*Electronic Journal of Mathematical Analysis and Applications*
Vol. 10(2) July 2022, pp. 75-82.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

# INITIAL BOUNDS FOR CERTAIN P-VALENT ANALYTIC FUNCTIONS ASSOCIATED WITH Q-P-VALENT BERNARDI INTEGRAL OPERATOR AND COMPLEX ORDER

Z. M. SALEH AND A. O. MOSTAFA

Abstract. In this paper we introduce a new subclass of p-valent analytic functions of complex order defined by using q-p-valent Bernardi integral operator. Also we obtain initial bounds for functions in this class.

## 1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$F(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and $p-$valent in the open unit disc $\mathbb{U}$=$\{z \in \mathbb{C} : |z| < 1\}$. We note that $\mathcal{A}(1) = \mathcal{A}$. Also, let $\mathcal{T}(p)$ denote the subclass of $\mathcal{A}(p)$ consisting of analytic and $p-$valent functions which can expressed in the form:

$$F(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k > 0), \tag{2}$$

with $\mathcal{T}(1) = \mathcal{T}$. For $F(z) \in \mathcal{A}(p)$ given by (1) and $0 < q < 1$, the $q-$derivative operator $\nabla_q$ of $F(z)$ is given by [9, 11] (see also [1, 3, 4, 24, 26, 27])

$$\nabla_{p,q} F(z) = \begin{cases} \dfrac{F(z) - F(qz)}{(1-q)z}, & z \neq 0 \\ F^{'}(0), & z = 0 \end{cases}, \tag{3}$$

provided that $F^{'}(0)$ exists. From (1) and (3), we deduce that

$$\nabla_{p,q} F(z) = [p]_q z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q a_k z^{k-1}, \tag{4}$$

---

1991 *Mathematics Subject Classification.* 30C45.

*Key words and phrases.* Analytic functions, p-valent functions, Bernardi integral operator, Hankel determinant, Fekete–Szego problem.

Submitted Feb. 24, 2021.

where

$$[j]_q = \frac{1-q^j}{1-q}, \ [0]_q = 0. \tag{5}$$

As $q \to 1^-$, $[j]_q = j$ and $\nabla_{p,q}F(z) = F'(z)$.

For a function $F$ which is differentiable in a given subset of $\mathbb{C}$. Further, for $p = 1$, we have $\nabla_{1,q}F(z) = \nabla_q F(z)$ (see [24, 25]). The $q-$Jackon definite integral of the function $F(z)$ is defined by

$$\int_0^z F(t)\ d_qt = z(1-q)\sum_{k=0}^{\infty} q^k F(zq^k), \quad (z \in \mathbb{C}), \tag{6}$$

provided that the series converges (see [11]). For a function $F$ given by (1), we observe that

$$\int_0^z F(t)\ d_qt = \frac{z^{p+1}}{[p+1]_q} + \sum_{k=p+1}^{\infty} \frac{a_k z^{k+1}}{[k+1]_q},$$

and

$$\lim_{q\to 1^-} \int_0^z F(t)\ d_qt = \frac{z^{p+1}}{p+1} + \sum_{k=p+1}^{\infty} \frac{a_k z^{k+1}}{k+1} = \int_0^z F(t)\ dt,$$

where $\int_0^z F(t)\ dt$ is the ordinary integral.

We use the $q-$Jackon definite integral of the function $F(z) \in \mathcal{A}(p)$ to define the $q-p-$valent Bernardi integral operator $\mathcal{M}_{c,p,q}$ in the following definition.

