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BEREZIN NUMBER INEQUALITIES VIA OPERATOR CONVEX FUNCTIONS

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Abstract. The Berezin symbol $\widetilde{T}$ and the Berezin number of an operator T on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel $K_{\mathcal{H},\xi}$ are defined, respectively, by

$$\widetilde{T}(\xi)=\left\langle T\frac{K_{\mathcal{H},\xi}}{\left\|K_{\mathcal{H},\xi}\right\|},\frac{K_{\mathcal{H},\xi}}{\left\|K_{\mathcal{H},\xi}\right\|}\right\rangle,\ \xi\in\Omega\ \text{ and }\ \operatorname{ber}(T):=\sup_{\xi\in\Omega}\left|\widetilde{T}(\xi)\right|.$$

We study some inequalities by using this bounded function $\widetilde{T}$, involving powers of the Berezin number and the Berezin norms of reproducing kernel Hilbert space operators. Namely, we applying the Hermite-Hadamard inequality and some other recent results by using the concept of operator convexity.

1. Introduction

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions. Recall that a reproducing kernel Hilbert space (shortly, RKHS) is the Hilbert space $\mathcal{H}=\mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that:

(a) The evaluation functionals

$$\varphi_\xi(f)=f(\xi),\ \xi\in\Omega,$$

are continuous on $\mathcal{H}$.

(b) for every $\xi\in\Omega$ there exists a function $f_\xi\in\mathcal{H}$ such that $f_\xi(\xi)\neq 0$.

Then, according to the Riesz representation theorem, for each $\xi\in\Omega$ there exists a unique function $K_{\mathcal{H},\xi}\in\mathcal{H}$ such that $f(\xi)=\langle f,K_{\mathcal{H},\xi}\rangle$ for all $f\in\mathcal{H}$. The family $\{K_{\mathcal{H},\xi}:\xi\in\Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. The normalized reproducing kernel $k_{\mathcal{H},\xi}$ is defined by $k_{\mathcal{H},\xi}=\frac{K_{\mathcal{H},\xi}}{\|K_{\mathcal{H},\xi}\|}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. For an operator $T\in\mathcal{B}(\mathcal{H})$, the Berezin transform (symbol) of T, denoted by $\widetilde{T}$, is the complex-valued function on Ω defined by

$$\widetilde{T}(\xi):=\langle Tk_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle.$$

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For each bounded operator T on $\mathcal{H}$, the Berezin transform $\widetilde{T}$ is a bounded real-analytic function on Ω. Properties of the operator T are often reflected in properties of the Berezin transform $\widetilde{T}$. The Berezin transform is named in honour of F. Berezin, who introduced this concept in [6].

It is obvious that the Berezin symbol $\widetilde{T}$ is a bounded function on Ω and $\sup_{\xi\in\Omega}\left|\widetilde{T}(\xi)\right|$, which is called the Berezin number of operator T (see Karaev [25, 26]), does not exceed $\|T\|$, i.e.,

$$\operatorname{ber}(T) := \sup_{\xi\in\Omega}\left|\widetilde{T}(\xi)\right| \leq \|T\|.$$

It is also clear from the definition of Berezin symbol that the range of the Berezin symbol $\widetilde{T}$, which is called the Berezin set of operator T, lies in the numerical range $W(T)$ of operator T, i.e.,

$$\operatorname{Ber}(T) := \operatorname{Range}\left(\widetilde{T}\right) = \left\{\widetilde{T}(\xi) : \xi \in \Omega\right\} \subset W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

which implies that $\operatorname{ber}(T) \leq w(T) := \sup_{\|x\|=1}|\langle Tx, x\rangle|$ (numerical radius of operator T) (for more information, see [1, 8, 10, 16, 22, 23, 24, 28, 29, 31, 37]).

Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [25]. For the basic properties and facts on these new concepts, see [3, 4, 27, 32, 33].

It is well-known that

$$\operatorname{ber}(T) \leq w(T) \leq \|T\| \tag{1}$$

for any $T \in B(\mathcal{H})$ .Also, Berezin number inequalities were given by using the other inequalities in [11, 12, 13, 14, 15, 30, 34, 35, 36].

