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ROUGH STATISTICAL CONVERGENCE FOR GENERALIZED DIFFERENCE SEQUENCES

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Abstract. In this work, we introduce the rough statistical convergence for normed linear spaces with generalized difference sequences. We also define the set of $r-$statistical limit points for generalized difference sequences and discuss some topological properties of this set.

1. Introduction

Although it is said that the notion of statistical convergence, which has remarkable applications in many parts of mathematics, was put forward independently by Fast [6] and Steinhaus[9], in many studies today, the starting point of the idea was called "almost convergence" in the book "Trigonometric Series" published by Zygmund [2] in 1935. After these studies, Schoenberg [12] revised the definition of statistical convergence and expressed this concept as a summability method. Salat [28] first studied the statistical convergent sequence space and some topological properties of this space and obtained some results. Freedman and Sember [1] defined the concept of density, which is closely related to statistical convergence. Fridy [14] defined the notion of "Statistical Limit Points" and Fridy and Orhan [16] defined the notions of "Statistical limit superior and limit inferior". Many mathematicians, especially Connor [17], Kolk [4], Fridy [13], and Fridy and Orhan [15], have contributed to the development of statistical convergence.

Kızmaz [8] defined $c_0(\Delta), c(\Delta)$ and $l_\infty(\Delta)$ sequence spaces, $\Delta x_k = (x_k - x_{k+1})$ for (x_k) real number sequence, and showed that the considered spaces were Banach spaces according to $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$ norm. Et and Colak [21] defined the generalized difference sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ for a positive number m, formed by generalizing these sequence spaces to Δ^m-sequence spaces, l_∞, c, and c_0 being bounded, convergent and null convergent sequence spaces, respectively. Besides, Et and Nuray [19] introduced the notion of Δ^m- statistical convergence by combining the notion of generalized difference sequences with statistical convergence. Aside from the mentioned authors, many others, such as Aydın and Başarır [3], Gümüş and Nuray [7], Et [18], and Et and Başarır [20], researched various properties of this concept.

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Rough convergence in finite-dimensional normed space was conceptualized by Phu [10]. With the help of this concept he gave in finite-dimensional normed spaces, Phu defined the rough limit set of a series and stated that this set is bounded, closed, and convex. In another work, Phu [11] extended rough convergence and related properties to infinite-dimensional normed spaces. Additionally, Aytar [25] defined the concept of rough statistical convergence in normed spaces and examined the basic properties of the set of rough statistical limit points. In his other work Aytar [26] revealed the relations between the rough convergence of a (x_k) real number sequence and its classical kernel. Moreover, Aytar [27] studied rough limit set and rough cluster points. Demir [24] and Demir and Gümüş [22] have examined the idea of rough convergence in difference sequences. On the other hand, Demir and Gümüş [23] defined the rough statistical convergence of the (Δx_k) sequences and examined some topological and algebraic properties of the obtained set of rough statistical limit points. Finally, Karabacak and Or [5] introduced the concept of rough convergence for generalized difference sequences.

In this work, $\mathbb{X} = \mathbb{R}^n$ means the real n-dimensional space with the $\|.\|$. Consider a generalized difference sequence $(\Delta^m x_k)$ such that $x_k \in \mathbb{X}$, $k \in \mathbb{N}$.

2. Preliminaries

Definition 2.1. [1] The natural density of the set $A \subseteq \mathbb{N}$ is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n}$$

where $|\{k \in A : k \leq n\}|$ denotes the number of elements of A that do not exceed n. It can be observed that if the set A is finite, then $\delta(A) = 0$.

Definition 2.2. [6] Let $(\mathbb{R}, |.|)$ be a metric space and (x_k) be a sequence in $\mathbb{R}$ and $x_* \in \mathbb{R}$. Then, a sequence (x_k) is called statistical convergent to x_*, if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - x_*| \geq \varepsilon\}|}{n} = 0$$

and is denoted by $st - \lim_{k\to\infty} x_k = x_*$.

