

## D- GAUSSIAN JACOBSTHAL, D- GAUSSIAN JACOBSTHAL- LUCAS POLYNOMIALS AND THEIR MATRIX REPRESENTATIONS

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ABSTRACT. In this paper, we define  $d$ - Gaussian Jacobsthal polynomials and  $d$ -Gaussian Jacobsthal-Lucas polynomials. We present the sum, generating functions and Binet formulas of these polynomials. We give the matrix representations of them. We present these matrices as binary representation according to the Riordan group matrix representation. By using Riordan method, we give factorizations of Pascal matrix involving  $d$ -Gaussian Jacobsthal polynomials and  $d$ -Gaussian Jacobsthal-Lucas polynomials. We give the inverse of matrices of these polynomials.

### 1. INTRODUCTION

Fibonacci numbers, which emerged with the solution of the famous rabbit problem, have been made many generalizations until today and still find application in many scientific fields [8, 9, 10, 11, 15]. Many generalizations of number sequences were then described and studied. One of the most well-known number sequences is the Jacobsthal numbers [4, 15, 19]. One of the most important of these generalizations is those about Gaussian [13, 14, 16, 18]. Özkan et al. defined Gauss Fibonacci polynomials, Gauss Lucas polynomials and gave their applications in [12].

Asci et al. defined the Gaussian Jacobsthal and the Gaussian Jacobsthal Lucas sequences [1] and the Gaussian Jacobsthal Polynomials and the Gaussian Jacobsthal Lucas Polynomials sequences [2].

Shapiro et al. described Riordan matrices and the Riordan group as a set of matrices  $M = (m_{ij})$ ,  $i, j \geq 0$  whose elements are complex numbers [20].

One of the latest works in this area is [17] where it is introduced  $d$ -Fibonacci and  $d$ -Lucas polynomials.

In this work, we give  $d$ - Gaussian Jacobsthal polynomials and  $d$ -Gaussian Jacobsthal-Lucas polynomials. We find the matrix representations, the sum, generating functions and Binet formulas of these polynomials. By using Riordan method, we introduce the factorizations of Pascal matrix involving  $d$ -Gaussian Jacobsthal polynomials and  $d$ -Gaussian Jacobsthal-Lucas polynomials. We also give the inverse of matrices of these polynomials. Now, let us give some basic definitions for this paper in this section.

**Definition 1.1.** The Jacobsthal numbers  $J_n$  are defined by

$$J_{n+2} = J_{n+1} + 2J_n$$

for  $n \geq 0$  with  $J_0 = 0$  and  $J_1 = 1$ . [5] Similarly, the Jacobsthal-Lucas numbers  $j_n$ ,

$$j_{n+2} = j_{n+1} + 2j_n$$

for  $n \geq 0$  with  $j_0 = 2$  and  $j_1 = 1$ . [5]

**Definition 1.2.**  $k$  – Jacobsthal numbers are defined by

$$j_{k,n+1} = k j_{k,n} + 2 j_{k,n-1}$$

for  $n \geq 2$  with  $j_{k,0} = 2$  and  $j_{k,1} = k$ . [7]

Similarly,  $k$  – Jacobsthal-Lucas numbers have been introduced in [3] and given some properties.

**Definition 1.3.** Jacobsthal polynomials were studied in [6] by Horodam and defined by the following recurrence relation,

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x)$$

for  $n \geq 2$  with  $J_0(x) = 0$  and  $J_1(x) = 1$ .

**Definition 1.4.** Jacobsthal-Lucas Polynomials have been defined in [6] by Horodam following the recurrence relation,

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x)$$

for  $n \geq 2$  with  $j_0(x) = 2$  and  $j_1(x) = 1$ .

