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NOTE ON THE DISTRIBUTION OF THE DIRICHLET L-FUNCTIONS AT THE a-POINTS OF THE CORRESPONDING Δ-FUNCTIONS

KAMEL MAZHOUDA AND MOHAMMED MEKKAOUI

Abstract. Let $L(s,\chi)$ be a Dirichlet L-function associated with a primitive character χ mod q and a be a non zero complex number. We denote by $\Delta(s,\chi)$ the function which appears in the functional equation $L(s,\chi) = \Delta(s,\chi)L(1-s,\overline{\chi})$ and $\delta_{a,\chi} = \beta_{a,\chi} + i\gamma_{a,\chi}$ the solutions of the equation $\Delta(s,\chi) = a$ which are called a-points of $\Delta(s,\chi)$. In this note, we will prove that for every complex number $a \neq 0$ the mean of the values $L(\delta_{a,\chi},\chi)$ on the sequence of a-points $\delta_{a,\chi}$ of the function $\Delta(s,\chi)$ exists and equals $a+1$.

1. Introduction and main result

Let q be a positive integer and χ be a Dirichlet character modulo q associated with the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}.$$

The series $L(s,\chi)$ converges absolutely and uniformly in the region $Re(s) > 1+\epsilon$, for any $\epsilon > 0$. It therefore represents a holomorphic function on the half-plane $Re(s) > 1$, which further extends to a meromorphic function in the complex plane $\mathbb{C}$. In particular, for the principal character $\chi = 1$, we get back the Riemann zeta function $\zeta(s)$. The function $L(s,\chi)$ has only real zeros in the half plane $Re(s) < 0$, these zeros are called the trivial zeros. If $\chi(-1) = 1$, the trivial zeros of $L(s,\chi)$ are $s = -2n$ for all non-negative integers n. If $\chi(-1) = -1$, the trivial zeros of $L(s,\chi)$ are $s = -2n-1$ for all non-negative integers n. Beside the trivial zeros of $L(s,\chi)$, there are infinitely many non-trivial zeros lying in the strip $0 < Re(s) < 1$.

Let

$$\Delta(s,\chi) = \frac{2\tau(\chi)}{i^\kappa q}\left(\frac{2\pi}{q}\right)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi}{2}(s+\kappa)\right),$$

with $\tau(\chi) = \sum_{r=1}^{q}\chi(r)e^{\frac{2\pi i r}{q}}$ and $\kappa = \frac{1}{2}(1-\chi(-1))$. The function $\Delta(s,\chi)$ appears in the functional equation $L(s,\chi) = \Delta(s,\chi)L(1-s,\overline{\chi})$. Let denote by $\delta_{a,\chi} =$

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$\beta_{a,\chi} + i\gamma_{a,\chi}$ the solutions of the equation $\Delta(s,\chi) = a$ which are called a-points of $\Delta(s,\chi)$.

In this paper, we will prove that for every complex number $a \neq 0$ the mean of the values $L(\delta_{a,\chi},\chi)$ on the sequence of a-points $\delta_{a,\chi}$ of the function $\Delta(s,\chi)$ exists and equals $a+1$; the case $a = 0$ is related to the trivial zeros of $L(s,\chi)$. Therefore, these averages of these $L(s,\chi)$-values attain all but one possible complex limit. This indicates an interesting link between the distribution of $a+1$-points of the Dirichlet L-functions and a-points of $\Delta(s,\chi)$. To do so, we give an asymptotic formula for the sum

$$\sum_{\substack{\delta_{a,\chi}\,:\,0<\gamma_{a,\chi}<T\\ \beta_{a,\chi}>-1}} L(\eta+\delta_{a,\chi},\chi), \qquad \text{as } T \to \infty$$

