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EXISTENCE OF NONTRIVIAL WEAK SOLUTIONS FOR DISCRETE NONLINEAR PROBLEMS IN n-DIMENSIONAL HILBERT SPACE

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Abstract. In this manuscript, we are concerned with the existence results for weak solutions to some p-Laplacian problem in n-dimensional Hilbert space. Our approach is based on the method of the critical point theory. Under suitable conditions we obtain the existence of nontrivial solutions for an energy functional.

1. Introduction

For $a, b \in \mathbb{N}$ with $a \leq b$, we define $\mathbb{N}[a, b]$ as the discrete interval $\{a, a+1, \cdots, b\}$. We investigate the existence of solutions for the following nonlinear discrete Dirichlet boundary value problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \Delta_i\big(\phi_p(\Delta_i u(k_1, \cdots, k_{i-1}, k_i - 1, k_{i+1}, \cdots, k_n))\big) = f(k, u(k)), \\ \hfill k \in \prod_{i=1}^{n} \mathbb{N}[1, T_i], \\ \begin{cases} u(k_1, \cdots, k_{i-1}, 0, k_{i+1}, \cdots, k_n) = 0 \\ u(k_1, \cdots, k_{i-1}, T_i + 1, k_{i+1}, \cdots, k_n) = 0 \end{cases} \quad \forall\ i \in \mathbb{N}[1, n], \end{cases} \tag{1.1}$$

where, for any $i \in \mathbb{N}[1, n]$

$$\begin{aligned} \Delta_i u(k_1, \cdots, k_i, \cdots, k_n) &= u(k_1, \cdots, k_{i-1}, k_i + 1, k_{i+1}, \cdots, k_n) \\ &\quad - u(k_1, \cdots, k_{i-1}, k_i, k_{i+1}, \cdots, k_n) \end{aligned}$$

is the forward difference operator, $f(k, .) : \mathbb{R} \longrightarrow \mathbb{R}$, for all $k \in \prod_{i=1}^{n} \mathbb{N}[1, T_i]$ is a continuous function that will be defined through assumptions and ϕ_p is the p-Laplacian operator given by $\phi_p(s) = |s|^{p-2}s,\ 2 < p < +\infty$.

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The study of this type of problem is motivated by its various applications, for example, computer science, economics, neural networks, ecology, cybernetics, etc. In recent years, many authors have been interested in the study of nonlinear difference equations in one dimensional or two dimensional spaces.
For background and recent results, we refer to [1]-[8] and the references therein. For example, in [6] the authors proved by using critical point theory, the existence of a continuous spectrum of eigenvalues for the problem

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) = \lambda|u(k)|^{q(k)-2}u(k), \ k\in\mathbb{Z}[1,T], \\ \\ u(0)=u(T+1)=0, \end{cases} \tag{1.2}$$

where $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator.
The main purpose of the present paper is to extend the study of difference equations in two dimensions. These models are of independent interest since their mathematical structure has a different nature. So, our goal is to contribute to the generalization of the study of difference equations in higher dimensions. The main obstacle to this generalization is related to the forward difference operator. In two dimensions, there are several ways to overcome this problem (see S. Du and Z. Zhou in [3], I. Ibrango et all in [4], $\cdots$). One of them is to use the definition in [3] where the authors considered the following problem:

$$\Delta_1(\phi_p(\Delta_1 x(i-1,j))) + \Delta_2(\phi_p(\Delta_2 x(i,j-1))) + \lambda f((i,j),x(i,j)) = 0 \tag{1.3}$$

for any $(i,j)\in\mathbb{N}[1,m]\times\mathbb{N}[1,n]$ with

$$\Delta_1 x(i,j) = x(i+1,j)-x(i,j) \quad \text{and} \quad \Delta_2 x(i,j) = x(i,j+1)-x(i,j).$$

Consequently, let us point out that in the literature, to our best knowledge, there were no such existence results for our problem in this situation (dimension $n>2$) which is nevertheless discrete variants of the anisotropic or isotropic partial differential equations (see [5]).