**Definition 1.5.** Let  $p_i(x)$  be a real coefficient for  $i = 1, \dots, d + 1$ . Then  $d$ – Fibonacci polynomials are defined by

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \dots + p_{d+1}(x)F_{n-d}(x)$$

with  $F_n(x) = 0$  for  $n \leq 0$  and  $F_1(x) = 1$ . [17]

## 2. GENERALIZATION OF GAUSSIAN JACOBSTHAL AND GAUSSIAN JACOBSTHAL-LUCAS POLYNOMIALS

### 2.1. Generalization of Gaussian Jacobsthal Polynomials.

We introduce a new generalization of Gaussian Jacobsthal polynomials. Let  $p_i(x)$  be a real polynomial for  $i = 1, \dots, d + 1$ . Then  $d$ – Gaussian Jacobsthal polynomials are defined by

$$GJ_n(x) = p_1(x)GJ_{n-1}(x) + p_2(x)GJ_{n-2}(x) + \dots + p_{d+1}(x)GJ_{n-d-1}(x) \quad (1)$$

with  $GJ_n(x) = \frac{x}{2}$  and  $GJ_n(x) = 0$  for  $n < 0$ . We give a few terms of  $d$ –Gaussian Jacobsthal polynomials in Table 1.

From Equation (1), the characteristic equation of  $d$ –Gaussian Jacobsthal polynomials is given by

$$r^{d+1} - p_1(x)r^d - p_2(x)r^{d-1} - \dots - p_{d+1}(x) = 0.$$

The roots of this equation are  $\{\alpha_1(x), \alpha_2(x), \dots, \alpha_{d+1}(x)\}$ . Thus, we give the Generating function of these polynomials as follows.

TABLE 1. Some values of  $d$ -Gaussian Jacobsthal polynomials

$n$	$GJ_n(x)$
0	$\frac{i}{2}$
1	$p_1(x) \frac{i}{2}$
2	$p_1^2(x) \frac{i}{2} + \frac{i}{2} p_2(x)$
3	$p_1^3(x) \frac{i}{2} + p_1(x) p_2(x) i + p_3(x) \frac{i}{2}$
4	$p_1^4(x) \frac{i}{2} + \frac{3}{2} p_1^2(x) p_2(x) i + p_1(x) p_3(x) i + p_1^2(x) \frac{i}{2} + p_4(x) \frac{i}{2}$

**Theorem 2.1.** *Generating function of  $GJ_n(x)$  is given as follows*

$$G(x, t) = \sum_{n=0}^{\infty} GJ_n(x) t^n = \frac{\frac{i}{2}}{(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}.$$

*Proof.* We have

$$G(x, t) = GJ_0(x) + GJ_1(x)t + GJ_2(x)t^2 + \dots + GJ_n(x)t^n + \dots \quad (2)$$

Let us multiply Equation (2) by  $p_1(x)t, p_2(x)t^2, \dots, p_{d+1}(x)t^{d+1}$ , respectively. So, the following equations are obtained.

$$G(x, t) = GJ_0(x) + GJ_1(x)t + GJ_2(x)t^2 + \dots + GJ_n(x)t^n + \dots$$

$$p_1(x)tG(x, t) = p_1(x)tGJ_0(x) + p_1(x)t^2GJ_1(x) + p_1(x)t^3GJ_2(x) + \dots$$

$$p_2(x)t^2G(x, t) = p_2(x)t^2GJ_0(x) + p_2(x)t^3GJ_1(x) + p_2(x)t^4GJ_2(x) + \dots \vdots$$

$$p_{d+1}(x)t^{d+1}G(x, t) = p_{d+1}(x)GJ_0(x) + p_{d+1}(x)t^{d+2}GJ_1(x) + p_{d+1}(x)t^{d+3}GJ_2(x) + \dots$$

If we take the necessary calculations to take advantage of the recurrence relation, we obtain the following equation,

$$G(x, t) (1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1}) = GJ_0(x) + GJ_1(x)t - p_1(x)tGJ_0(x)$$

$$G(x, t) = \frac{\frac{i}{2}}{(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}.$$

In this case, the desired formula is obtained.  $\square$

Binet formula of  $GJ_n(x)$  has the following form

$$GJ_n(x) = \sum_{i=1}^{d+1} D_i(x) [\alpha_i(x)]^n.$$

Let's write the following equations for some values of  $n$  for the equation.

$$GJ_0(x) = \sum_{i=1}^{d+1} D_i(x),$$

$$GJ_1(x) = \sum_{i=1}^{d+1} D_i(x) \alpha_i(x),$$

$$GJ_2(x) = \sum_{i=1}^{d+1} D_i(x) [\alpha_i(x)]^2,$$

$$\begin{aligned}
 GJ_3(x) &= \sum_{i=1}^{d+1} D_i(x) [\alpha_i(x)]^3, \\
 &\vdots \\
 GJ_n(x) &= \sum_{i=1}^{d+1} D_i(x) [\alpha_i(x)]^n.
 \end{aligned}$$