where $\eta \in (-\epsilon, 1)$ and ϵ is arbitrary. The proof of Lemma 2.1 below will show that for $a \neq 0$ the a-points of $\Delta(s,\chi)$ are clustered around the critical line or, in other words, with increasing imaginary part $\gamma_{a,\chi}$ the real part $\beta_{a,\chi}$ is tending to 1/2. Hence, the critical line $1/2 + i\mathbb{R}$ is the unique vertical Julia line for $\delta_{a,\chi}$ [1]. There are further a-points of $\Delta(s,\chi)$ in the left half-plane, close to zeros of $\Delta(s,\chi)$, the condition $\beta_a > -1$ excludes them with at most finitely many exceptions. Notice that $\Delta(s,\chi)$ is regular except for simple poles at the positive integers $s = 2n+1$, if $\chi(-1) = 1$ and $s = 2n$, if $\chi(-1) = -1$; moreover, $\Delta(s,\chi)$ vanishes exactly for the non-positive integers $s = -2n$ if $\chi(-1) = 1$ and $s = -2n-1$, if $\chi(-1) = -1$. Both, 0 and ∞ are thus deficient values for in the language of value-distribution theory. It appears that the distribution of values of both, $\Delta(s,\chi)$ and $L(s,\chi)$ in the left half-plane is pretty similar (except for the value 0 when $\chi(-1) = 1$). In this context the formula in this lemma should be compared with the (in principle) identical counterpart for $L(s,\chi)$.

The main result is stated in the flowing theorem which extend Steuding & Suriajaya work [8] to the Dirichlet L-functions.

[1] Julia improved the Big Picard-theorem by showing that if the analytic function f has an essential singularity at b, then there exist a real θ_0 and a complex z such that for every sufficiently small $\epsilon > 0$

$$\mathbb{C} - \{z\} \subset f\left(\{a + r\exp(i\theta) : |\theta - \theta_0| < \epsilon, 0 < r < \epsilon\}\right).$$

The ray $\{b + r\exp(i\theta_0) : r > 0\}$ is called Julia line. Steuding in [7] remarked that the distribution of the a-points close to the real axis is quite regularly and it can be shown that there is always a a-point in a neighborhood of any trivial zero of $L(s,\chi)$ (and for any function in the Selberg class), and with finitely many exceptions there are no other in the left half-plane. Moreover, he indicated that the extraordinary value distribution shows that the critical line is a so-called Julia line.

Theorem 1.1. *Let χ be a primitive character modulo q and a be a non zero complex number. Then as $T \to \infty$, we have*

$$\begin{aligned} &\sum_{\substack{\delta_{a,\chi}:0<\gamma_{a,\chi}<T\\ \beta_{a,\chi}>-1}} L(\eta+\delta_{a,\chi},\chi) \\ =\ & \frac{T}{2\pi}\log\left(\frac{qT}{2\pi e}\right)+\frac{a}{q(1-\eta)}\left(\frac{qT}{2\pi}\right)^{1-\eta}\log\left(\frac{qT}{2\pi}\right) \\ & -\frac{a}{q(1-\eta)^2}\left(\frac{qT}{2\pi}\right)^{1-\eta}+O_a((qT)^{\frac{1}{2}+\epsilon}), \qquad (1)\end{aligned}$$

where $\eta \in (-\epsilon, 1)$ and ϵ is arbitrary.

From Theorem 1.1 and Lemma 2.1 below, we deduce the average value of $L(\delta_{a,\chi},\chi)$ over the a-points $\delta_{a,\chi}$ of $\Delta(s,\chi)$ with $0<\mathrm{Im}(\delta_{a,\chi})<T$, i.e.,

$$\lim_{T\to+\infty}\frac{1}{N_{a,\chi}(T)}\sum_{\substack{\delta_{a,\chi}:0<\gamma_{a,\chi}<T\\ \beta_{a,\chi}>-1}} L(\delta_{a,\chi},\chi)=a+1,$$

where $N_{a,\chi}(T)$ is the number of a-points $\delta_{a,\chi}=\beta_{a,\chi}+i\gamma_{a,\chi}$ of $\Delta(s,\chi)$ satisfying $\beta_{a,\chi}>-1$ and $0<\gamma_{a,\chi}<T$.

2. Preliminary lemmas and equations

In this section, we give some lemmas and formulas useful for the proof of our Theorem which its proof uses the same argument as in [8]. We start with well-known results on the Dirichlet L-function $L(s,\chi)$ (see Davenport book [1]).