Definition 2.3. [14] Let $(\mathbb{R}, |.|)$ be a metric space and (x_k) be a sequence in $\mathbb{R}$ and $c \in \mathbb{R}$. Then, the number c is referred to as a statistical cluster point of the sequence (x_k), if for all $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - c| < \varepsilon\}) \neq 0$$

and is denoted by Γ_x.

Definition 2.4. [19] Let $(\mathbb{R}, |.|)$ be a metric space and $(\Delta^m x_k)$ be a generalized difference sequence in $\mathbb{R}$. Then, the sequence $(\Delta^m x_k)$ is called Δ^m-statistically convergent to x_*, if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \leq n : |\Delta^m x_k - x_*| \geq \varepsilon\}|}{n} = 0$$

Definition 2.5. [10] Let $(\mathbb{X}, \|.\|)$ be a normed linear space, (x_k) be a sequence in $\mathbb{X}$, $x_* \in \mathbb{X}$ and $r \geq 0$. Then, the sequence (x_k) is said to be $r-$ convergent to x_*, if for all $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $k \geq k_\varepsilon$ implies

$$\|x_k - x_*\| < r + \varepsilon,$$

or equivalently

$$\limsup \|x_k - x_*\| \leq r,$$

and is denoted by $x_k \xrightarrow{r} x_*$. The number x_* is called the $r-$ limit point of the sequence (x_k) and r is referred to as roughness degree. The set

$$LIM^r_{x_k} = \{x_* \in \mathbb{X} : x_k \xrightarrow{\text{r}} x_*\}$$

is referred to as r-limit set of the sequence (x_k).

Definition 2.6. [23] Let $(\mathbb{X}, \|.\|)$ be a normed linear space, (Δx_k) be a difference sequence in $\mathbb{X}$, $x_* \in \mathbb{X}$ and $r \geq 0$. Then, (Δx_k) is said to be rough statistically convergent to x_*, if for all $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : \|\Delta x_k - x_*\| \geq r + \varepsilon\}) = 0$$

or

$$st - \limsup \|\Delta x_k - x_*\| \leq r$$

and is denoted by $\Delta x_k \xrightarrow{\text{r-st}} x_*$. The set

$$st - LIM^r_{\Delta x_k} = \{x_* \in \mathbb{X} : \Delta x_k \xrightarrow{\text{r-st}} x_*\}$$

is referred to as r-statistical limit set of the sequence (Δx_k).

Based on these, let's define the concept of rough statistical convergence for generalized difference sequences and have a look at a number of its properties.

3. Main results

In this part, we defined rough statistical convergence for generalized difference sequences. In addition, we examined some properties of the set of r-statistical limit points of a generalized difference sequence.

Definition 3.1. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$, $x_* \in \mathbb{X}$ and $r \geq 0$. Then, $(\Delta^m x_k)$ is said to be rough statistically convergent to x_*, if for all $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : \|\Delta^m x_k - x_*\| \geq r + \varepsilon\}) = 0$$

or

$$st - \limsup \|\Delta^m x_k - x_*\| \leq r$$

and is denoted by $\Delta^m x_k \xrightarrow{\text{r-st}} x_*$. The set

$$st - LIM^r_{\Delta^m x_k} = \{x_* \in \mathbb{X} : \Delta^m x_k \xrightarrow{\text{r-st}} x_*\}$$

is called to be r-statistical limit set of $(\Delta^m x_k)$.

In Definition 3.1, statistical convergence is obtained when $r = 0$, r is meant the roughness degree. The following examples examine the relationship between statistical convergence and rough statistical convergence for generalized difference sequences.