If we multiply both sides of these equations by the coefficients of  $t^n$ ,

$$\begin{aligned}
 GJ_0(x) &= \sum_{i=1}^{d+1} D_i(x), \\
 tGJ_1(x) &= \sum_{i=1}^{d+1} D_i(x) \alpha_i(x) t, \\
 t^2GJ_2(x) &= \sum_{i=1}^{d+1} D_i(x) [\alpha_i(x)]^2 t^2, \\
 &\vdots \\
 t^nGJ_n(x) &= \sum_{i=1}^{d+1} D_i(x) [\alpha_i(x)]^n t^n.
 \end{aligned}$$

If we add up the left side of the equation, we have,

$$\sum_{n=0}^{\infty} GJ_n(x) t^n = \frac{t^{\frac{i}{2}}}{(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}.$$

If we add up the right side of the equation,

$$\sum_{i=1}^{d+1} D_i(x) (1 + \alpha_i(x)t + [\alpha_i(x)]^2 t^2 + \dots) = \sum_{i=1}^{d+1} D_i(x) \left( \frac{1}{1 - \alpha_i(x)t} \right).$$

So, we get the following equation

$$\frac{t^{\frac{i}{2}}}{(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})} = \sum_{i=1}^{d+1} \left( \frac{D_i(x)}{1 - \alpha_i(x)t} \right).$$

More precisely, the coefficients allow us to give the explicit form of  $d$ -Gaussian Jacobsthal polynomials. Actually

**Theorem 2.2.** *For  $n \geq 0$ , the following equality is true.*

$$GJ_n(x) = \frac{i}{2} \sum_{\substack{n_1, n_2, \dots, n_{d+1} \\ 1+n_1+2n_2+\dots+(d+1)n_{d+1}=n}} \left[ \binom{n_1+n_2+\dots+n_{d+1}}{n_1, n_2, \dots, n_{d+1}} p_1^{n_1}(x) p_2^{n_2}(x) \dots p_{d+1}^{n_{d+1}}(x) \right] t^n.$$

*Proof.* Let's use the generating function for proof.

$$\begin{aligned} \sum_{n=0}^{\infty} GJ_n(x)t^n &= \frac{\frac{i}{2}}{(1-p_1(x)-p_2(x)t^2-\dots-p_{d+1}t^{d+1})} \\ &= \frac{i}{2} \sum_{n=0}^{\infty} (p_1(x)t + p_2(x)t^2 + \dots + p_{d+1}(x)t^{d+1})^n \\ &= \frac{i}{2} \sum_{n_1+n_2+\dots+n_{d+1}=n}^{\infty} \left[ \binom{n}{n_1, n_2, \dots, n_{d+1}} p_1^{n_1}(x) p_2^{n_2}(x) \dots p_{d+1}^{n_{d+1}}(x) \right] t^{n_1+2n_2+\dots+(d+1)n_{d+1}} \\ &= \frac{i}{2} \sum_{\substack{n_1, n_2, \dots, n_{d+1} \\ n_1+2n_2+\dots+(d+1)n_{d+1}=n}}^{\infty} \left[ \binom{n_1+n_2+\dots+n_{d+1}}{n_1, n_2, \dots, n_{d+1}} p_1^{n_1}(x) p_2^{n_2}(x) \dots p_{d+1}^{n_{d+1}}(x) \right] t^n. \end{aligned}$$

Now if we substitute  $n-1$  for  $n$  we get what we want.

$$GJ_n(x) = \frac{i}{2} \sum_{\substack{n_1, n_2, \dots, n_{d+1} \\ 1+n_1+2n_2+\dots+(d+1)n_{d+1}=n}} \left[ \binom{n_1+n_2+\dots+n_{d+1}}{n_1, n_2, \dots, n_{d+1}} p_1^{n_1}(x) p_2^{n_2}(x) \dots p_{d+1}^{n_{d+1}}(x) \right] t^n.$$

□

**Theorem 2.3.** Let  $SGJ_n(x)$  be sum of the  $d$ -Gaussian Jacobsthal polynomials. Then we have

$$SGJ_n(x) = \sum_{n=0}^{\infty} GJ_n(x) = \frac{\frac{i}{2}}{1-p_1(x)-p_2(x)-\dots-p_{d+1}(x)}.$$