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle ., .\rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{B}(\mathcal{H})$. If $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with complex entries. An operator $T \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Tk_{\mathcal{H},\xi}, k_{\mathcal{H},\xi}\rangle \geq 0$ for all $k_{\mathcal{H},\xi} \in \mathcal{H}$, and we then write $T \geq 0$. The absolute value of T is denoted by $|T|$, that is $|T| = (T^*T)^{\frac{1}{2}}$. The Gelfand map $f(t) \longmapsto f(T)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(sp(T))$ of continuous functions on the spectrum $sp(T)$ of a self-adjoint operator T and the C^*-algebra generated by T and I. If $f, g \in C(sp(T))$, then $f(t) \geq g(t)$ $(t \in sp(T))$ implies that $f(T) \geq g(T)$. A linear map Φ on $\mathcal{B}(\mathcal{H})$ is positive if $\Phi(T) \geq 0$ whenever $T \geq 0$. It is said to be normalized if $\Phi(1_{\mathcal{H}}) = 1_{\mathcal{H}}$.

For a bounded linear operator T on a Hilbert space $\mathcal{H}$ let $w(T)$ and $\|T\|$ denote the numerical radius and the usual operator norm of T, respectively. It is well known that $w(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|.\|$. The celebrated inequality

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \tag{2}$$

for every $T \in \mathcal{B}(\mathcal{H})$, was generalized by Kittaneh [23] where he refined the right-hand side of (1), where he proved

$$w(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right)$$

for any $T \in \mathcal{B}(\mathcal{H})$. For basic properties of the numerical radius, we refer to [16] and [18].

It has been shown in [19] that if $T \in B(\mathcal{H})$, then

$$\frac{1}{4}\|T^*T + TT^*\| \le (\operatorname{ber}(T))^2 \le \frac{1}{2}\|T^*T + TT^*\|. \tag{3}$$

A function $f : J \to \mathbb{R}$ on the interval J is convex if

$$f(\beta s + (1-\beta)t) \le \beta f(s) + (1-\beta) f(t)$$

for all $s, t \in J$ and $\beta \in [0,1]$. The Hermite-Hadamard inequality for a convex function f on an interval J asserts that

$$f\left(\frac{s+t}{2}\right) \le \int_0^1 f(\beta s + (1-\beta)t)\, d\beta \le \frac{f(s)+f(t)}{2},$$

in which $s, t \in J$. There are several refinements and generalizations of the Hermite Hadamard inequality for convex functions; see [9] and references therein. We say that a function $f : J \to \mathbb{R}$ is an operator convex function on the interval J, if

$$f(\beta T + (1-\beta)S) \le \beta f(T) + (1-\beta) f(S),$$

where $T, S \in \mathcal{B}(\mathcal{H})$ are self-adjoint operators with spectra in J and $\beta \in [0,1]$.
To prove our results, we need the following sequence of lemmas.
Lemma 1 Every unital positive map on a commutative C^*-algebra is completely positive.
Theorem 1 ([7]) Let Φ be a unital completely positive linear map from a C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}_n(\mathbb{C})$ into $\mathcal{M}_n(\mathbb{C})$. Then, there exists a Hilbert space $\mathcal{K}$, an isometry $V : \mathbb{C}^m \to \mathcal{K}$ and a unital $*$-homomorphism π from $\mathcal{A}$ into the C^*-algebra $\mathcal{B}(\mathcal{K})$ such that $\Phi(A) = V^*\pi(A)V$.
Theorem 2 ([17]) Let $f : [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function and let $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a unital positive linear map. Then,

$$f\left(\widetilde{\Phi(A)}(\xi)\right) \le \Phi\widetilde{(f(A))}(\xi) - \left(\Phi\left(f\left(\left|A - \widetilde{\Phi(A)}(\xi) I_n\right|\right)\right)\right)^{\sim}(\xi),$$

for every positive matrix $A \in \mathcal{M}_n(\mathbb{C})$ and every $\lambda \in \xi$.
Lemma 2 ([21]) Let $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors. If g, h are nonnegative continuous functions on $[0,\infty)$ which are satisfying the relation $g(t)h(t) = t$ $(t \in [0,\infty))$, then

$$|\langle Tx, y\rangle|^2 \le \langle g(|T|)x, g(|T|)x\rangle \langle h(|T^*|)y, h(|T^*|)y\rangle.$$