Example 3.2. Suppose that a generalized difference sequence $(\Delta^m y_k)$ is statistical convergent to y_*. For sufficiently large k, it is impossible to calculate $\Delta^m y_k$ exactly by computer. In addition, $(\Delta^m x_k)$ satisfying $\|\Delta^m x_k - \Delta^m y_k\| \leq r$ for all $k \in \mathbb{N}$. Hence, for any $\varepsilon > 0$ and $r \geq 0$

$$\delta(\{k \in \mathbb{N} : \|\Delta^m y_k - y_*\| \geq \varepsilon\}) = 0$$

and

$$\delta(\{k \in \mathbb{N} : \|\Delta^m x_k - \Delta^m y_k\| > r\}) = 0$$

Because of these, $(\Delta^m x_k)$ is not statistical convergent. However, as the inclusion

$$\{k \in \mathbb{N} : \|\Delta^m y_k - y_*\| \geq \varepsilon\} \supseteq \{k \in \mathbb{N} : \|\Delta^m x_k - y_*\| \geq r + \varepsilon\}$$

provides and $\delta(\{k \in \mathbb{N} : \|\Delta^m y_k - y_*\| \geq \varepsilon\}) = 0$,

$$\delta(\{k \in \mathbb{N} : \|\Delta^m x_k - y_*\| \geq r + \varepsilon\}) = 0$$

is obtained. Consequently, $(\Delta^m x_k)$ is $r-$statistical convergent.

Remark 3.3. If $st - LIM^r_{\Delta^m x_k} \neq \emptyset$, then

$$st - LIM^r_{\Delta^m x_k} = [st - \limsup \Delta^m x_k - r, st - \liminf \Delta^m x_k + r]$$

Example 3.4. The unbounded sequence $(\Delta^m x_k)$ may not rough converge however may be rough statistically convergent. For example, define

$$\Delta^m x_k := \begin{cases} k, & \exists k \in \mathbb{N} \ni k = n^2 \\ 0, & \forall k \in \mathbb{N}, k \neq n^2 \end{cases}$$

in $\mathbb{R}$.

$$\begin{aligned} A(\varepsilon) &= \{k \in \mathbb{N} : |\Delta^m x_k - 0| \geq r + \varepsilon\} \\ &= \{k \in \mathbb{N} : |\Delta^m x_k| \geq r + \varepsilon > 0\} \\ &= \{k \in \mathbb{N} : \Delta^m x_k = k\} \\ &= \{1, 4, 9, 16, \dots\} \end{aligned}$$

and $\delta(A(\varepsilon)) = 0$. Therefore, $0 \in st - LIM^r_{\Delta^m x_k}$ and we obtain $st - LIM^r_{\Delta^m x_k} \neq \emptyset$. Since $st - \limsup \Delta^m x_k = 0$ and $st - \liminf \Delta^m x_k = 0$, by Remark 3.3, we obtain

$$st - LIM^r_{\Delta^m x_k} = [-r, r]$$

In addition, $LIM^r_{\Delta^m x_k} = \emptyset$ for all $r \geq 0$ because $(\Delta^m x_k)$ is the unbounded sequence.

In the previous example, it appears that $st - LIM^r_{\Delta^m x_k} \neq \emptyset$ does not imply $LIM^r_{\Delta^m x_k} \neq \emptyset$. However, the converse is correct, i.e,

$$LIM^r_{\Delta^m x_k} \subseteq st - LIM^r_{\Delta^m x_k}$$

and

$$diam\left(LIM^r_{\Delta^m x_k}\right) \leq diam\left(st - LIM^r_{\Delta^m x_k}\right)$$

where

$$diam\left(LIM^r_{\Delta^m x_k}\right) := \sup\left\{\|y - z\| : y, z \in LIM^r_{\Delta^m x_k}\right\}$$

and

$$diam\left(st - LIM^r_{\Delta^m x_k}\right) := \sup\left\{\|y - z\| : y, z \in st - LIM^r_{\Delta^m x_k}\right\}$$

Theorem 3.5. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$ and $r \geq 0$. Then, $diam\left(st - LIM^r_{\Delta^m x_k}\right) \leq 2r$. In general, $diam\left(st - LIM^r_{\Delta^m x_k}\right)$ has no smaller bound.