The rest of this paper is arranged as follows. In the next section, we give some basic preliminaries and an illustration. In section 3, we provide our main results that contains several theorem. We end with a conclusion in the last section.

2. Preliminaries

To simplify the notations, in what follows, for any $k=(k_1,\cdots,k_n)\in\prod_{i=1}^{n}\mathbb{N}[0,T_i+1]$, let us put

$$\begin{aligned} k^{i+} &= (k_1,\cdots,k_{i-1},k_i+1,k_{i+1},\cdots,k_n), \\ k^{i-} &= (k_1,\cdots,k_{i-1},k_i-1,k_{i+1},\cdots,k_n), \\ k_0^i &= (k_1,\cdots,k_{i-1},0,k_{i+1},\cdots,k_n), \\ k_{T_i+1}^i &= (k_1,\cdots,k_{i-1},T_i+1,k_{i+1},\cdots,k_n). \end{aligned}$$

Then, the problem (1.1) becomes

$$\begin{cases} -\sum_{i=1}^{n}\Delta_i\left(\phi_p(\Delta_i u(k^{i-})\right) = f(k,u(k)), \ k\in\prod_{i=1}^{n}\mathbb{N}[1,T_i] \\ \\ u(k_0^i) = 0 = u(k_{T_i+1}^i), \quad \forall\ i\in\{1,2,\cdots,n\}, \end{cases} \tag{2.1}$$

with, for any $i \in \mathbb{N}[1,n]$,
$$\Delta_i u(k) = u(k^{i+}) - u(k).$$
In order to facilitate the manipulation of expressions we note
$$\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i} = \sum_{k_1=1}^{T_1}\sum_{k_2=1}^{T_2}\cdots\sum_{k_n=1}^{T_n} \quad \text{and} \quad X = \prod_{i=1}^{n}\mathbb{N}[0, T_i+1].$$
Let us introduce the following Hilbert space
$$E = \left\{u : X \longrightarrow \mathbb{R} \ \text{ such that } \ u(k_0^i) = 0 = u(k_{T_i+1}^i), \ \forall\ i \in \mathbb{N}[1,n]\right\}$$
with the norm
$$\|u\| = \left(\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i u(k^{i-})|^p\right)^{\frac{1}{p}}$$
and the equivalent norm
$$\|u\|_m = \left(\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}|u(k)|^m\right)^{1/m}, \quad \forall\ m \ge 2.$$
For each $k \in X$, the function $f(k,.) : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a function $\alpha : X \longrightarrow [1,+\infty)$ such that
$$|f(k,\xi)| \le \alpha(k)|\xi|^{r(k)-1}, \tag{2.2}$$
where $2 \le r\,(k) < p$ with the notations
$$r^- = \min_{k\in X} r(k) \quad \text{and} \quad r^+ = \max_{k\in X} r(k)$$
that will be used in the rest of the work.
We denote
$$F(k,\xi) = \int_0^{\xi} f(k,s)ds \ \text{ for } \ (k,\xi) \in X \times \mathbb{R} \tag{2.3}$$
and we deduce that
$$|F(k,\xi)| \le \beta(k)|\xi|^{r(k)}, \tag{2.4}$$
with $\beta : X \longrightarrow (0,+\infty)$ is such that for all $k \in X$,
$$0 < \beta^- = \min_{k\in X}(\beta(k)) \le \beta(k) \le \beta^+ = \max_{k\in X}(\beta(k)) < +\infty. \tag{2.5}$$
We need the following assumption to show that the weak solution is non-trivial: there exist constants $C_1, C_2 > 0$ and $\mu > p$ such that
$$F(k,\xi) \ge C_1|\xi|^{\mu} - C_2. \tag{2.6}$$
By analogous arguments as in [3], the following proposition is checked.