In particular, $|\langle Tx, y\rangle|^2 \le \langle |T|^v x, |T|^v x\rangle \left\langle |T^*|^{(1-v)} y, |T^*|^{(1-v)} y\right\rangle$ $(0 \le v \le 1)$.
Lemma 3 ([9]) Let $f : J \to \mathbb{R}$ be an operator convex function on the interval J. Then

$$\begin{aligned} f\left(\frac{T+S}{2}\right) &\le (1-\beta) f\left[\frac{(1-\beta)T + (1+\beta)S}{2}\right] + \beta f\left[\frac{(2-\beta)T + \beta S}{2}\right] \\ &\le \int_0^1 f((1-t)T + tS)\, dt, \\ &\le \frac{f(T)+f(S)}{2}, \end{aligned}$$

where $T, S \in \mathcal{B}(\mathcal{H})$ are self-adjoint operators with spectra in J and $\beta \in [0,1]$.

Recently, Alomari et al. in [2] have obtained some numerical radius inequalities for Hilbert space operators. In this work, some Berezin number inequalities for Hilbert space operators are proven. Namely, some refinements of the inequality in [19], the result of Dragomir [9] regarding the Hermite-Hadamard inequality, and the result of [5].

2. Main results

In the this section, we prove some Berezin number inequalities via operator convex functions.

Theorem 3 ([5, Th. 1]) Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $T\in\mathcal{B}(\mathcal{H})$ and $f:[0,\infty)\to[0,\infty)$ is an increasing operator convex function, then

$$f(\operatorname{ber}(T))\leq\left\|\int_0^1 f(t\,|T|+(1-t)\,|T^*|)\,dt\right\|_{\operatorname{ber}}\leq\frac{1}{2}\left\|f(|T|)+f(|T^*|)\right\|_{\operatorname{ber}}.$$

The next result is a further refinement and generalization of Theorem 3.

Theorem 4 Let $T\in\mathcal{B}(\mathcal{H})$. If $f:[0,+\infty)\to\mathbb{R}$ is an increasing operator convex function and $0\leq\beta\leq1$, then we have

$$\begin{aligned}f(\operatorname{ber}(T))&\leq\left\|(1-\beta)f\left(\frac{(1-\beta)g^2(|T|)+(1+\beta)h^2(|T^*|)}{2}\right)\right.\\&\quad\left.+\beta f\left(\frac{(2-\beta)g^2(|T|)+\beta h^2(|T^*|)}{2}\right)\right\|_{\operatorname{ber}}\\&\leq\left\|\int_0^1 f\left(tg^2(|T|)+(1-t)\,h^2(|T^*|)\right)dt\right\|_{\operatorname{ber}}\\&\leq\left\|\frac{f\left(g^2(|T|)\right)+f\left(h^2(|T^*|)\right)}{2}\right\|_{\operatorname{ber}},\end{aligned}$$

where g and h are non-negative continuous functions on $[0,\infty)$ which are satisfying the relation $g(t)\,h(t)=t$ $(t\in[0,\infty))$.

Proof. Let $k_{\mathcal{H},\xi}$ be normalized reproducing kernel. Since f is increasing on $[0,\infty)$, then

$$\begin{aligned}f(|\langle Tk_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle|)&\leq f\left(\sqrt{\langle g^2(|T|)\,k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle\langle h^2(|T^*|)\,k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle}\right)\\&\quad\text{(by Lemma 2)}\\&\leq f\left(\frac{\left\langle g^2(|T|)\,k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle+\left\langle h^2(|T^*|)\,k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle}{2}\right)\\&\quad\text{(by the AM-GM inequality)}\\&=f\left(\left\langle\left(\frac{g^2(|T|)+h^2(|T^*|)}{2}\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\right)\\&\leq(1-\beta)\left[\left\langle f\left(\frac{(1-\beta)g^2(|T|)+(1+\beta)h^2(|T^*|)}{2}\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\right]\end{aligned}$$