Proof. Suppose that $diam\left(st - LIM^r_{\Delta^m x_k}\right) > 2r$. There exists $g, h \in st - LIM^r_{\Delta^m x_k}$ such that

$$\|g-h\| > 2r$$

and get $\varepsilon \in \left(0, \frac{\|g-h\|}{2} - r\right)$. $g, h \in st - LIM^r_{\Delta^m x_k}$ implies that $\delta(K) = \emptyset$ and $\delta(L) = \emptyset$, where

$$K := \{k \in \mathbb{N} : \|\Delta^m x_k - g\| \geq r + \varepsilon\}$$

and

$$L := \{k \in \mathbb{N} : \|\Delta^m x_k - h\| \geq r + \varepsilon\}$$

From the feature of natural density, $\delta(K^c \cap L^c) = 1$. Therefore,

$$\|g-h\| \leq \|\Delta^m x_k - g\| + \|\Delta^m x_k - h\| < 2r + 2\varepsilon < 2r + 2\left(\frac{\|g-h\|}{2} - r\right) = \|g-h\|$$

for all $k \in \delta(K^c \cap L^c)$ which is a contradictions. Therefore, assumption is wrong.

It can easy be to show that $diam(\overline{B}_r(x_*)) = 2r$ where $\overline{B}_r(x_*) := \{y \in \mathbb{X} : \|y - x_*\| \leq r\}$. Consider a sequence $(\Delta^m x_k)$ such that statistically convergent to x_*. For arbitrary $y \in \overline{B}_r(x_*)$,

$$\|\Delta^m x_k - y\| \leq \|\Delta^m x_k - x_*\| + r, \forall k \in \{k \in \mathbb{N} : \|\Delta^m x_k - x_*\| < \varepsilon\}$$

Since the sequence $(\Delta^m x_k)$ is statistically convergent to x_*, $\overline{B}_r(x_*) \subseteq st - LIM^r_{\Delta^m x_k}$ is obtained. Conversely, it is easily show that $st - LIM^r_{\Delta^m x_k} \subseteq \overline{B}_r(x_*)$. Consequently, $st - LIM^r_{\Delta^m x_k} = \overline{B}_r(x_*)$. This says that in general upper bound $2r$ of $diam\left(st - LIM^r_{\Delta^m x_k}\right)$ can not be decreased anymore. □

By Phu [10], there exists a non-negative real number r such that $LIM^r_{\Delta^m x_k} \neq \emptyset$ for a bounded sequence. Since the fact $LIM^r_{\Delta^m x_k} \neq \emptyset$ implies $st - LIM^r_{\Delta^m x_k} \neq \emptyset$, we have following result.

Corollary 3.6. Let$(\mathbb{X}, \|.\|)$ be a normed linear space and $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$. If a sequence $(\Delta^m x_k)$ is bounded, then there exists a $r \geq 0$ such that $st - LIM^r_{\Delta^m x_k} \neq \emptyset$.

The opposite of this end result isn't always proper. If the sequence is statistically bounded, the opposite may correct it. Therefore, the next theorem is given:

Theorem 3.7. Let$(\mathbb{X}, \|.\|)$ be a normed linear space and $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$. Then, a sequence $(\Delta^m x_k)$ is statistically bounded if and only if there is a $r \geq 0$ such that $st - LIM^r_{\Delta^m x_k} \neq \emptyset$.

Proof. Assume that $(\Delta^m x_k)$ is statistically bounded. Therefore, there exists a $M > 0$ such that $\delta(K) = 0$, where $K := \{k \in \mathbb{N} : \|\Delta^m x_k\| \geq M\}$. Define $b := \sup\{\|\Delta^m x_k\| : k \in K^c\}$. Then, $st - LIM^b_{\Delta^m x_k}$ include the orgin of $\mathbb{X}$. Hence, $st - LIM^r_{\Delta^m x_k} \neq \emptyset$.