If χ mod q is a primitive character, then

$$L(s,\chi)=\Delta(s,\chi)L(1-s,\overline{\chi}), \qquad (2)$$

where

$$\Delta(s,\chi)=\frac{2\tau(\chi)}{i^{\kappa}q}\left(\frac{2\pi}{q}\right)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi}{2}(s+\kappa)\right), \qquad (3)$$

with $\tau(\chi)=\sum_{r=1}^{q}\chi(r)e^{\frac{2\pi i r}{q}}$ and $\kappa=\frac{1}{2}(1-\chi(-1))$. The function $\Delta(s,\chi)$ is a meromorphic function with only real zeros and poles satisfying the functional equation

$$\Delta(s,\chi)\Delta(1-s,\overline{\chi})=1.$$

From (3) and by stirling's formula (see[4, page 13]), we get

$$\begin{aligned} &\Delta(s,\chi) \\ =\ & \frac{\tau(\chi)}{i^{\kappa}\sqrt{q}}\exp\left\{-it\log\left(\frac{q|t|}{2\pi e}\right)+sgn(t)\left(\frac{i\pi}{2}\right)\left(\frac{1}{2}-\kappa\right)\right\} \\ & \times\left(\frac{q|t|}{2\pi}\right)^{\frac{1}{2}-\sigma}\left(1+O\left(\frac{1}{|t|}\right)\right) \qquad (4)\end{aligned}$$

in any fixed halfstrip $\alpha\le\sigma\le\beta, |t|\ge 1$. Moreover, for any fixed σ and $|t|\ge 1$, we have

$$\frac{\Delta'}{\Delta}(s,\chi)=-\log\left(\frac{q|t|}{2\pi}\right)+O\left(\frac{1}{|t|}\right). \qquad (5)$$

By the functional equation (2) and the Phragmén-Lindelöf principle, we deduce that[2]

$$L(s,\chi) \ll_\epsilon \begin{cases} |qt|^{\frac{1}{2}-\sigma+\epsilon} & \sigma < 0, \\ |qt|^{\frac{1}{2}(1-\sigma)+\epsilon} & 0 \le \sigma \le 1, \\ |qt|^{\epsilon} & \sigma > 1, \end{cases} \tag{6}$$

as $|t| \to \infty$ and where ϵ is an arbitrarily small positive number.
For a non zero complex number a. We write

$$\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a} = \frac{\Delta'}{\Delta}(s,\chi)\frac{1}{1-\frac{a}{\Delta(s,\chi)}}. \tag{7}$$

From equations (4) and (5), we obtain, for $\sigma > \frac{1}{2}$ and $t \ge t_a > 1$ (t_a is defined below in Lemma 2.1)

$$\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a} \ll_a (qt)^{\frac{1}{2}-\sigma}\log(qt+1). \tag{8}$$

Furthermore, we find for $\sigma < \frac{1}{2}$ that

$$\begin{aligned} \frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a} &= \frac{\Delta'}{\Delta}(s,\chi)\left(1+\sum_{n\ge 1}\left(\frac{a}{\Delta(s,\chi)}\right)^n\right) \\ &= -\log\left(\frac{qt}{2\pi}\right) + O\left(\frac{1}{t}\right) + O_a\left((qt)^{\sigma-\frac{1}{2}}\log(qt+1)\right). \end{aligned} \tag{9}$$

Moreover, for an a-point $\delta_{a,\chi} = \beta_{a,\chi} + i\gamma_{a,\chi}$ of $\Delta(s,\chi)$, it follows from equation (4) that

$$|a| = \left(\frac{q\gamma_{a,\chi}}{2\pi}\right)^{\frac{1}{2}-\beta_{a,\chi}}\left(1+O\left(\frac{1}{\gamma_{a,\chi}}\right)\right) \tag{10}$$

and

$$\phi = \gamma_{a,\chi}\left(\log\left(\frac{2\pi e}{q\gamma_{a,\chi}}\right)\right) + \frac{\pi}{4} + \theta_0 + O\left(\frac{1}{\gamma_{a,\chi}}\right) \mod 2\pi, \tag{11}$$

where $a = \Delta(\delta_{a,\chi},\chi) = |a|\exp(i\phi)$ and $\tau(\chi) = \sqrt{q}\exp(i\theta_0)$.
This shows that

$$\beta_{a,\chi} \to \frac{1}{2} \qquad as \qquad \gamma_{a,\chi} \to \infty. \tag{12}$$

Hence, there exists a positive real number $t_a > 1$, depending only on a, such that all a-points $\delta_{a,\chi} = \beta_{a,\chi} + i\gamma_{a,\chi}$ have a real part $\beta_{a,\chi} \in (-1,2)$ whenever $\gamma_{a,\chi} > t_a$.

