a+1.5] \\ 2a+4, & \xi \in [2a+1.5, \ 2a+3] \end{cases}, \qquad a \in \mathbb{N}_0.$$

$$\sigma_2(\xi) = \sigma_1(\xi) + 2.$$

Then,

$$\varpi(\xi) = \min_{1 \leq i \leq 2} \{\varpi_i(\xi)\} = \varpi_1(\xi)$$

and

$$M = \limsup_{|z| \to \infty} \frac{z(\varpi_1(\xi))}{z(\varpi_1(\xi)) \ln\left(z(\varpi_1(\xi)) z(\varpi_2(\xi)) + 3\right)} = 0.$$

Now, at $\xi = 2a + 1.5$, $a \in \mathbb{N}_0$, *we obtain*

$$\liminf_{\xi \to \infty} \int_{\xi}^{\varpi(\xi)} \rho(s) ds = \liminf_{\xi \to \infty} \int_{\xi}^{\xi + 2.5} \frac{3}{e} ds = \frac{7.5}{e} > 0 = \frac{M}{e},$$

then all solutions of this equation oscillate.

Example 2.6. *Consider the first order nonlinear advanced differential equation.*

$$z'(\xi) - \frac{2}{e} z(\varpi_1(\xi)) \ln\left(e^{-z(\varpi_2(\xi))} + 4\right) = 0, \ \xi \geq 0, \tag{2.25}$$

where $\varpi_1(\xi) = \xi + 1$, $\varpi_2(\xi) = \xi + 2$ *and* $\varpi(\xi) = \min_{1 \leq i \leq 2} \{\varpi_i(\xi)\} = \varpi_1(\xi)$. *Then, we have*

$$M = \limsup_{|z| \to \infty} \frac{z(\varpi_1(\xi))}{z(\varpi_1(\xi)) \ln\left(e^{-z(\varpi_2(\xi))} + 4\right)} = \frac{1}{\ln 4} \cong 0.72134.$$

Finally, we observe that

$$\limsup_{\xi \to \infty} \int_{\xi}^{\varpi(\xi)} \rho(s) ds = \limsup_{\xi \to \infty} \int_{\xi}^{\xi+1} \frac{2}{e} ds = \frac{2}{e} \cong 0.73575 > 0.72134 = M,$$

then all solutions of this equation oscillate.

3. Conclusion

In this article, a first order nonlinear differential equation of advanced type is considered. Some sufficient conditions for the oscillatory solutions of these equations are established under the assumption that advanced arguments are not necessarily monotone. We profited from the lemmas in order to prove the main results. These results essentially complement and extend some well-known results in the literature. Two examples are presented to illustrate the importance of the main results.

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