Suppose that $st - LIM^r_{\Delta^m x_k} \neq \emptyset$. Then, there exists $x_* \in st - LIM^r_{\Delta^m x_k}$. Hence, almost all k $\Delta^m x_k$'s are contained in some ball with any radius greater than r. Therefore, the sequence $(\Delta^m x_k)$ is statistically bounded. □

By Phu [10], $(\Delta^m x') = (\Delta^m x_{k_n})$ is a subsequence $(\Delta^m x_k)$ such that

$$LIM^r_{\Delta^m x_k} \subseteq LIM^r_{\Delta^m x'}$$

However, this condition doesn't realize in the statistical convergence. For example, let

$$\Delta^m x_k := \begin{cases} k, & \exists k \in \mathbb{N} \ni n = k^2 \\ 0, & \forall k \in \mathbb{N}, n \neq k^2 \end{cases}$$

Then, $(\Delta^m x_{k_n}) := ({k_n}^2)$ is a subsequence $(\Delta^m x_k)$. However, $st - LIM^r_{\Delta^m x_k} = [-r, r]$ and $st - LIM^r_{\Delta^m x'} = \emptyset$.

This condition only happens when the subsequence is non-thin. Now, the definition of non-thin subsequence is given:

Definition 3.8. Let$(\mathbb{X}, \|.\|)$ be a normed linear space and $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$. Then, $(\Delta^m x') = (\Delta^m x_{k_n})$ is a non-thin subsequence of $(\Delta^m x_k)$ if $\delta(\{k_n : n \in \mathbb{N}\}) \neq 0$.

Theorem 3.9. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$ and $r \geq 0$. If $(\Delta^m x') = (\Delta^m x_{k_n})$ is a non-thin subsequence of $(\Delta^m x_k)$, then

$$st - LIM^r_{\Delta^m x_k} \subseteq st - LIM^r_{\Delta^m x'}$$

Proof. The proof is straight forward. □

Theorem 3.10. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$ and $r \geq 0$. Then, $st - LIM^r_{\Delta^m x_k}$ is closed.

Proof. Let $y_* \in \overline{st - LIM^r_{\Delta^m x_k}}$. Then, there exists an arbitrary sequence (y_j) in $st - LIM^r_{\Delta^m x_k}$ which converges to some point y_*. For all $\varepsilon > 0$, by definition, there is a $j_{\frac{\varepsilon}{2}}$ such that

$$\|y_{j_{\frac{\varepsilon}{2}}} - y_*\| < \frac{\varepsilon}{2} \text{ and } \delta\left(\left\{k \in \mathbb{N} : \|\Delta^m x_k - y_{j_{\frac{\varepsilon}{2}}}\| \geq r + \frac{\varepsilon}{2}\right\}\right) = 0$$

Take $i \in \left\{k \in \mathbb{N} : \|\Delta^m x_k - y_{j_{\frac{\varepsilon}{2}}}\| < r + \frac{\varepsilon}{2}\right\}$. Then, $\|\Delta^m x_k - y_{j_{\frac{\varepsilon}{2}}}\| < r + \frac{\varepsilon}{2}$ and hence

$$\|\Delta^m x_k - y_*\| \leq \|\Delta^m x_k - y_{j_{\frac{\varepsilon}{2}}}\| + \|y_{j_{\frac{\varepsilon}{2}}} - y_*\| < r + \varepsilon$$

That is, $i \in \{k \in \mathbb{N} : \|\Delta^m x_k - y_*\| < r + \varepsilon\}$, which proves

$$\left\{k \in \mathbb{N} : \|\Delta^m x_k - y_{j_{\frac{\varepsilon}{2}}}\right\} \subseteq \{k \in \mathbb{N} : \|\Delta^m x_k - y_*\| < r + \varepsilon\}$$

$y_{j_{\frac{\varepsilon}{2}}} \in st - LIM^r_{\Delta^m x_k}$ implies that the natural density of the set on the right-hand side of last inequality is equal to 1. Therefore,

$$\delta\left(\left\{k \in \mathbb{N} : \|\Delta^m x_k - y_*\| \geq r + \frac{\varepsilon}{2}\right\}\right) = 0$$

i.e., $y_* \in st - LIM^r_{\Delta^m x_k}$ Consequently, $st - LIM^r_{\Delta^m x_k}$ is closed. □

Theorem 3.11. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$ and $r \geq 0$. Then, $st - LIM^r_{\Delta^m x_k}$ is convex.