*Proof.* We get the following equation

$$SGJ_n(x) = \sum_{n=0}^{\infty} GJ_n(x) = GJ_0(x) + GJ_1(x) + GJ_2(x) + \dots + GJ_n(x) + \dots$$

If we multiply the last equation by  $p_1(x), p_2(x), \dots, p_{d+1}(x)$  respectively then we obtain

$$p_1(x) SGJ_n(x) = p_1(x) GJ_1(x) + p_1(x) GJ_2(x) + \dots + p_1(x) GJ_n(x) + \dots$$

$$p_2(x) SGJ_n(x) = p_2(x) GJ_1(x) + p_2(x) GJ_2(x) + \dots + p_2(x) GJ_n(x) + \dots$$

$$\vdots$$

$$p_{d+1}(x) SGJ_n(x) = p_{d+1}(x) GJ_1(x) + p_{d+1}(x) GJ_2(x) + \dots + p_{d+1}(x) GJ_n(x) + \dots$$

If the necessary mathematical operations are done, we get

$$SGJ_n(x) (1-p_1(x)-p_2(x)-\dots-p_{d+1}(x)) = \frac{i}{2}.$$

Thus, we have

$$SGJ_n(x) = \sum_{n=0}^{\infty} GJ_n(x) = \frac{\frac{i}{2}}{1-p_1(x)-p_2(x)-\dots-p_{d+1}(x)}.$$

□

From [9], we know that the  $d$ - Fibonacci polynomials matrix  $Q_d$  is given by

$$Q_d = \begin{bmatrix} p_1(x) & p_2(x) & \cdots & p_{d+1}(x) \\ 1 & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & \\ 0 & & 0 & 1 & 0 \end{bmatrix} \tag{3}$$

where  $\det Q_d = (-1)^d p_{d+1}(x)$ . Now, we can give matrix representation for  $GJ_n(x)$  in the next theorem.

**Theorem 2.4.** *The representation for  $GJ_n(x)$  has the form*

$$\begin{bmatrix} GJ_n(x) & p_2(x)GJ_{n-1}(x) + \cdots + p_{d+1}(x)GJ_{n-d}(x) & \cdots & p_{d+1}(x)GJ_{n-1}(x) \\ GJ_{n-1}(x) & p_2(x)GJ_{n-2}(x) + \cdots + p_{d+1}(x)GJ_{n-d-1}(x) & \cdots & p_{d+1}(x)GJ_{n-2}(x) \\ \vdots & \vdots & \cdots & \vdots \\ GJ_{n-d}(x) & p_2(x)GJ_{n-d-1}(x) + \cdots + p_{d+1}(x)GJ_{n-2d}(x) & \cdots & p_{d+1}(x)GJ_{n-d-1}(x) \end{bmatrix} \tag{4}$$

*Proof.* To prove the theorem, let's use mathematical induction over  $n$ . Since  $GJ_0(x) = 0$  for  $n \leq 0$ , when  $n = 1$ , we get

$$\begin{aligned} \frac{i}{2}Q_d &= \frac{i}{2} \begin{bmatrix} p_1(x) & p_2(x) & \cdots & p_{d+1}(x) \\ 1 & 0 & & \\ 0 & \ddots & & \\ & & \ddots & \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} p_1(x)\frac{i}{2} & p_2(x)\frac{i}{2} & \cdots & p_{d+1}(x)\frac{i}{2} \\ \frac{i}{2} & 0 & & \\ 0 & \ddots & & \\ & & \ddots & \\ 0 & 0 & \frac{i}{2} & 0 \end{bmatrix} \end{aligned} \tag{5}$$

For  $n = 1$  on the right side of (1), we get the following matrix

$$\begin{bmatrix} GJ_1(x) & p_2(x)GJ_0(x) & \cdots & p_{d+1}(x)GJ_0(x) \\ GJ_0(x) & 0 & & 0 \\ 0 & p_2(x)GJ_{n-3}(x) + \cdots + p_{d+1}(x)GJ_{n-d-2}(x) & & 0 \\ 0 & 0 & & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ddots & 0 \end{bmatrix} \tag{6}$$

$$GJ_n(x) = p_1(x)GJ_{n-1}(x) + p_2(x)GJ_{n-2}(x) + \cdots + p_{d+1}(x)GJ_{n-d-1}(x)$$