[2] From [2] and an application of the Phragmén-Lindelöf principle yields the estimate

$$L(s,\chi) \ll (q(|t|+2))^{\frac{3}{16}+\epsilon} \text{ for } \frac{1}{2} \le \sigma \le 1 + \frac{1}{\log qt}$$

and

$$L(s,\chi) \ll (q(|t|+2))^{\frac{1}{2}}\log(q(|t|+2)) \text{ for } -\frac{1}{\log qt} \le \sigma \le \frac{1}{2}.$$

When we assume the Riemann hypothesis, the first bound can be replaced by $(q(|t|+2))^{\epsilon}$.

Lemma 2.1. *Let χ be a primitive character modulo q and a be a non zero complex number. Then for sufficiently large T, we have*

$$N_{a,\chi}(T) = \frac{T}{2\pi}\log\left(\frac{qT}{2\pi e}\right) + O_a\left(\log(qT)\right), \tag{13}$$

where $N_{a,\chi}(T)$ is the number of a-points $\delta_{a,\chi} = \beta_{a,\chi} + i\gamma_{a,\chi}$ of $\Delta(s,\chi)$ satisfying $\beta_{a,\chi} > -1$ and $0 < \gamma_{a,\chi} < T$.

Proof. To prove this lemma, we use the argument principle theorem to the function $\Delta(s,\chi) - a$ and integrate counterclockwise over the rectangular contour $\mathbf{R}$ determined by the vertices $-1 + it_a$, $2 + it_a$, $2 + iT$ and $-1 + iT$. We have

$$N_{a,\chi}(T) = \sum_{\substack{0 < \gamma_{a,\chi} < T \\ \beta_{a,\chi} > -1}} 1 = \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\Delta'(s,\chi)}{\Delta(s,\chi) - a} ds + O_a(1).$$

Hence, we have

$$\begin{aligned} N_{a,\chi}(T) &= \frac{1}{2\pi i}\left\{\int_{-1+it_a}^{2+it_a} + \int_{2+it_a}^{2+iT} + \int_{2+iT}^{-1+iT} + \int_{-1+iT}^{-1+it_a}\right\} \frac{\Delta'(s,\chi)}{\Delta(s,\chi) - a} ds + O(1) \\ &:= I_1 + I_2 + I_3 + I_4 + O_a(1). \end{aligned}$$

The integral I_1 is independent of T, so we have $I_1 = O_a(1)$. Next, using equations (8) and (9), we get

$$I_2 = \frac{1}{2\pi i}\int_{2+it_a}^{2+iT} \frac{\Delta'(s,\chi)}{\Delta(s,\chi) - a} ds = O_a\left(\int_{t_a}^{T} (qt)^{\frac{1}{2}-2}\log(qt)dt\right) = O_a(\log(qT))$$

and

$$\begin{aligned} I_3 &= \frac{1}{2\pi i}\int_{2+iT}^{-1+iT} \frac{\Delta'(s,\chi)}{\Delta(s,\chi) - a} ds = \frac{1}{2\pi i}\left\{\int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{-1+iT}\right\} \frac{\Delta'(s,\chi)}{\Delta(s,\chi) - a} ds \\ &= \int_{\frac{1}{2}}^{2} O_a\left((qT)^{\frac{1}{2}-\sigma}\log(qT)\right) d\sigma \\ &\quad + \int_{-1}^{\frac{1}{2}} \left\{\log\left(\frac{qT}{2\pi}\right) + O\left(\frac{1}{T}\right) + O_a\left((qT)^{\sigma-\frac{1}{2}}\log(qT)\right)\right\} d\sigma \\ &= O_a(\log(qT)). \end{aligned}$$

Finally, we estimate I_4. By equation (9), we have

$$\begin{aligned} I_4 &= -\frac{1}{2\pi i}\int_{-1+it_a}^{-1+iT} \frac{\Delta'(s,\chi)}{\Delta(s,\chi) - a} ds \\ &= \frac{1}{2\pi}\int_{t_a}^{T} \left(\log\left(\frac{qt}{2\pi}\right) + O\left(\frac{1}{t}\right)\right) dt + O_a(\log(qT)) \\ &= \frac{T}{2\pi}\log\left(\frac{qT}{2\pi}\right) - \frac{T}{2\pi} + O_a(\log(qT)). \end{aligned}$$