Proof. Assume that $y_0, y_1 \in st - LIM^r_{\Delta^m x_k}$. In this case, $\delta(A_1) = \delta(A_2) = 0$, where $A_1 := \{k \in \mathbb{N} : \|\Delta^m x_k - y_0\| \geq r + \varepsilon\}$ and $A_2 := \{k \in \mathbb{N} : \|\Delta^m x_k - y_1\| \geq r + \varepsilon\}$, and $\delta(A_1^c \cap A_2^c) = 1$. In addition,

$$\|\Delta^m x_k - [(1-\lambda)y_0 + \lambda y_1]\| = \|(1-\lambda)(\Delta^m x_k - y_0) + \lambda(\Delta^m x_k - y_1)\| < r + \varepsilon$$

for all $k \in A_1^c \cap A_2^c$ and $\lambda \in [0,1]$. Since $\delta(A_1^c \cap A_2^c) = 1$,

$$\delta(\{k \in \mathbb{N} : \|\Delta^m x_k - [(1-\lambda)y_0 + \lambda y_1]\| \geq r + \varepsilon\}) = 0.$$

Consequently, $st - LIM^r_{\Delta^m x_k}$ is convex. □

Theorem 3.12. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$, $x_* \in \mathbb{X}$ and $r \geq 0$. Then, $(\Delta^m x_k)$ is $r-$statistical convergent to x_* if and only if there exists a generalized difference sequence $(\Delta^m y_k)$ such that

$$\Delta^m y_k \xrightarrow{\text{st}} x_* \text{ and } \|\Delta^m x_k - \Delta^m y_k\| \leq r \ \ (k \in \mathbb{N})$$

Proof. Suppose that $\Delta^m y_k \xrightarrow{\text{st}} x_*$ and $\|\Delta^m x_k - \Delta^m y_k\| \leq r$. For all $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : \|\Delta^m y_k - x_*\| \geq \varepsilon\}) = 0.$$

The inclusion

$$\{k \in \mathbb{N} : \|\Delta^m x_k - x_*\| \geq r + \varepsilon\} \subseteq \{k \in \mathbb{N} : \|\Delta^m y_k - x_*\| \geq \varepsilon\}$$

provides. Since $\delta(\{k \in \mathbb{N} : \|\Delta^m y_k - x_*\| \geq \varepsilon\}) = 0$, $\delta(\{k \in \mathbb{N} : \|\Delta^m x_k - x_*\| \geq r + \varepsilon\}) = 0$. Therefore, the proof is completed.

Now, suppose that $\Delta^m x_k \xrightarrow{\text{r-st}} x_*$. Define

$$\Delta^m y_k := \begin{cases} x_*, & \text{if} \quad \|\Delta^m x_k - x_*\| \leq r \\ \Delta^m x_k + r\frac{x_* - \Delta^m x_k}{\|x_* - \Delta^m x_k\|}, & \text{if} \quad \|\Delta^m x_k - x_*\| > r \end{cases}$$

If $\|\Delta^m x_k - x_*\| \leq r$, then

$$\|\Delta^m y_k - x_*\| = \|x_* - x_*\| = 0$$

If $\|\Delta^m x_k - x_*\| > r$, then

$$\|\Delta^m y_k - x_*\| = \left\| \left(\Delta^m x_k + r\frac{x_* - \Delta^m x_k}{\|x_* - \Delta^m x_k\|}\right) - x_* \right\| \leq \|\Delta^m x_k - x_*\| + r$$