The following equation is obtained from the recurrence relation

$$p_2(x)GJ_{n-3}(x) + \cdots + p_{d+1}(x)GJ_{n-d-2}(x) = GJ_n(x) - p_1(x)GJ_{n-1}(x)$$

For  $n = 1$ , from (1), we obtain

$$\begin{aligned} p_2(x)GJ_{n-3}(x) + \cdots + p_{d+1}(x)GJ_{n-d-2}(x) &= GJ_1(x) = p_1(x)\frac{i}{2} \\ p_3(x)GJ_{n-3}(x) + \cdots + p_{d+1}(x)GJ_{n-d-1}(x) &= GJ_1(x) = p_1(x)\frac{i}{2} \end{aligned}$$

and if it continues like this then it will be seen that the matrices in (5) and (6) are equal. Now assume the matrix (5) satisfy for  $n$ . That is, we have

$$\frac{i}{2}Q_d^n = \begin{bmatrix} GJ_n(x) & p_2(x)GJ_{n-1}(x) + \dots + p_{d+1}(x)GJ_{n-d}(x) & \dots & p_{d+1}(x)GJ_{n-1}(x) \\ GJ_{n-1}(x) & p_2(x)GJ_{n-2}(x) + \dots + p_{d+1}(x)GJ_{n-d-1}(x) & \dots & p_{d+1}(x)GJ_{n-2}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GJ_{n-d}(x) & p_2(x)GJ_{n-d-1}(x) + \dots + p_{d+1}(x)GJ_{n-2d}(x) & \dots & p_{d+1}(x)GJ_{n-d-1}(x) \end{bmatrix}$$

Let show that it is true for  $n + 1$ . We know that

$$\begin{aligned} \frac{i}{2}Q_d^{n+1} &= \frac{i}{2}Q_d^n Q_d = \\ & \begin{bmatrix} GJ_n(x) & p_2(x)GJ_{n-1}(x) + \dots + p_{d+1}(x)GJ_{n-d}(x) & \dots & p_{d+1}(x)GJ_{n-1}(x) \\ GJ_{n-1}(x) & p_2(x)GJ_{n-2}(x) + \dots + p_{d+1}(x)GJ_{n-d-1}(x) & \dots & p_{d+1}(x)GJ_{n-2}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GJ_{n-d}(x) & p_2(x)GJ_{n-d-1}(x) + \dots + p_{d+1}(x)GJ_{n-2d}(x) & \dots & p_{d+1}(x)GJ_{n-d-1}(x) \end{bmatrix} \\ & \begin{bmatrix} p_1(x) & p_2(x) & \dots & p_{d+1}(x) \\ 1 & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} GJ_{n+1}(x) & p_2(x)GJ_n(x) + \dots + p_{d+1}(x)GJ_{n-d+1}(x) & \dots & p_{d+1}(x)GJ_n(x) \\ GJ_n(x) & p_2(x)GJ_{n-1}(x) + \dots + p_{d+1}(x)GJ_{n-d}(x) & \dots & p_{d+1}(x)GJ_{n-1}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GJ_{n-d+1}(x) & p_2(x)GJ_{n-d}(x) + \dots + p_{d+1}(x)GJ_{n-2d+1}(x) & \dots & p_{d+1}(x)GJ_{n-d}(x) \end{bmatrix} \end{aligned}$$

□

**Corollary 2.1.** For  $n, m \geq 0$ , the following equality is provided.

$$\begin{aligned} \frac{i}{2}GJ_{n+m}(x) &= GJ_{n+1}(x)GJ_{m+1}(x) + p_2(x)(GJ_{n-1}(x)GJ_{m-1}(x)) \\ &+ \dots + p_{d+1}(x)(GJ_{n-d+1}(x)GF_{m-1}(x)) \\ &+ \dots + GJ_{n-1}(x)GJ_{m-d}(x) \end{aligned}$$

*Proof.* For proof, we will use the product of matrices  $\frac{i}{2}Q_d^n$  and  $\frac{i}{2}Q_d^m$ . For this, writing

$$\frac{i}{2}Q_d^n \frac{i}{2}Q_d^m = \frac{i}{2}Q_d^{n+m}.$$

The result is the first row and the first column of matrix  $\frac{i}{2}Q_d^{n+m}$ . □

**Corollary 2.2.** For  $n \geq 1$  the following equality is true,

$$GJ_{n-1}(x) = F_n(x)(p_1(x) + i).$$

and here the  $F_n(x)$  polynomials are  $d$ - Fibonacci polynomials.