$$
\begin{aligned}
&+\beta\left[\left\langle f\left(\frac{(2-\beta)\,g^2(|T|)+\beta h^2(|T^*|)}{2}\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\right]\\
&=\left\langle\left[(1-\beta)\,f\left(\frac{(1-\beta)\,g^2(|T|)+(1+\beta)\,h^2(|T^*|)}{2}\right)\right.\right.\\
&\left.\left.+\beta f\left(\frac{(2-\beta)\,g^2(|T|)+\beta h^2(|T^*|)}{2}\right)\right]k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\\
&\leq\int_0^1\left\langle f\left(tg^2(|T|)+(1-t)\,h^2(|T^*|)\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle dt\\
&=\left\langle\left(\int_0^1 f\left(tg^2(|T|)+(1-t)\,h^2(|T^*|)\right)dt\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\\
&\leq\left\langle\left(\frac{f\left(g^2(|T|)\right)+f\left(h^2(|T^*|)\right)}{2}\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f\left(\left|\left\langle Tk_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\right|\right)&\leq\left\langle\left[(1-\beta)\,f\left(\frac{(1-\beta)\,g^2(|T|)+(1+\beta)\,h^2(|T^*|)}{2}\right)\right.\right.\\
&\left.\left.+\beta f\left(\frac{(2-\beta)\,g^2(|T|)+\beta h^2(|T^*|)}{2}\right)\right]k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\\
&\leq\left\langle\left(\int_0^1 f\left(tg^2(|T|)+(1-t)\,h^2(|T^*|)\right)dt\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\\
&\leq\left\langle\left(\frac{f\left(g^2(|T|)\right)+f\left(h^2(|T^*|)\right)}{2}\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle.
\end{aligned}
$$

By taking supremum over $\xi\in\Omega$, we have

$$
\begin{aligned}
f(\mathrm{ber}(T))&\leq\left\|(1-\beta)\,f\left(\frac{(1-\beta)\,g^2(|T|)+(1+\beta)\,h^2(|T^*|)}{2}\right)\right.\\
&\left.+\beta f\left(\frac{(2-\beta)\,g^2(|T|)+\beta h^2(|T^*|)}{2}\right)\right\|_{\mathrm{ber}}\\
&\leq\left\|\int_0^1 f\left(tg^2(|T|)+(1-t)\,h^2(|T^*|)\right)dt\right\|_{\mathrm{ber}}\\
&\leq\left\|\frac{f\left(g^2(|T|)\right)+f\left(h^2(|T^*|)\right)}{2}\right\|_{\mathrm{ber}}.
\end{aligned}
$$

Remark 1 (a) If we put $g(t)=t^v, h(t)=t^{1-v}$ in Theorem 4, we have

$$
f(\mathrm{ber}(T))\leq\left\|(1-\beta)\,f\left(\frac{(1-\beta)\,|T|^{2v}+(1+\beta)\left(|T^*|^{2(1-v)}\right)}{2}\right)\right.
$$

$$+\beta f\left(\frac{(2-\beta)|T|^{2v}+\beta|T^*|^{2(1-v)}}{2}\right)\Bigg\|_{\text{ber}}$$

$$\leq\left\|\int_0^1 f(t)|T|^{2v}+(1-t)|T^*|^{2(1-v)}\,dt\right\|_{\text{ber}}$$

$$\leq\left\|\frac{f\left(|T|^{2v}\right)+f\left(|T^*|^{2(1-v)}\right)}{2}\right\|_{\text{ber}}.$$