Hence,

$$\|\Delta^m y_k - x_*\| \leq \begin{cases} 0, & \text{if} \quad \|\Delta^m x_k - x_*\| \leq r \\ \|\Delta^m x_k - x_*\| + r, & \text{if} \quad \|\Delta^m x_k - x_*\| > r \end{cases}$$

and

$$\|\Delta^m x_k - \Delta^m y_k\| \leq r$$

for $k \in \mathbb{N}$. $\Delta^m x_k \xrightarrow{\text{r-st}} x_*$ implies

$$st - \limsup \|\Delta^m x_k - x_*\| \leq r$$

Therefore,

$$st - \limsup \|\Delta^m y_k - x_*\| = 0$$

□

Theorem 3.13. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$, $r \geq 0$ and $t \in \Gamma_{\Delta^m x_k}$, which be the set of all statistical cluster points of $\Delta^m x_k$. Then, $\|x_* - t\| \leq r$, for all $x_* \in st - LIM^r_{\Delta^m x_k}$.

Proof. Suppose that $t \in \Gamma_{\Delta^m x_k}$ and $x_* \in st - LIM^r_{\Delta^m x_k}$ such that $\|x_* - t\| > r$. Let's $\varepsilon := \frac{\|x_* - x\| - r}{3}$. In this case,

$$\{k \in \mathbb{N} : \|\Delta^m x_k - t\| < \varepsilon\} \subseteq \{k \in \mathbb{N} : \|\Delta^m x_k - x_*\| \geq r + \varepsilon\}$$

Since $t \in \Gamma_{\Delta^m x}$, $\delta(\{k \in \mathbb{N} : \|\Delta^m x_k - t\| < \varepsilon\}) \neq 0$. Therefore, by last inequation, take

$$\delta(\{k \in \mathbb{N} : \|\Delta^m x_k - x_*\| \geq r + \varepsilon\}) \neq 0$$

which contrast with $x_* \in st - LIM^r_{\Delta^m x_k}$. □

Theorem 3.14. Let$(\mathbb{X}, \|.\|)$ be a normed linear space, $(\Delta^m x_k)$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathbb{X}$, $x_* \in \mathbb{X}$ and $r \geq 0$. Then, a sequence $(\Delta^m x_k)$ statistical convergent to x_* if and only if $st - LIM^r_{\Delta^m x_k} = \overline{B}_r(x_*)$.

Proof. In Theorem 3.5, we proved that if $\Delta^m x_k \xrightarrow{st} x_*$, then $st - LIM^r_{\Delta^m x_k} = \overline{B}_r(x_*)$. Let $st - LIM^r_{\Delta^m x_k} = \overline{B}_r(x_*) \neq \emptyset$. In this case, by Theorem 3.7, $\Delta^m x_k$ is statistical bounded. Now, suppose that $(\Delta^m x_k)$ sequence has two different cluster point such as t_* and x_*. Then, the point

$$z := x_* + \frac{r}{\|x_* - t_*\|}(x_* - t_*)$$

implies

$$\|z - t_*\| = r + \|x_* - t_*\| > r$$

Since t_* is a statistical cluster point, $z \notin st - LIM^r_{\Delta^m x_k}$ is obtained from the last inequality and this contradicts with $\|z - t_*\| = r$ and $st - LIM^r_{\Delta^m x_k} = \overline{B}_r(x_*)$. Therefore, our hypothesis is incorrect. That is x_* is the single statistical cluster point of $\Delta^m x_k$. Consequently, $\Delta^m x_k \xrightarrow{st} x_*$ □

4. Conclusion

This paper studies the concept of rough statistical convergence of the $(\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ sequences, which are called generalized difference sequences. Some topological and algebraic properties of the rough statistical limit points are examined.

In future studies, similar properties can be examined for the concept of ideal convergence for generalized difference sequences. Again, instead of the convergence and statistical convergence concepts used in this study, the results obtained by using different types of convergence can be examined.

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