*Proof.* The proof can be easily seen on  $n$  by induction. □

**2.2. Generalization of Gaussian Jacobsthal-Lucas Polynomials.**

Now, we present a new generalization of Gaussian Jacobsthal-Lucas polynomials.  $d$ - Gaussian Jacobsthal-Lucas polynomials are defined by

$$GJL_n(x) = p_1(x)GJL_{n-1}(x) + p_2(x)GJL_{n-2}(x) + \dots + p_{d+1}(x)GJL_{n-d-1}(x) \tag{7}$$

with  $GJL_n(x) = 2 - \frac{i}{2}$  and  $GJL_n(x) = 0$  for  $n < 0$ . We give a few terms of  $d$ -Gaussian Jacobsthal-Lucas polynomials in Table 2.

TABLE 2. Some values of  $d$ -Gaussian Jacobsthal-Lucas polynomials

$n$	$GJL_n(x)$
0	$2 - \frac{i}{2}$
1	$p_1(x) \left(2 - \frac{i}{2}\right)$
2	$p_1^2(x) \left(2 - \frac{i}{2}\right) + p_2(x) \left(2 - \frac{i}{2}\right)$
3	$p_1^3(x) \left(2 - \frac{i}{2}\right) + 2p_1^2(x) + p_1(x)p_2(x) \left(2 - \frac{i}{2}\right) + p_3(x) \left(2 - \frac{i}{2}\right)$
4	$p_1^4(x) \left(2 - \frac{i}{2}\right) + 2p_1^3(x) + p_1^2(x)p_2(x) \left(4 - \frac{3i}{2}\right) + p_1(x)p_3(x) \left(4 - \frac{i}{2}\right) + p_1^2(x) \left(2 - \frac{i}{2}\right) + p_4(x) \left(2 - \frac{i}{2}\right)$

**Theorem 2.5.** *Generating function of  $GJL_n(x)$  is given as follows*

$$G(x, t) = \sum_{n=0}^{\infty} GJL_n(x) t^n = \frac{\left(2 - \frac{i}{2}\right)}{\left(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1}\right)}.$$

*Proof.* It is like that of Theorem 2.1 □

Binet formula of  $GJL_n(x)$  has the following form.

$$GJL_n(x) = \sum_{i=1}^{d+1} E_i(x) [\alpha_i(x)]^n.$$

If operations similar to section 2.1 are carried out, we have the following equations

$$\frac{\left(2 - \frac{i}{2}\right)}{\left(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1}\right)} = \sum_{i=1}^{d+1} \left(\frac{E_i(x)}{1 - \alpha_i(x)t}\right).$$

More precisely, the coefficients allow us to give the explicit form of  $d$ - Gaussian Jacobsthal- Lucas polynomials. Actually,

**Theorem 2.6.** *For  $n \geq 0$  the following equality is true.*

$$GJL_n(x) = \left(2 - \frac{i}{2}\right) \sum_{\substack{n_1, n_2, \dots, n_{d+1} \\ 1+n_1+2n_2+\dots+(d+1)n_{d+1}=n}} \left[ \binom{n_1+n_2+\dots+n_{d+1}}{n_1, n_2, \dots, n_{d+1}} p_1^{n_1}(x) p_2^{n_2}(x) \dots p_{d+1}^{n_{d+1}}(x) \right] t^n.$$

*Proof.* The proof is like that of Theorem 2.2 □

**Theorem 2.7.** *Let  $SGJL_n(x)$  be sum of the  $d$ -Gaussian Jacobsthal-Lucas polynomials. Then we have*

$$SGJL_n(x) = \sum_{n=0}^{\infty} GJL_n(x) = \frac{\left(2 - \frac{i}{2}\right)}{1 - p_1(x) - p_2(x) - \dots - p_{d+1}(x)}.$$