(b) For $g(t)=t^{\frac{1}{2}}, h(t)=t^{\frac{1}{2}}$, we have

$$\begin{aligned} f(\text{ber}(T)) &\leq \left\|(1-\beta)f\left(\frac{(1-\beta)|T|+(1+\beta)|T^*|}{2}\right)+\beta f\left(\frac{(2-\beta)|T|+\beta|T^*|}{2}\right)\right\|_{\text{ber}} \\ &\leq \left\|\int_0^1 f(t|T|+(1-t)|T^*|)\,dt\right\|_{\text{ber}} \\ &\leq \left\|\frac{f(|T|)+f(|T^*|)}{2}\right\|_{\text{ber}}. \end{aligned} \tag{4}$$

In particular for $\beta=\frac{1}{2}$;

$$\begin{aligned} f(\text{ber}(T)) &\leq \frac{1}{2}\left\|f\left(\frac{|T|+3|T^*|}{4}\right)+f\left(\frac{3|T|+|T^*|}{4}\right)\right\|_{\text{ber}} \\ &\leq \left\|\int_0^1 f(t|T|+(1-t)|T^*|)\,dt\right\|_{\text{ber}} \\ &\leq \left\|\frac{f(|T|)+f(|T^*|)}{2}\right\|_{\text{ber}}. \end{aligned} \tag{5}$$

It should mention here that the inequality (4) is refinement of Theorem 3 and inequality (5) is already in [20]. Based on these results one can observe that Theorem 4 is more general than Theorem 3.

The function $f(t)=t^r$ $(1\leq r\leq 2)$ is an increasing operator convex function. The following result is a refinement and generalization of (3).

Corollary 1 Let $T\in\mathcal{B}(\mathcal{H})$. Then for any $0\leq\beta\leq 1$ and $1\leq r\leq 2$, we have

$$\begin{aligned} \text{ber}^r(T) &\leq \Bigg\|(1-\beta)\left(\frac{(1-\beta)g^2(|T|)+(1+\beta)h^2(|T^*|)}{2}\right)^r \\ &\quad+\beta\left(\frac{(2-\beta)g^2(|T|)+\beta h^2(|T^*|)}{2}\right)^r\Bigg\|_{\text{ber}} \\ &\leq \left\|\int_0^1\left(tg^2(|T|)+(1-t)h^2(|T^*|)\right)^r dt\right\|_{\text{ber}} \\ &\leq \left\|\frac{\left(g^2(|T|)\right)^r+\left(h^2(|T^*|)\right)^r}{2}\right\|_{\text{ber}}. \end{aligned}$$

In particular

$$\begin{aligned}\mathrm{ber}^{r}(T) &\leq \left\| (1-\beta)\left(\frac{(1-\beta)|T|^{2v}+(1+\beta)|T^{*}|^{2(1-v)}}{2}\right)^{r} \right.\\ &\quad \left. +\beta\left(\frac{(2-\beta)|T|^{2v}+\beta|T^{*}|^{2(1-v)}}{2}\right)^{r}\right\|_{\mathrm{ber}}\\ &\leq \left\| \int_{0}^{1}\left(t|T|^{2v}+(1-t)|T^{*}|^{2(1-v)}\right)^{r}dt\right\|_{\mathrm{ber}}\\ &\leq \left\|\frac{\left(|T|^{2v}\right)^{r}+\left(|T^{*}|^{2(1-v)}\right)^{r}}{2}\right\|_{\mathrm{ber}},\end{aligned}$$

where $v\in[0,1]$.

Proposition 1 ([5, Prop. 2]) Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $T,S\in\mathcal{B}(\mathcal{H})$ and $f:[0,\infty)\to[0,\infty)$ is an increasing operator convex function, then

$$f(\mathrm{ber}(S^{*}T))\leq\left\|\int_{0}^{1}f\left(\left(t|T|^{2}+(1-t)|S|^{2}\right)\right)dt\right\|_{\mathrm{ber}}\leq\frac{1}{2}\left\|\left(f\left(|T|^{2}\right)+f\left(|S|^{2}\right)\right)\right\|_{\mathrm{ber}}.$$

In particular, for any $1\leq r\leq 2$

$$\mathrm{ber}^{r}(S^{*}T)\leq\left\|\int_{0}^{1}\left(t|T|^{2}+(1-t)|S|^{2}\right)^{r}dt\right\|_{\mathrm{ber}}\leq\frac{1}{2}\left\||T|^{2r}+f\left(|S|^{2r}\right)\right\|_{\mathrm{ber}}.$$

In the next theorem, we show a refinement and generalization of Proposition 1.