**Definition 1** Let $c$ be a real number such that $c > -p$ $(p \in \mathbb{N})$. The $q-p-$valent Bernardi integral operator $\mathcal{M}_{c,p,q}(z)$ is defined by

$$\mathcal{M}_{c,p,q}(z) = \frac{[c+p]_q}{z^c} \int_0^z t^{c-1} F(t)\ d_qt \quad (c > -p;\ F(z) \in \mathcal{A}(p)). \tag{7}$$

For a function $F$ given by (1), we have

$$\mathcal{M}_{c,p,q}(z) = z^p + \sum_{k=p+1}^{\infty} \frac{[c+p]_q}{[c+k]_q} a_k z^k \quad (c > -p;\ p \in \mathbb{N}). \tag{8}$$

We note that:

(1) $\lim_{q\to 1^-} \mathcal{M}_{c,p,q}(z) = \mathcal{M}_{c,p}(z)$ $(c > -p)$, where $\mathcal{M}_{c,p}(z)$ is the $p-$valent Bernardi integral operator (see Saitoh [22] and Saitoh et al. [23]);

(2) $\mathcal{M}_{c,1,q}(z) = \mathcal{M}_{c,q}(z)$ (see Noor et al. [20] and Aldweby and M. Darus [2]);

(3) $\lim_{q\to 1^-} \mathcal{M}_{c,1,q}(z) = \mathcal{M}_c(z)$ $(c > -1)$ (see Bernardi [5] and Libera [15]).

By using the operator $\mathcal{M}_{c,p,q}(z)$ we define the class $\mathcal{S}_q(c,p,\tau)$ as follows.

**Definition 2** Let $\tau \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $c > -p$, $p \in \mathbb{N}$, $0 < q < 1$ and $F \in \mathcal{A}(p)$, such that $\mathcal{M}_{c,p,q}F(z) \neq 0$ for $z \in \mathbb{U}/\{0\}$. We say that $F \in \mathcal{S}_q(c,p,\tau)$ if

$$\operatorname{Re}\left\{1 + \frac{1}{[p]_q \tau}\left(\frac{z\nabla_q(\mathcal{M}_{c,p,q}F(z))}{\mathcal{M}_{c,p,q}F(z)} - [p]_q\right)\right\} > 0. \tag{9}$$

Note that:

(i) $\lim_{q\to 1-} \mathcal{S}_q(c,p,\tau) = \mathcal{S}(c,p,\tau) = \left\{F(z) : \operatorname{Re}\left\{1 + \frac{1}{p\tau}\left(\frac{z(\mathcal{M}_{c,p}F(z))'}{\mathcal{M}_{c,p}F(z)} - p\right)\right\} > 0\right\}$;

(ii) $\mathcal{S}_q(c,1,\tau) = \mathcal{S}_q(c,\tau) = \left\{F(z) : \operatorname{Re}\left\{1 + \frac{1}{\tau}\left(\frac{z\nabla_q(\mathcal{M}_{c,q}F(z))}{\mathcal{M}_{c,q}F(z)} - 1\right)\right\} > 0\right\}$;

(iii) $\lim_{q\to 1-} \mathcal{S}_q(c,1,\tau) = \mathcal{S}(c,\tau) = \left\{F(z) : \operatorname{Re}\left\{1 + \frac{1}{\tau}\left(\frac{z(\mathcal{M}_c F(z))'}{\mathcal{M}_c F(z)} - 1\right)\right\} > 0\right\}$;

(iv) $\mathcal{S}_q(c,1,1)=\mathcal{S}_q(c)=\left\{F(z):\operatorname{Re}\left\{\frac{z\nabla_q(\mathcal{M}_{c,q}F(z))}{\mathcal{M}_{c,q}F(z)}\right\}>0\right\}$;

(v) $\lim_{q\to 1-}\mathcal{S}_q(c,p,1)=\mathcal{S}(c)=\left\{F(z):\operatorname{Re}\left\{1+\frac{1}{p}(\frac{z(\mathcal{M}_{c,p}F(z))'}{\mathcal{M}_{c,p}F(z)}-p)\right\}>0\right\}$;