Proposition 2.1. *Let $u \in E$. There exists a positive constant C which depends on $T = (T_1, \cdots, T_n)$ and p such that the following inequality*
$$\max_{k\in X}|u(k)| \le C\,(T,p)\,\|u\| \tag{2.7}$$
is verified.

Remark 2.2. (Illustration dimension $n = 3$)
Let $O(0,0,0)$, $A(T_1+1,0,0)$, $C(0,T_2+1,0)$ and $H(0,0,T_3+1)$.

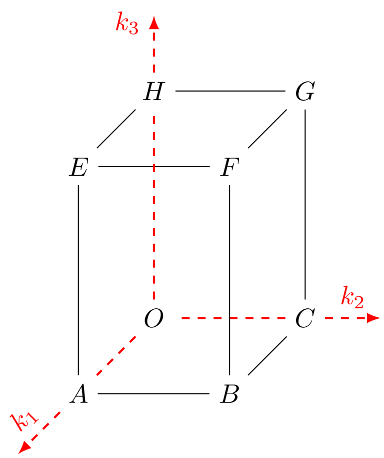

FIGURE 1. Domain $X = \prod_{i=1}^{3} \mathbb{N}[0, T_i + 1]$, $\partial X = X_1 \cup X_2 \cup X_3$

with

$$X_1 = \text{Face}(ABFE) \cup \text{Face}(CGHO) = \{0, T_1+1\} \times \mathbb{N}[0, T_2+1] \times \mathbb{N}[0, T_3+1],$$

$$X_2 = \text{Face}(BCGF) \cup \text{Face}(AOHE) = \mathbb{N}[0, T_1+1] \times \{0, T_2+1\} \times \mathbb{N}[0, T_3+1]$$

and

$$X_3 = \text{Face}(ABCO) \cup \text{Face}(FGHE) = \mathbb{N}[0, T_1+1] \times \mathbb{N}[0, T_2+1] \times \{0, T_3+1\}.$$

We can write the Dirichlet condition as follow:

$$u(k) = 0, \quad \forall\ k \in \partial X.$$

3. Existence of nontrivial weak solutions

Throughout what follows all constants are positive.

Definition 3.1. By a weak solution for problem (2.1) we understand a function $u \in E$ such that

$$\sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i u(k^{i-})|^{p-2} \Delta_i u(k^{i-}) \Delta_i v(k^{i-}) - \sum_{\substack{k_i=1 \\ 1\le i\le n}}^{T_i} f(k, u(k)) v(k) = 0 \quad (3.1)$$

for any $v \in E$.

Theorem 3.2. *Assume that conditions (2.2) -(2.6) are satisfied. Then, there exists a nontrivial weak solution of the problem (2.1).*

Proof. The energy functional $J : E \longrightarrow \mathbb{R}$, corresponding to problem (2.1), is given by $J = I - L$ where

$$I(u) = \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i+1} \frac{1}{p} |\Delta_i u(k^{i-})|^p \quad \text{and} \quad L(u) = \sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} F(k, u(k))$$

for any $u \in E$.

The functional J is well defined on E, it is of class $C^1\big(E, \mathbb{R}\big)$ and for any $u, v \in E$ we have

$$\begin{aligned}
\langle I'(u), v\rangle &= \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i u(k^{i-})|^{p-2} \Delta_i u(k^{i-}) \Delta_i v(k^{i-}) \\
&= \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i+1} \phi_p\left(\Delta_i u(k^{i-})\right) \Delta_i v(k^{i-}) \\
&= -\sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i} \Delta_i \phi_p\left(\Delta_i u(k^{i-})\right) v(k) \\
&= -\sum_{i=1}^{n} \sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} \Delta_i \phi_p\left(\Delta_i u(k^{i-})\right) v(k) \qquad (3.2)
\end{aligned}$$

and

$$\langle L'(u), v\rangle = \sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} f(k, u(k)) v(k). \qquad (3.3)$$