Theorem 5 Let $T,S\in\mathcal{B}(\mathcal{H})$. If $f:[0,+\infty)\to\mathbb{R}$ is an increasing operator convex function and $\beta\in[0,1]$, then

$$\begin{aligned}f(\mathrm{ber}(S^{*}T)) &\leq \left\|(1-\beta)f\left(\frac{(1-\beta)|T|^{2}+(1+\beta)|S|^{2}}{2}\right)\right.\\ &\quad\left.+\beta f\left(\frac{(2-\beta)|T|^{2}+\beta|S|^{2}}{2}\right)\right\|_{\mathrm{ber}}\\ &\leq\left\|\int_{0}^{1}f\left(t|T|^{2}\right)+(1-t)|S|^{2}dt\right\|_{\mathrm{ber}}\\ &\leq\left\|\frac{f\left(|T|^{2}\right)+f\left(|S|^{2}\right)}{2}\right\|_{\mathrm{ber}}.\end{aligned}$$

In particular, for any $1 \leq r \leq 2$, we have

$$\begin{aligned}
\operatorname{ber}^r(S^*T) &\leq \left\| (1-\beta)\left(\frac{(1-\beta)|T|^2+(1+\beta)|S|^2}{2}\right)^r + \beta\left(\frac{(2-\beta)|T|^2+\beta|S|^2}{2}\right)^r \right\|_{\mathrm{ber}} \\
&\leq \left\| \int_0^1 \left(t|T|^2+(1-t)|S|^2\right)^r dt \right\|_{\mathrm{ber}} \\
&\leq \left\| \frac{|T|^{2r}+|S|^{2r}}{2} \right\|_{\mathrm{ber}}.
\end{aligned}$$

Proof Let $k_{\mathcal{H},\xi}$ be normalized reproducing kernel. Since f is increasing on $[0,+\infty)$ we have

$$\begin{aligned}
f\left(|\langle S^*Tk_{\mathcal{H},\xi}, k_{\mathcal{H},\xi}\rangle|\right) &= f\left(|\langle Tk_{\mathcal{H},\xi}, Sk_{\mathcal{H},\xi}\rangle|\right) \qquad (6)\\
&\leq f\left(\sqrt{\langle |T|\,k_{\mathcal{H},\xi}, |T|\,k_{\mathcal{H},\xi}\rangle\,\langle |S|\,k_{\mathcal{H},\xi}, |S|\,k_{\mathcal{H},\xi}\rangle}\right) \\
&= f\left(\sqrt{\left\langle |T|^2 k_{\mathcal{H},\xi}, k_{\mathcal{H},\xi}\right\rangle \left\langle |S|^2 k_{\mathcal{H},\xi}, k_{\mathcal{H},\xi}\right\rangle}\right) \\
&\leq f\left(\frac{\left\langle |T|^2 k_{\mathcal{H},\xi}, k_{\mathcal{H},\xi}\right\rangle + \left\langle |S|^2 k_{\mathcal{H},\xi}, k_{\mathcal{H},\xi}\right\rangle}{2}\right) \\
&\text{(by the AM-GM inequality)} \\
&f\left(\left\langle\left(\frac{|T|^2+|S|^2}{2}\right) k_{\mathcal{H},\xi}, k_{\mathcal{H},\xi}\right\rangle\right).
\end{aligned}$$

Applying inequalities (6) and a similar method in the proof of Theorem 2, we can obtain the desired results.