*Proof.* The proof is done similar to the proof of Theorem 2.3 □



**Theorem 2.8.** *We have the following representation for  $GJL_n(x)$  as follows.*

$$\left(2 - \frac{i}{2}\right) Q_d^n = \begin{bmatrix} GJL_n(x) & p_2(x)GJL_{n-1}(x) + \dots + p_{d+1}(x)GJL_{n-d}(x) & \dots & p_{d+1}(x)GJL_{n-1}(x) \\ GJL_{n-1}(x) & p_2(x)GJL_{n-2}(x) + \dots + p_{d+1}(x)GJL_{n-d-1}(x) & \dots & p_{d+1}(x)GJL_{n-2}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GJL_{n-d}(x) & p_2(x)GJL_{n-d-1}(x) + \dots + p_{d+1}(x)GJL_{n-2d}(x) & \dots & p_{d+1}(x)GJL_{n-d-1}(x) \end{bmatrix}$$

*Proof.* Like the proof of Theorem 2.4, it is easily demonstrated by induction over  $n$ . □

**Corollary 2.3.** *For  $n, m \geq 0$ , the following equality is provided.*

$$\begin{aligned} \left(2 - \frac{i}{2}\right) GJL_{n+m}(x) = & GJL_{n+1}(x) GJL_{m+1}(x) + p_2(x) (GJL_{n-1}(x) GFL_{m-1}(x)) \\ & + \dots + p_{d+1}(x) (GJL_{n-d+1}(x) GJL_{m-1}(x)) \\ & + \dots + GJL_{n-1}(x) GJL_{m-d}(x) \end{aligned}$$

*Proof.* Proof is done similar to the proof of Corollary 2.1. □

**Corollary 2.4.** *For  $n \geq 1$  the following equality is true.*

$$GJL_{n-1}(x) = F_n(x) \left(2 - \frac{i}{2}\right).$$

and here the  $F_n(x)$  polynomials are  $d$ -Fibonacci polynomials.

*Proof.* The proof can be easily seen on  $n$  by induction. □

**Lemma 2.9.** *For  $n \geq 1$  the following equality is true.*

$$GJ_n(x) + GJL_n(x) = 2F_n(x).$$

and here the  $F_n(x)$  polynomials are  $d$ -Fibonacci polynomials.

*Proof.* The proof can be easily seen on  $n$  by induction. □

### 3. THE INFINITE $d$ -GAUSSIAN JACOBSTHAL AND THE INFINITE $d$ - GAUSSIAN JACOBSTHAL- LUCAS POLYNOMIALS MATRIX

**3.1. The Infinite  $d$ - Gaussian Jacobsthal Polynomials Matrix.** The  $d$ -Gaussian Jacobsthal matrix polynomials is denoted by  $G\mathcal{J}(x) = [G\mathcal{J}_{P_1, P_2, \dots, P_{d+1}, i, j}(x)]$  and defined as follows.

$$\begin{aligned} G\mathcal{J}(x) &= \begin{bmatrix} \frac{i}{2} & 0 & 0 & \dots \\ p_1(x) \left(2 - \frac{i}{2}\right) & \frac{i}{2} & 0 & \vdots \\ p_1^2(x) \frac{i}{2} + p_2(x) \frac{i}{2} & p_1(x) \frac{i}{2} & \frac{i}{2} & \vdots \\ k_1(x) & k_2(x) & p_1(x) \frac{i}{2} & \frac{i}{2} \\ \vdots & \dots & \dots & \ddots \end{bmatrix} \\ &= (g_{G\mathcal{J}(x)}(t), f_{G\mathcal{J}(x)}(t)), \end{aligned}$$

where  $k_1(x) = p_1^3(x) + ip_1^2(x) + p_1(x)p_2(x) + ip_2(x)$  and  $k_2(x) = p_1^2(x) \frac{i}{2} + p_2(x) \frac{i}{2}$ .