(vi) $\mathcal{S}_q(c,p,(1-\frac{\alpha}{[p]_q})e^{-i\theta}\cos\theta)=\mathcal{S}_q(c,p,\alpha,\theta)$ $(0\le\alpha<[p]_q,\ |\theta|<\frac{\pi}{2})$, where

$$\operatorname{Re}\left\{e^{i\theta}(\frac{z\nabla_q(\mathcal{M}_{c,p,q}F(z))}{\mathcal{M}_{c,p,q}F(z)})\right\}>\alpha\cos\theta;$$

(vii) $\lim_{q\to 1-}\mathcal{S}_q(c,p,(1-\frac{\alpha}{[p]_q})e^{-i\theta}\cos\theta)=\mathcal{S}(c,\alpha,\theta)$ $(0\le\alpha<[p]_q,\ |\theta|<\frac{\pi}{2})$, where

$$\operatorname{Re}\left\{e^{i\theta}(\frac{z(\mathcal{M}_{c,p,q}F(z))'}{\mathcal{M}_{c,p,q}F(z)})\right\}>\alpha\cos\theta.$$

Pommerenke [21] (see also [18]) defined the Hankel determinent for $F(z)\in\mathcal{A}$, $\eta\ge 1$, $\gamma\ge 0$ as

$$G_\eta(\gamma)=\begin{vmatrix} a_\gamma & a_{\gamma+1} & a_{\gamma+\eta-1}\\ a_{\gamma+1} & a_{\gamma+2} & a_{\gamma+\eta}\\ & & \\ a_{\gamma+\eta-1} & a_{\gamma+\eta} & a_{\gamma+2\eta-2}\end{vmatrix}\ (a_1=1), \tag{10}$$

This determinant has also been considered by several authors, for example $G_2(1)=a_3-a_2^2$, is known as the Fekete-Szego functional ( see Fekete-Szego [8] who generalized the estimate to $\left|a_3-\mu a_2^2\right|$ where $\mu$ is real ).

For more studies of $G_\eta(\gamma)$ see [7, 14, 19].

Also Hankel determinant for various subclasses of $p-$valent functions was investigated by various authors including Krishna and Ramreddy [12] and Hayami and Owa [10].

We consider the Hankel determinant in the case of $\eta=3$ and $\gamma=p$:

$$G_3(p)=\begin{vmatrix} a_p & a_{p+1} & a_{p+2}\\ a_{p+1} & a_{p+2} & a_{p+3}\\ a_{p+2} & a_{p+3} & a_{p+4}\end{vmatrix}.$$

For $F\in\mathcal{A}(p)$, $a_p=1$, we have

$$G_3(p)=a_{p+2}(a_{p+1}a_{p+3}-a_{p+2}^2)-a_{p+3}(a_{p+3}-a_{p+1}a_{p+2})+a_{p+4}(a_{p+2}-a_{p+1}^2).$$

and by applying the triangle inequality, we obtain

$$|G_3(p)|\le|a_{p+2}|\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|+|a_{p+3}|\left|a_{p+3}-a_{p+1}a_{p+2}\right|+|a_{p+4}|\left|a_{p+2}-a_{p+1}^2\right|. \tag{11}$$

Incidentally, the sharp upper bound for the functional $\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|$ on the right hand side of the inequality (11) for the class of functions which is of our interest in this paper was obtained by Vamshee Krishna and Ramreddy [13]. Thus, in this paper we obtain upper bounds to the functionals $|a_{p+3}-a_{p+1}a_{p+2}|$ and $\left|a_{p+2}-a_{p+1}^2\right|$, then the sharp upper bound on $G_3(p)$.

## 2. Main Results

Unless indicated, let $\tau\in\mathbb{C}^*$, $\mu\in\mathbb{C}$, $0<q<1$, $c>-p$, $p\in\mathbb{N}$ and $F(z)$ given by (1).