In the following result, we prove some inequalities involving the operator norm.
Proposition 2 Let $T, S \in \mathcal{B}(\mathcal{H})$. If $f : I \to \mathbb{R}$ be an increasing operator convex function on the interval I and for any $\beta \in [0,1]$, we have

$$\begin{aligned}
f\left(\left\|\frac{T+S}{2}\right\|_{\mathrm{ber}}\right) &\leq \left\| (1-\beta) f\left(\frac{(1-\beta)|T|+(1+\beta)|S|}{2}\right) \right. \\
&\quad \left. +\beta f\left(\frac{(2-\beta)|T|+\beta|S|}{2}\right) \right\|_{\mathrm{ber}} \\
&\leq \left\| \int_0^1 f\left(t|T|\right)+(1-t)|S|\,dt \right\|_{\mathrm{ber}} \\
&\leq \frac{1}{2}\left\| f(|T|)+f(|S|) \right\|_{\mathrm{ber}}.
\end{aligned}$$

In particular for $\beta = \frac{1}{2}$, we have

$$f\left(\left\|\frac{T+S}{2}\right\|_{\mathrm{ber}}\right) \leq \left\|f\left(\frac{|T|+3|S|}{4}\right)+f\left(\frac{3|T|+|S|}{4}\right)\right\|_{\mathrm{ber}}$$
$$\leq \left\|\int_0^1 f(t|T|)+(1-t)|S|\,dt\right\|_{\mathrm{ber}}$$
$$\leq \frac{1}{2}\left\|f(|T|)+f(|S|)\right\|_{\mathrm{ber}}.$$

Proof Let $k_{\mathcal{H},\xi}$ be normalized reproducing kernel. Then we have

$$f\left(\left|\left\langle\left(\frac{T+S}{2}\right)k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\right|\right) = f\left(\left|\frac{\langle Tk_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle+\langle Sk_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle}{2}\right|\right)$$
$$\leq f\left(\frac{|\langle Tk_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle|+|\langle Sk_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle|}{2}\right)$$
$$\leq f\left(\frac{\langle|T|\,k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle+\langle|S|\,k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\rangle}{2}\right)$$
$$= f\left(\left\langle\frac{|T|+|S|}{2}k_{\mathcal{H},\xi},k_{\mathcal{H},\xi}\right\rangle\right).$$

Using a similar method in the proof of Theorem 2, we get our results.

If $f:[0,\infty)\to[0,\infty)$ an increasing operator convex function, then by using the functional calculus for positive operator $\frac{|T|+|S|}{2}$ we have

$$f\left(\left\|\frac{|T|+|S|}{2}\right\|\right) = \left\|f\left(\frac{|T|+|S|}{2}\right)\right\|.$$

Now, if we replace T by $|T|$ and S by $|S|$ in Proposition 2, then we have

$$\left\|f\left(\frac{|T|+|S|}{2}\right)\right\|_{\mathrm{ber}}$$
$$\leq \left\|(1-\beta)f\left(\frac{(1-\beta)|T|+(1+\beta)|S|}{2}\right)+\beta f\left(\frac{(2-\beta)|T|+\beta|S|}{2}\right)\right\|_{\mathrm{ber}}$$
$$\leq \left\|\int_0^1 f(t|T|+(1-t)|S|)\,dt\right\|_{\mathrm{ber}}$$
$$\leq \frac{1}{2}\left\|f(|T|)+f(|S|)\right\|_{\mathrm{ber}}. \tag{7}$$

Theorem 6 ([20]) Let $T\in\mathcal{B}(\mathcal{H})$ have the Cartesian decomposition $T=B+iC$. If $f:[0,\infty)\to[0,\infty)$ is an increasing operator convex function, then

$$\left\|f\left(\frac{T^*T+TT^*}{4}\right)\right\|_{\mathrm{ber}} \leq \left\|\int_0^1 f\left((1-\alpha)B^2+\alpha C^2\right)d\alpha\right\|_{\mathrm{Ber}}$$
$$\leq \frac{1}{2}\left\|f\left(B^2\right)+f\left(C^2\right)\right\|_{\mathrm{ber}}$$
$$\leq f\left(\mathrm{ber}^2(T)\right).$$

For the last result of this section, we give the following refinement of Theorem 6.