Hence, Lemma 2.1 follows from estimates of I_1, I_2, I_3 and I_4. □

Lemma 2.2. *Let χ be a primitive character modulo q and a be a non zero complex number. Then, for $-1 \le \sigma \le 2$ and $t \ge 1$, we have*

$$\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a} = \sum_{|t-\gamma_{a,\chi}|\le 1} \frac{1}{s-\delta_{a,\chi}} + O_a(\log(qt)). \tag{14}$$

Proof. Recall that $\Delta(s,\chi)$ is analytic except for simple poles at $s = 2n+1+\kappa$. Thus, $(\Delta(s,\chi)-a)\Gamma\left(\frac{1-s+\kappa}{2}\right)^{-1}$ is an entire function of order one. Hence, by the Hadamard factorization theorem, we have

$$\Delta(s,\chi)-a = \exp\left(A(\chi)+B(\chi)s\right)\prod_{\delta_{a,\chi}}\left(1-\frac{s}{\delta_{a,\chi}}\right)\exp\left(\frac{s}{\delta_{a,\chi}}\right),$$

where $A(\chi)$ and $B(\chi)$ are certain complex constants and the product is taken over all a-points $\delta_{a,\chi}$ of $\Delta(s,\chi)$. Hence, taking the logarithmic derivative, we get

$$\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a} = -\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\kappa}{2}\right) + B(\chi) + \sum_{\delta_{a,\chi}} \frac{1}{s-\delta_{a,\chi}} + \frac{1}{\delta_{a,\chi}}.$$

It follows from Stirling's formula that

$$\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\kappa}{2}\right) \ll \log(t)$$

and from equation (8), we have

$$\frac{\Delta'(2+it,\chi)}{\Delta(2+it,\chi)-a} \ll_a 1.$$

Using last estimates, we obtain

$$\begin{aligned}
\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a} &= \sum_{\delta_{a,\chi}} \frac{1}{s-\delta_{a,\chi}} - \frac{1}{2+it-\delta_{a,\chi}} + O(\log t) \\
&= \left\{\sum_{|\gamma_{a,\chi}-t|\le 1} + \sum_{\gamma_{a,\chi}>t+1} + \sum_{\gamma_{a,\chi}<t-1}\right\}\left(\frac{1}{s-\delta_{a,\chi}} - \frac{1}{2+it-\delta_{a,\chi}}\right) \\
&\quad + O(\log t) \\
&:= S_1 + S_2 + S_3 + O(\log t).
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
S_1 &= \sum_{|\gamma_{a,\chi}-t|\le 1} \frac{1}{s-\delta_{a,\chi}} - \sum_{|\gamma_{a,\chi}-t|\le 1} \frac{1}{2+it-\delta_{a,\chi}} \\
&= \sum_{|\gamma_{a,\chi}-t|\le 1} \frac{1}{s-\delta_{a,\chi}} + O_a\left(\sum_{|\gamma_{a,\chi}-t|\le 1}\right) \\
&= \sum_{|\gamma_{a,\chi}-t|\le 1} \frac{1}{s-\delta_{a,\chi}} + O_a\left(\log qt\right).
\end{aligned}$$

Moreover, for any positive integer n,

$$\sum_{t+n<\gamma_{a,\chi}\le t+n+1} \frac{1}{s-\delta_{a,\chi}} - \frac{1}{2+it-\delta_{a,\chi}} \ll_a \sum_{t+n<\gamma_{a,\chi}\le t+n+1} \frac{1}{n^2} \ll_a \frac{\log(t+n)}{n^2}.$$

This yields $S_2 = O_a(\log t)$. By the same argument we can estimate the sum S_3 using the same bounds. Then, Lemma 2.2 follows from estimates of S_1, S_2 and S_3. □