To prove our main results we shall need the following lemmas. Let $P$ be the family of all functions $p$ analytic in $\mathbb{U}$ for which $R\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + .... \tag{12}$$

**Lemma 1** [6] Let $p \in P$, then $|c_k| \leq 2,\ k = 1, 2, ...$ and the inequality is sharp.
**Lemma 2** [16] Let $p \in P$, then

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2xc_1(4 - c_1^2) - x^2 c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \end{aligned} \tag{13}$$

for some $x$ and $y$ such that $|x| \leq 1$, $|y| \leq 1$.
**Lemma 3** [17] If $p \in P$ is of the form (12) and $\nu$ is a complex number, then

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1; |2\nu - 1|\}.$$

**Theorem 1** Let $F(z) \in \mathcal{S}_q(c, p, \tau)$, then

$$\left|a_{p+1} a_{p+3} - a_{p+2}^2\right| \leq \frac{4([p]_q)^2 \left|\tau\right|^2}{\mathbb{B}^2([p+2]_q - [p]_q)^2}, \tag{14}$$

where $\mathbb{A} = \frac{[c+p]_q}{[c+p+1]_q}$, $\mathbb{B} = \frac{[c+p]_q}{[c+p+2]_q}$ and $\mathbb{E} = \frac{[c+p]_q}{[c+p+3]_q}$.
**Proof.** Let $\mathcal{M}_{c,p,q}F(z) = z^p + \beta_{p+1} z^{p+1} + \beta_{p+2} z^{p+2} + \beta_{p+3} z^{p+3} + ...$, then

$$\beta_{p+1} = \mathbb{A} a_{p+1},\ \beta_{p+2} = \mathbb{B} a_{p+2},\ \beta_{p+3} = \mathbb{E} a_{p+3}. \tag{15}$$

By (9), there exists $p \in P$ such that

$$\frac{z\nabla_q(\mathcal{M}_{c,p,q}F(z))}{\mathcal{M}_{c,p,q}F(z)} = [p]_q(1 + \tau(p(z) - 1)). \tag{16}$$

so that

$$\begin{aligned} &\frac{[p]_q + \beta_{p+1}[p+1]_q z + \beta_{p+2}[p+2]_q z^2 + \beta_{p+3}[p+3]_q z^3 + ...}{1 + \beta_{p+1} z + \beta_{p+2} z^2 + \beta_{p+3} z^3 + ...} \\ = &\ [p]_q(1 + \tau c_1 z + \tau c_2 z^2 + \tau c_3 z^3 + ...), \end{aligned} \tag{17}$$

which implies

$$\begin{aligned} &[p]_q + \beta_{p+1}[p+1]_q z + \beta_{p+2}[p+2]_q z^2 + \beta_{p+3}[p+3]_q z^3 + ... \\ = &\ [p]_q + [p]_q(\tau c_1 + \beta_{p+1})z + [p]_q(\beta_{p+2} + \tau c_1 \beta_{p+1} + \tau c_2)z^2 \\ &+ [p]_q(\tau c_3 + \tau c_1 \beta_{p+2} + \tau c_2 \beta_{p+1} + \beta_{p+3})z^3 + .... \end{aligned} \tag{18}$$

Equating the coefficients of both sides we have

$$\begin{aligned} \beta_{p+1} &= \frac{[p]_q \tau c_1}{[p+1]_q - [p]_q},\ \beta_{p+2} = \frac{[p]_q \tau c_2}{([p+2]_q - [p]_q)} + \frac{([p]_q)^2 \tau^2 c_1^2}{([p+1]_q - [p]_q)([p+2]_q - [p]_q)} \quad \text{and} \\ \beta_{p+3} &= \frac{[p]_q \tau c_3}{([p+3]_q - [p]_q)} + \frac{([p]_q)^2 \tau^2 c_1 c_2([p+1]_q + [p+2]_q - 2[p]_q)}{([p+1]_q - [p]_q)([p+2]_q - [p]_q)([p+3]_q - [p]_q)} \\ &+ \frac{([p]_q)^3 \tau^3 c_1^3}{([p+1]_q - [p]_q)([p+2]_q - [p]_q)([p+3]_q - [p]_q)}, \end{aligned} \tag{19}$$