Theorem 7 Let $T \in \mathcal{B}(\mathcal{H})$ with the Cartesian decomposition $T = B + iC$. If $f : [0, \infty) \to [0, \infty)$ is an increasing operator convex function, then

$$\begin{aligned}\left\| f\left(\frac{T^*T + TT^*}{4}\right)\right\|_{\text{ber}} &\leq \left\|(1-\beta) f\left(\frac{(1-\beta)B^2 + (1+\beta)C^2}{2}\right) + \beta f\left(\frac{(2-\beta)B^2 + \beta C^2}{2}\right)\right\|_{\text{ber}} \\ &\leq \left\|\int_0^1 f\left(tC^2 + (1-t)B^2\right) dt\right\|_{\text{ber}} \\ &\leq \frac{1}{2}\left\|f\left(B^2\right) + f\left(C^2\right)\right\|_{\text{ber}} \leq f\left(\text{ber}^2(T)\right),\end{aligned}$$

where $\beta \in [0, 1]$.

Proof If $T = B + iC$ is the Cartesian decomposition, then $B^2 + C^2 = \frac{T^*T + TT^*}{2}$. Now, if we replace $|T|$ by B^2 and $|B|$ by C^2 in inequalities (7), we have

$$\begin{aligned}\left\| f\left(\frac{T^*T + TT^*}{4}\right)\right\|_{\text{ber}} &= \left\| f\left(\frac{B^2 + C^2}{2}\right)\right\|_{\text{ber}} \\ &\leq \left\|(1-\beta) f\left(\frac{(1-\beta)B^2 + (1+\beta)C^2}{2}\right) + \beta f\left(\frac{(2-\beta)B^2 + \beta C^2}{2}\right)\right\|_{\text{ber}} \\ &\leq \left\|\int_0^1 f\left(tC^2 + (1-t)B^2\right) dt\right\|_{\text{ber}} \\ &\leq \frac{1}{2}\left\|f\left(B^2\right) + f\left(C^2\right)\right\|_{\text{ber}} \qquad (8)\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{1}{2}\left\|f\left(B^2\right) + f\left(C^2\right)\right\|_{\text{ber}} &\leq \frac{1}{2}\left(\left\|f\left(B^2\right)\right\|_{\text{ber}} + \left\|f\left(C^2\right)\right\|_{\text{ber}}\right) \\ &= \frac{1}{2}\left(f\left(\left\|B^2\right\|_{\text{ber}}\right) + f\left(\left\|C^2\right\|_{\text{ber}}\right)\right) \\ &\leq f\left(\text{ber}^2(T)\right). \qquad (9)\end{aligned}$$

Using (8) and (9) we get our desired result.

Let $f(t) = t^r$, $1 \leq r \leq 2$ is an increasing operator convex function. The previous theorem implies the following extension and refinement of the first inequality in (3).

Corollary 2 Let $T \in \mathcal{B}(\mathcal{H})$ with the Cartesian decomposition $T = B + iC$. Then for every $\beta \in [0, 1]$ and $1 \leq r \leq 2$,

$$\begin{aligned}\frac{1}{4^r}\left\|T^*T + TT^*\right\|_{\text{ber}}^r &\leq \frac{1}{2^r}\left\|(1-\beta)\left((1-\beta)B^2 + (1+\beta)C^2\right)^r + \beta\left((2-\beta)B^2 + \beta C^2\right)^r\right\|_{\text{ber}} \\ &\leq \frac{1}{2}\left\|B^{2r} + C^{2r}\right\|_{\text{ber}} \\ &\leq \text{ber}^{2r}(T).\end{aligned}$$

In particular,

$$\begin{aligned}\frac{1}{4}\left\|T^*T+TT^*\right\|_{\text{ber}} &\leq \frac{1}{4\sqrt{2}}\left\|\left(B^2+3C^2\right)^2+\left(3B^2+C^2\right)^2\right\|_{\text{ber}}^{1/2}\\ &\leq \left\|\int_0^1\left(tC^2+(1-t)\,B^2\right)^2 dt\right\|_{\text{ber}}^{1/2}\\ &\leq \frac{1}{\sqrt{2}}\left\|B^4+C^4\right\|_{\text{ber}}^{1/2}\leq \text{ber}^2\left(T\right).\end{aligned}$$

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