3. Proof of Theorem 1.1

The basic idea of the proof is to interpret the sum of $L(\eta+\delta_{a,\chi},\chi)$ as a sum of residues. By Cauchy's theorem, we have

$$\sum_{\substack{\delta_{a,\chi}:0<\gamma_{a,\chi}<T\\ \beta_{a,\chi}>-1}} L(\eta+\delta_{a,\chi},\chi)=\frac{1}{2\pi i}\int_{\mathbf{R}}\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a}L(s,\chi)ds+O_a(1).$$

where the integration is taken over a rectangular contour in counterclockwise direction denoted by $\mathbf{R}$ with vertices $1+\eta+\epsilon+it_a$, $1+\eta+\epsilon+iT$, $-\eta-\epsilon+iT$ and $-\eta-\epsilon+it_a$. Hence,

$$\begin{aligned}
&\sum_{\substack{\delta_{a,\chi}:0<\gamma_{a,\chi}<T\\ \beta_{a,\chi}>-1}} L(\eta+\delta_{a,\chi},\chi)\\
=\;& \frac{1}{2\pi i}\int_{\mathbf{R}}\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a}L(\eta+s,\chi)ds+O_a(1)\\
=\;& \frac{1}{2\pi i}\left\{\int_{-\eta-\epsilon+it_a}^{1+\eta+\epsilon+it_a}+\int_{1+\eta+\epsilon+it_a}^{1+\eta+\epsilon+iT}+\int_{1+\eta+\epsilon+iT}^{-\eta-\epsilon+iT}+\int_{-\eta-\epsilon+iT}^{-\eta-\epsilon+it_a}\right\}\frac{\Delta'(s,\chi)}{\Delta(s,\chi)-a}L(\eta+s,\chi)ds\\
&+O_a(1)\\
:=\;& I_1+I_2+I_3+I_4+O_a(1).
\end{aligned}$$

The integral I_1 is independent of T, so one has $I_1 = O(1)$. Next, we consider I_2. Using equation (8) and the fact that $L(s,\chi)\ll 1$, we get

$$\begin{aligned}
I_2 &\ll_a \int_{t_a}^{T}(qt)^{\frac{-1}{2}-\eta-\epsilon}\log(qt)dt\\
&\ll_a (qT)^{-\frac{1}{2}-\eta-\epsilon}\log(qT).
\end{aligned}$$

From Lemma 2.2, we have

$$I_3 = \frac{1}{2\pi i}\int_{1+\eta+\epsilon+iT}^{-\eta-\epsilon+iT}\sum_{|\gamma_{a,\chi}-T|<1}\frac{L(\eta+s,\chi)}{s-\delta_{a,\chi}}ds+O_a\left(\int_{1+\eta+\epsilon+iT}^{-\eta-\epsilon+iT}\log(qT)L(\eta+s,\chi)ds\right).$$

Now, we change the path of integration. If $\gamma_{a,\chi}<T$, we change the path to the upper semicircle with center $\delta_{a,\chi}$ and radius 1. If $\gamma_{a,\chi}>T$, we change the path to the lower semicircle with center $\delta_{a,\chi}$ and radius 1. Then, we have

$$\frac{1}{s-\delta_{a,\chi}}\ll 1$$

on the new path. This estimate and the bound (6) yields

$$I_3 = O_a\left((qT)^{\frac{1}{2}+\epsilon}\sum_{|\gamma_{a,\chi}^{(k)}-T|<1}1\right)+O_a\left((qT)^{\frac{1}{2}+\epsilon}\log qT\right).$$

By Lemma 2.1, we obtain

$$I_3 = O_a\left((qT)^{\frac{1}{2}+\epsilon}\log qT\right).$$

Finally, we estimate I_4. Using equation (9) and the fact that $\Delta(s,\chi)\Delta(1-s,\overline{\chi}) = 1$, we get

$$\begin{aligned} I_4 &= -\frac{1}{2\pi i}\int_{-\eta-\epsilon+it_a}^{-\eta-\epsilon+iT}\frac{\Delta'}{\Delta}(s,\chi)\left(1+\frac{a}{\Delta(s,\chi)}+\sum_{m\geq 2}\left(\frac{a}{\Delta(s,\chi)}\right)^m\right)L(\eta+s,\chi)ds \\ &= \frac{1}{2\pi i}\int_{1+\eta+\epsilon-it_a}^{1+\eta+\epsilon-iT}\frac{\Delta'}{\Delta}(1-s,\chi)\left(1+a\Delta(s,\overline{\chi})+\sum_{m\geq 2}(a\Delta(s,\overline{\chi}))^m\right)L(1+\eta-s,\chi)ds \\ &:= J_1+J_2+J_3. \end{aligned}$$