so that, on account of (15) and (19)

$$\begin{aligned} a_{p+1} &= \frac{[p]_q\tau c_1}{\mathbb{A}([p+1]_q-[p]_q)},\ a_{p+2}=\frac{[p]_q\tau}{\mathbb{B}([p+2]_q-[p]_q)}(c_2+\frac{[p]_q\tau c_1^2}{([p+1]_q-[p]_q)}) \text{ and} \\ a_{p+3} &= \frac{[p]_q\tau c_3}{\mathbb{E}([p+3]_q-[p]_q)}+\frac{([p]_q)^3\tau^3c_1^3}{\mathbb{E}([p+3]_q-[p]_q)([p+1]_q-[p]_q)([p+2]_q-[p]_q)}) \\ &\quad +\frac{([p]_q)^2\tau^2c_1c_2([p+1]_q+[p+2]_q-2[p]_q)}{\mathbb{E}([p+3]_q-[p]_q)([p+1]_q-[p]_q)([p+2]_q-[p]_q)}. \end{aligned} \tag{20}$$

From (20), we have

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| = \left|\begin{array}{c} \frac{([p]_q)^4\tau^4c_1^4}{\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)^2([p+2]_q-[p]_q)}+\frac{([p]_q)^3\tau^3c_1^2c_2([p+1]_q+[p+2]_q-2[p]_q)}{\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)^2([p+2]_q-[p]_q)} \\ +\frac{([p]_q)^2\tau^2c_1c_3}{\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)}-\left[\frac{[p]_q\tau}{\mathbb{B}([p+2]_q-[p]_q)}\left(c_2+\frac{[p]_q\tau c_1^2}{([p+1]_q-[p]_q)}\right)\right]^2 \end{array}\right|. \tag{21}$$

By using Lemma 2,

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| = \left|\begin{array}{c} \frac{([p]_q)^4\tau^4c_1^4}{\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)^2([p+2]_q-[p]_q)}+\frac{([p]_q)^3\tau^3c_1^2([p+1]_q+[p+2]_q-2[p]_q)[\frac{c_1^2+x(4-c_1^2)}{2}]}{\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)^2([p+2]_q-[p]_q)} \\ +\frac{([p]_q)^2\tau^2c_1[\frac{c_1^3+2xc_1(4-c_1^2)-x^2c_1(4-c_1^2)+2y(1-|x|^2)(4-c_1^2)}{4}]}{\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)}-\frac{([p]_q)^4\tau^4c_1^4}{\mathbb{B}^2([p+2]_q-[p]_q)^2([p+1]_q-[p]_q)^2} \\ -\frac{([p]_q)^2\tau^2[\frac{c_1^2+x(4-c_1^2)}{2}]^2}{\mathbb{B}^2([p+2]_q-[p]_q)^2}-\frac{2([p]_q)^3\tau^3c_1^2[\frac{c_1^2+x(4-c_1^2)}{2}]}{\mathbb{B}^2([p+2]_q-[p]_q)^2([p+1]_q-[p]_q)} \end{array}\right|. \tag{22}$$