Then, we obtain that the functional J is differentiable in sense of Gâteaux and its Gâteaux derivative is as follows

$$\langle J'(u), v\rangle = -\sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} \left[\sum_{i=1}^{n} \Delta_i \phi_p\left(\Delta_i u(k^{i-})\right) + f(k, u(k))\right] v(k). \qquad (3.4)$$

For any fixed v in E, we see that the critical point u to J satisfies the problem (2.1). Also, note that, since E is a finite dimensional space, the weak solution coincide with the classical solution of the problem (2.1).
Now let us show that J in coercive on E and bounded from below. To prove the coercivity of J, we may assume that $\|u\| > 1$, and we have

$$
\begin{aligned}
J(u) &= \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} \frac{1}{p}|\Delta_i u(k^{i-})|^p - \sum_{\substack{k_i=1 \\ 1\le i\le n}}^{T_i} F(k,u(k)) \\
&\geq \frac{1}{p}\sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i u(k^{i-})|^p - \sum_{\substack{k_i=1 \\ 1\le i\le n}}^{T_i} \beta(k)|u(k)|^{r(k)} \\
&\geq \frac{1}{p}\|u\|^p - \beta^+ \sum_{\substack{k_i=1 \\ 1\le i\le n}}^{T_i} |u(k)|^{r(k)} \\
&\geq \frac{1}{p}\|u\|^p - \beta^+ \left(\sum_{K_1} |u(k)|^{r^+} + \sum_{K_2} |u(k)|^{r^-} \right) \\
&\geq \frac{1}{p}\|u\|^p - \beta^+ \sum_{K_1} |u(k)|^{r^+} - C_3 \\
&\geq \frac{1}{p}\|u\|^p - \beta^+ \sum_{\substack{k_i=1 \\ 1\le i\le n}}^{T_i} |u(k)|^{r^+} - C_3 \\
&\geq \frac{1}{p}\|u\|^p - \beta^+ \left(C(T,p)\right)^{r^+} \left(\prod_{i=1}^{n} T_i \right) \|u\|^{r^+} - C_3 \\
&\geq \frac{1}{p}\|u\|^p - C_4\|u\|^{r^+} - C_3.
\end{aligned}
$$

Since $p > r^+$, we have $\lim\limits_{\|u\|\to+\infty} J(u) = +\infty$. Thus, J is coercive on E and bounded from below.
Besides, for $||u|| \leq 1$, we have

$$
J(u) \geq \frac{1}{p}\|u\|^p - \beta^+ \left(C(T,p)\right)^{r^+} \left(\prod_{i=1}^{n} T_i \right) \|u\|^{r^+} - C_3 \geq -C_5 - C_3 > -\infty
$$

namely, J is bounded from below.

Since J is continuous, bounded from below and coercive on E, using the relation between critical points of J and problem (2.1), we deduce that J has a minimizer which is a weak solution of problem (2.1).
In what follows, we prove that the solution u is nontrivial. For $u \in E \setminus \{0\}$ and

$t > 1$, we have

$$\begin{aligned} J(tu) &= \frac{1}{p}\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\le j\neq i\le n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i\left(tu(k^{i-})\right)|^p - \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}F(k,tu(k))\\ &\le \frac{t^p}{p}\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\le j\neq i\le n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i\left(u(k^{i-})\right)|^p - \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\left(C_1|tu(k)|^\mu - C_2\right)\\ &\le \frac{t^p}{p}\|u\|^p - C_1t^\mu\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}|u(k)|^\mu + C_2\prod_{i=1}^{n}T_i. \end{aligned}$$

Since $\mu > p$, for sufficiently large $t > 1$ we assert that $J(tu) < 0$. □

4. Conclusion

In this paper, we study the existence of nontrivial weak solutions for discrete nonlinear problems in an n-dimensional Hilbert space. The minimization technique allows us to show that the energy functional admits at least one nontrivial critical point which is a weak solution of the associated problem. The originality of this work lies in the generalization of the space (dimension $n > 2$).

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