By equations (2) and (5), we obtain

$$\begin{aligned} \overline{J_1} &= -\frac{1}{2\pi}\int_{t_a}^{T}\frac{\Delta'}{\Delta}(-\eta-\epsilon-it,\overline{\chi})L(-\epsilon-it,\overline{\chi})dt \\ &= \frac{1}{2\pi}\int_{t_a}^{T}\Delta(-\epsilon-it,\overline{\chi})\log\left(\frac{qT}{2\pi}\right)L(1+\epsilon+it,\chi)dt \\ &\quad+\int_{t_a}^{T}O\left(\frac{\Delta(-\epsilon-it,\overline{\chi})L(1+\epsilon+it,\chi)}{t}\right)dt. \end{aligned}$$

Using [3, Lemma 2.14], we get

$$\overline{J_1} = \frac{\tau(\overline{\chi})}{q}\sum_{1\leq n\leq\frac{qT}{2\pi}}\chi(n)e^{-\frac{2\pi in}{q}}\log(n)+O_a\left((qT)^{\frac{1}{2}+\epsilon}\log qT\right).$$

Recall that (see [1, page 146])

$$e^{-\frac{2\pi in}{q}} = \frac{1}{\phi(q)}\sum_{\chi'\equiv q}\tau(\overline{\chi'})\chi'(-n),$$

when $(n,q)=1$. The last formula yields to

$$\begin{aligned} \frac{\tau(\overline{\chi})}{q}\sum_{1\leq n\leq\frac{qT}{2\pi}}\chi(n)e^{-\frac{2\pi in}{q}}\log n &= \frac{\tau(\overline{\chi})}{q\phi(q)}\sum_{\chi'\equiv q}\tau(\overline{\chi'})\chi'(-1)\sum_{1\leq n\leq\frac{qT}{2\pi}}\chi(n)\chi'(n)\log n \\ &= \sum_{\chi'\neq\overline{\chi}}\frac{\tau(\overline{\chi})\tau(\overline{\chi'})\chi'(-1)}{q\phi(q)}\sum_{1\leq n\leq\frac{qT}{2\pi}}\chi(n)\chi'(n)\log n \\ &+ \frac{\tau(\overline{\chi})\tau(\chi)\overline{\chi(-1)}}{q\phi(q)}\sum_{1\leq n\leq\frac{qT}{2\pi}}\chi_0(n)\log n \\ &= K_1+K_2. \end{aligned}$$

Using Pólya-Vinogradov inequality

$$\sum_{n\leq x}\chi(n)\ll 2\sqrt{q}\log q$$

for every non principal character modulo q and partial summation, we obtain $K_1 \ll \log(qT)$. By the Eratosthenes-Legendre sieve [5, Theorem 3.1], we know that

$$\sum_{k\le x}\chi_0(k)=\frac{\phi(q)}{q}x+O(q^{\epsilon}).$$

Then, partial summation gives

$$\begin{aligned}\sum_{k\le x}\chi_0(k)\log k &= \log x\left(\sum_{k\le x}\chi_0(k)\right)-\int_1^x\left(\sum_{k\le t}\chi_0(k)\right)\frac{1}{t}dt\\ &= \frac{\phi(q)}{q}x(\log x)-\frac{\phi(q)}{q}x+O\left(q^{\epsilon}\log x\right).\end{aligned}$$

Using last estimate and that

$$\tau(\overline{\chi})\tau(\chi)\overline{\chi(-1)}=|\tau(\chi)|^2=q,$$

we get

$$K_2=\frac{T}{2\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{2\pi}+O(\log(qT)).$$

Combining K_1 and K_2, we obtain

$$J_1=\frac{T}{2\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{2\pi}+O\left((qT)^{\frac{1}{2}+\epsilon}\log qT\right).$$

From equations (4) and (5), we obtain for J_2

$$\begin{aligned}\overline{J_2} &= -\frac{a}{2\pi i}\int_{1+\eta+\epsilon+it_a}^{1+\eta+\epsilon+iT}\frac{\Delta'}{\Delta}(1-s,\overline{\chi})\frac{\Delta(s,\chi)}{\Delta(s-\eta,\overline{\chi})}L(s-\eta,\overline{\chi})ds\\ &= \frac{a}{2\pi}\int_{t_a}^{T}\left(\log\left(\frac{qt}{2\pi}\right)+O\left(\frac{1}{t}\right)\right)\left(\left(\frac{qt}{2\pi}\right)^{-\eta}+O\left(\frac{1}{t}\right)\right)\sum_{n\ge1}\frac{\overline{\chi}(n)}{n^{1+\epsilon+it}}dt\\ &= \frac{a}{2\pi}\int_{t_a}^{T}\left(\frac{qt}{2\pi}\right)^{-\eta}\log\left(\frac{qt}{2\pi}\right)dt\\ &\quad +O\left(\sum_{n\ge2}\frac{\overline{\chi}(n)}{n^{1+\epsilon}}\int_{t_a}^{T}\left(\frac{qt}{2\pi}\right)^{-\eta}\log\left(\frac{qt}{2\pi}\right)\exp(-it\log n)dt\right).\end{aligned}$$