Substituting for $c_2$ and $c_3$ from (13) and since $|c_1|\leq 2$ by Lemma 1, let $c_1=c$ and assuming without restriction that $c\in[0,2]$ we obtain, by triangle inequality,

$$\begin{aligned} \left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \leq\ & \frac{([p]_q)^4\left|\tau\right|^4c^4}{\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)^2([p+2]_q-[p]_q)} \\ & +\frac{([p]_q)^3\left|\tau\right|^3c^4([p+1]_q+[p+2]_q-2[p]_q)}{2\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)^2([p+2]_q-[p]_q)} \\ & +\frac{([p]_q)^3\varepsilon c^2\left|\tau\right|^3([p+1]_q+[p+2]_q-2[p]_q)(4-c^2)}{2\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)^2([p+2]_q-[p]_q)} \end{aligned}$$

$$\begin{aligned} & +\frac{([p]_q)^4\left|\tau\right|^4c^4}{\mathbb{B}^2([p+2]_q-[p]_q)^2([p+1]_q-[p]_q)^2} \\ & +\frac{([p]_q)^2\left|\tau\right|^2\left[c^4+2\varepsilon c^2(4-c^2)-\varepsilon^2c^2(4-c^2)+2c(4-c^2)(1-\varepsilon^2)\right]}{4\mathbb{E}\mathbb{A}([p+3]_q-[p]_q)([p+1]_q-[p]_q)} \\ & +\frac{([p]_q)^2\left|\tau\right|^2\left[c^4+2\varepsilon c^2(4-c^2)+\varepsilon^2(4-c^2)^2\right]}{4\mathbb{B}^2([p+2]_q-[p]_q)^2}+\frac{([p]_q)^3\left|\tau\right|^3c^2(c^2+\varepsilon(4-c^2)}{\mathbb{B}^2([p+2]_q-[p]_q)^2([p+1]_q-[p]_q)} \\ \leq\ & N(\varepsilon), \end{aligned} \tag{23}$$

with $\varepsilon = |x| \leq 1$. Furthermore,

$$\begin{aligned} N^{'}(p) \leq\ & \frac{c^2 |\tau|^3 ([p]_q)^3([p+1]_q + [p+2]_q - 2[p]_q)(4-c^2)}{2\mathbb{E}\mathbb{A}([p+3]_q - [p]_q)([p+1]_q - [p]_q)^2([p+2]_q - [p]_q)} \\ & + \frac{|\tau|^2 ([p]_q)^2 \left[2c^2(4-c^2) - 2\varepsilon c^2(4-c^2) - 4c(4-c^2)\varepsilon\right]}{4\mathbb{E}\mathbb{A}([p+3]_q - [p]_q)([p+1]_q - [p]_q)} \\ & + \frac{([p]_q)^3 |\tau|^3 c^2(4-c^2)}{\mathbb{B}^2([p+2]_q - [p]_q)^2([p+1]_q - [p]_q)} \\ & + \frac{([p]_q)^2 |\tau|^2 \left[2c^2(4-c^2) + 2\varepsilon(4-c^2)^2\right]}{4\mathbb{B}^2([p+2]_q - [p]_q)^2}. \end{aligned} \tag{24}$$

By elementary calculations, we can show that $N^{'}(\varepsilon) \geq 0$ for $\varepsilon > 0$, which implies that $N$ is an increasing function and thus the upper bound for (21) corresponds to $\varepsilon = 1$ & $c = 0$, we have (14).

**Theorem 2** Let $F(z) \in \mathcal{S}_q(c, p, \tau)$, then

$$|a_{p+1}| \leq \frac{2[p]_q |\tau|}{\mathbb{A}([p+1]_q - [p]_q)} \tag{25}$$

$$|a_{p+2}| \leq \frac{2[p]_q |\tau|}{\mathbb{B}([p+2]_q - [p]_q)} \max\left\{1; \left|1 + \frac{2[p]_q \tau}{([p+1]_q - [p]_q)}\right|\right\} \tag{26}$$

and

$$\begin{aligned} \left|a_{p+2} - \mu a_{p+1}^2\right| \leq\ & \frac{2[p]_q |\tau|}{\mathbb{B}([p+2]_q - [p]_q)} \max \\ & \left\{1; \left|1 + \frac{2[p]_q \tau}{([p+1]_q - [p]_q)} \left[1 - \frac{\mathbb{B}([p+2]_q - [p]_q)}{\mathbb{A}^2([p+1]_q - [p]_q)} \mu\right]\right|\right\}, \end{aligned} \tag{27}$$

where $\mu \in \mathbb{R}$, $\mathbb{A} = \frac{[c+p]_q}{[c+p+1]_q}$ and $\mathbb{B} = \frac{[c+p]_q}{[c+p+2]_q}$.