From [9, Lemma 4.3], we deduce that the error term is $\ll_a 1$. Then, we have

$$J_2=\frac{a}{q(1-\eta)}\left(\frac{qT}{2\pi}\right)^{1-\eta}\log\left(\frac{qT}{2\pi}\right)-\frac{a}{q(1-\eta)^2}\left(\frac{qT}{2\pi}\right)^{1-\eta}+O_a(1).$$

Now, using equations (4) and (6), we get

$$J_3\ll_a\int_{t_a}^{T}\log(qt)\sum_{m\ge2}(qt)^{-m(\frac{1}{2}+\eta+\epsilon)}(qt)^{\frac{1}{2}+\epsilon}\ll_a(qT)^{\frac{1}{2}+\epsilon}\log(qT).$$

Combining J_1, J_2 and J_3, we obtain

$$\begin{aligned}I_4 &= \frac{T}{2\pi}\log\left(\frac{qT}{2\pi e}\right)+\frac{a}{q(1-\eta)}\left(\frac{qT}{2\pi}\right)^{1-\eta}\log\left(\frac{qT}{2\pi}\right)-\frac{a}{q(1-\eta)^2}\left(\frac{qT}{2\pi}\right)^{1-\eta}\\ &\quad +O_a\left((qT)^{\frac{1}{2}+\epsilon}\log(qT)\right).\end{aligned}$$

Finally, Theorem 1.1 follows from estimates of I_1, I_2, I_3 and I_4.

4. Concluding remarks

The Selberg class $\mathcal{S}$ has been introduced by Selberg [6]. It consists of Dirichlet series

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}, \qquad Re(s) > 1,$$

satisfying

- **Ramanujan hypothesis:** $a(n) = O(n^\epsilon)$.
- **Euler product:** for s with sufficiently large real part,

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\right)$$

  with suitable coefficients $b(p^k)$ satisfying $b(p^k) = O(p^{k\theta})$ for some $\theta < \frac{1}{2}$.
- **Analytic continuation:** there exists a non-negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order (and in the sequel m_F denotes the smallest integer m with this property).
- **Functional equation:** for $1 \le j \le r$, there exist positive real numbers Q_F, λ_j, and complex numbers μ_j, ω with $Re(\mu_j) \ge 0$ and $|\omega| = 1$, such that

$$\phi_F(s) = \omega \overline{\phi_F(1-\overline{s})},$$

  where

$$\phi_F(s) = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j).$$

The degree of $F \in \mathcal{S}$ is defined by

$$d_F = 2 \sum_{j=1}^{r} \lambda_j.$$

The logarithmic derivative of $F(s)$ has a Dirichlet series expansion

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n) n^{-s} \qquad Re(s) > 1,$$

where $\Lambda_F(n) = b(n)\log n$ is the generalized von Mangoldt function (supported on the prime powers). In view of our investigations the functional equation is of special interest. We rewrite the functional equation as

$$F(s) = \Delta_F(s) \overline{F(1-\overline{s})},$$

where

$$\Delta_F(s) = \omega Q^{1-2s} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s) + \overline{\mu_j})}{\Gamma(\lambda_j s + \mu_j)}.$$

It is an interesting question to extend Theorem 1.1 to other class of Dirichlet L-functions (the Selberg class with some further condition). This problem will be considered in a sequel to this paper.

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Kamel Mazhouda
Faculty of Sciences, University of Monastir, Monastir 5000, Tunisia
E-mail address: kamel.mazhouda@fsm.rnu.tn

Mohammed Mekkaoui
École Supérieure de commerce, Kolea, Tipaza, Algeria
E-mail address: m_mekkaoui@esc-alger.dz