**Proof.** Since if $F(z) \in \mathcal{S}_q(c, p, \tau)$, then $a_{p+1}$ and $a_{p+2}$ are given by (20) by Lemma 1, we obtain

$$|a_{p+1}| = \left|\frac{[p]_q \tau c_1}{\mathbb{A}([p+1]_q - [p]_q)}\right| \leq \frac{2[p]_q |\tau|}{\mathbb{A}([p+1]_q - [p]_q)}.$$

Therefore,

$$\begin{aligned} |a_{p+2}| &= \left|\frac{[p]_q \tau}{\mathbb{B}([p+2]_q - [p]_q)}(c_2 - \frac{c_1^2}{2([p+1]_q - [p]_q)} + \frac{(1 + 2[p]_q \tau)}{2([p+1]_q - [p]_q)} c_1^2)\right| \\ &= \left|\frac{[p]_q \tau}{\mathbb{B}([p+2]_q - [p]_q)}\left(c_2 - \nu c_1^2\right)\right|, \end{aligned}$$

where

$$\nu = \frac{-[p]_q \tau}{([p+1]_q - [p]_q)}.$$

Our result now follows by an application of Lemma 3.

Then, we have

$$\begin{aligned} a_{p+2}-\mu a_{p+1}^2 &= \frac{[p]_q\tau}{\mathbb{B}([p+2]_q-[p]_q)}(c_2+\frac{[p]_q\tau c_1^2}{([p+1]_q-[p]_q)})-\mu\frac{([p]_q)^2\tau^2c_1^2}{\mathbb{A}^2([p+1]_q-[p]_q)^2}. \\ &= \frac{[p]_q\tau}{\mathbb{B}([p+2]_q-[p]_q)}(c_2-\frac{c_1^2}{2([p+1]_q-[p]_q)}+\frac{(1+2[p]_q\tau)}{2([p+1]_q-[p]_q)}c_1^2) \\ &\quad -\mu\frac{([p]_q)^2\tau^2c_1^2}{\mathbb{A}^2([p+1]_q-[p]_q)^2}. \end{aligned} \tag{28}$$

Therefore,

$$\left|a_{p+2}-\mu a_{p+1}^2\right| = \left|\frac{[p]_q\tau}{\mathbb{B}([p+2]_q-[p]_q)}\{c_2-\nu c_1^2\}\right|, \tag{29}$$

where

$$\nu = \frac{[p]_q\tau}{([p+1]_q-[p]_q)}\left[\frac{\mathbb{B}([p+2]_q-[p]_q)}{\mathbb{A}^2([p+1]_q-[p]_q)}\mu-1\right]. \tag{30}$$

Our result now follows by an application of Lemma 3.

This completes the proof of Theorem 2.

**Remark 1** Letting $q \to 1^-$ in Theorems 1, 2, we obtain new results for the class $\mathcal{S}(c,p,\tau)$.

**Remark 2** Taking $p=1$ in Theorems 1, 2, we obtain new results for the class $\mathcal{S}_q(c,\tau)$.

**Remark 3** Letting $q \to 1^-$ and taking $p=1$ in Theorems 1, 2, we obtain new results for the class $\mathcal{S}(c,\tau)$.

## References


[1] M. H. Abu-Risha, M. H. Annaby, M. H. Ismail MH and et al., Linear $q-$difference equations, Z Anal Anwed., 26 (2007), 481–494.

[2] H. Aldweby and M. Darus, A note on $q-$integral operators, Electronic Notes in Discrete Math., 67(2018), 25-30.

[3] M. H. Annby and Z. S. Mansour, $q-$Fractional Calculus Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Berlin (2012).

[4] M. K. Aouf and T. M. Seoudy, Convolution properties for classes of bounded analytic functions with complex order defined by $q-$derivative operator, Rev Real Acad Cienc Exactas Fis Nat Ser A-Mat., 113 (2019), 1279–1288.

[5] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135(1969), 429–446.

[6] P. L. Duren, Univalent Functions, Grundlehren der Mathematishen Wissenschaften. New York: Springer, 1983.

[7] R. Ehrenborg, The Hankel determinant of exponantial polynomials, American Mathematical Monthly., 107 (2000), 557–560.

[8] M. Fekete and G. Szego, Eine Bemerkung uber ungerade schlichte Funktionen, J. London Math. Soc., 8 (1933), 85–89.

[9] G. Gasper and M. Rahman, Basic hypergeometric series, Cambridge, New York: Cambridge University Press, 1990.

[10] T. Hayami and S. Owa, Hankel determinant for $p-$valently starlike and convex functions of order, General Math., 17 (4) (2009), 29-44.

[11] F. H. Jackson, On $q-$functions and a certain difference operator, Trans. R. Soc. Edinb., 46 (1908), 253–281.

[12] D. V. Krishna and T. Ramreddy, Hankel determinant for $p-$valent starlike and convex functions of order, Novi Sad J. Math., 42 (2) (2012), 89-96.

[13] D. V. Krishna and T. Ramreddy, Coefficient inequality for certain subclass of $p-$valent functions, Pales. Poly. Univ., 2(2015), 223-228.

[14] J. W. Layman, The Hankel transform and some of its properties, J. of integer sequences., 4 (2001), 1–11.

[15] R. J. Libera, Some radius of convexity problems, Duke Math. J., 31(1964), 143–158.

[16] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P, Proc. Amer. Math. Soc., 87 (1983), no. 2, 251- 257.

[17] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis, Internat. Press, Cambridge, Ma (Tianjin, 1992), 157–169.

[18] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a really mean $p-$valent functions, Trans. Amer. Math. Soc., 223 (1976), no.2, 337–346.

[19] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Et Appl., 28 (1983), no. 8, 731–739.

[20] K. I. Noor, S. Riazi and M. A. Noor, On $q-$Bernardi integral operator, TWMS J. Pure Appl. Math., 8(2017), no.1, 3-11.

[21] Ch. Pommerenke, Univalent Function, Vandenhoek and Ruprecht, Gottingen, 1975.

[22] H. Saitoh, On certain class of mulivalent functions, Math. Japon., 37(1992), 871-875.

[23] H. Saitoh, S. Owa, T. Sekine, M. Nunokawa and R. Yamakawa, An application of certain integral operator, Appl. Math. Lett., 5(1992), 21-24.

[24] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of $q-$starlike and $q-$convex functions of complex order, J. Math. Inequal., 10 (2016), no. 1, 135–145.

[25] H. M. Srivastava, Operators of basic (or $q-$) calculus and fractional $q-$calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. Sci., 44(2020), 327–344.

[26] H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, Basic and fractional $q-$calculus and associated Fekete–Szego problem for $p-$valently $q-$starlike functions and $p-$valently $q-$convex functions of complex order, Miskolc Math. Notes., 20 (2019), no. 1, 489–509.

[27] H. M. Zayed and M. K. Aouf, subclasses of analytic functions of complex order associated with $q-$Mittage Leffler function, J.Egypt.Math Soc., 26 (2) (2018), 278–286.



Z. M. Saleh, Department of Mathematics, Faculty of Science, Mansoura University, Egypt

*E-mail address*: zeinabnsar2@gmail.com

A. O. Mostafa, Department of Mathematics, Faculty of Science, Mansoura University, Egypt

*E-mail address*: adelaeg254@yahoo.com