We can write the  $d$ -Gaussian Jacobsthal polynomial matrix as follows,

$$G\mathcal{J}(x) = \begin{bmatrix} GJ_0(x) & 0 & 0 & \cdots \\ GJ_1(x) & GJ_0(x) & 0 & \cdots \\ GJ_2(x) & GJ_1(x) & GJ_0(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

**Theorem 3.1.** *The first column of  $G\mathcal{J}(x)$  matrix has the form*

$$\left( \frac{i}{2}, p_1(x) \frac{i}{2}, p_1^2(x) \frac{i}{2} + p_2(x) \frac{i}{2}, k_1(x), \dots \right)^T.$$

The generator function of the first column is as follows,

$$g_{G\mathcal{J}(x)}(t) = \sum_{n=0}^{\infty} G\mathcal{J}_{P_1, P_2, \dots, P_{d+1}, i, j}(x) t^n = \frac{\frac{i}{2}}{(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}.$$

*Proof.* Let's write Generating functions of the first column of  $G\mathcal{J}(x)$  matrix as follow,

$$\begin{aligned} & \frac{i}{2} + \left( p_1(x) \frac{i}{2} \right) t + \left( p_1^2(x) \frac{i}{2} + p_2(x) \frac{i}{2} \right) t^2 + \dots \\ & = G\mathcal{J}_0(x) + G\mathcal{J}_1(x)t + G\mathcal{J}_2(x)t^2 + \dots \end{aligned}$$

From the generator function of  $G\mathcal{J}_n(x)$

$$\begin{aligned} G(x, t) &= GJ_0(x) + GJ_1(x)t + GJ_2(x)t^2 + \dots + GJ_n(x)t^n + \dots \\ &= \frac{\frac{i}{2}}{(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}. \end{aligned}$$

Thus, the desired expression is obtained. So,

$$g_{G\mathcal{J}(x)}(t) = \sum_{n=0}^{\infty} G\mathcal{J}_{P_1, P_2, \dots, P_{d+1}, i, j}(x) t^n = \frac{\frac{i}{2}}{(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}.$$

□

From the Riordan matrix, we have

$$f_{G\mathcal{J}(x)}(t) = t.$$

Then we write  $G\mathcal{J}(x)$  as following.

$$G\mathcal{J}(x) = (g_{G\mathcal{J}(x)}(t), f_{G\mathcal{J}(x)}(t)) = \left( \frac{\frac{i}{2}}{1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1}}, t \right).$$

If this Gaussian Jacobsthal polynomial matrix  $G\mathcal{J}(x)$  is finite, then the matrix is

$$G\mathcal{J}_f(x) = \begin{bmatrix} GJ_0(x) & 0 & 0 & \cdots \\ GJ_1(x) & GJ_0(x) & 0 & \cdots \\ GJ_2(x) & GJ_1(x) & GJ_0(x) & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ GJ_n(x) & GJ_{n-1}(x) & \cdots & GJ_0(x) \end{bmatrix}$$

and

$$\det G\mathcal{J}_f(x) = |G\mathcal{J}_f(x)| = (GJ_0(x))^n = \left( \frac{i}{2} \right)^n.$$

We present two factorization of Pascal matrix including the  $d$ -Gaussian Jacobsthal polynomials matrix. For that, let's define an infinite  $C(x)$  as follows.

$$C(x) = \begin{bmatrix} \frac{2}{i} & 0 & 0 & \dots \\ \frac{2(1-p_1(x))}{i} & \frac{2}{i} & 0 & \dots \\ \frac{2(1-p_1(x)-p_2(x))}{i} & \frac{2(2-p_1(x))}{i} & \frac{2}{i} & \dots \\ \frac{2(1-p_1(x)-p_2(x)-p_3(x))}{i} & \frac{2(3-2p_1(x)-p_2(x))}{i} & \frac{2(3-p_1(x))}{i} & \dots \\ \vdots & \vdots & \frac{2(6-3p_1(x)-p_2(x))}{i} & \dots \\ c_1(x) & c_3(x) & \dots & \dots \\ c_2(x) & c_4(x) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (8)$$

where  $c_1(x) = \frac{2(1-p_1(x)-p_2(x)-\dots-p_d(x))}{i}$ ,  $c_2(x) = \frac{2(d-(d-1)p_1(x)-(d-2)p_2(x)-\dots-p_{d-1}(x))}{i}$ ,  
 $c_3(x) = \frac{2(1-p_1(x)-p_2(x)-\dots-p_{d+1}(x))}{i}$  and  $c_4(x) = \frac{2((d+1)-dp_1(x)-(d-1)p_2(x)-\dots-p_d(x))}{i}$ .

By using the infinite  $d$ - Gaussian Jacobsthal matrix and the infinite  $C(x)$  matrix as in(8), we present the first factorization of the infinite Pascal matrix with the following theorem.

**Theorem 3.2.** *The factorization of the infinite Pascal matrix is as follows*

$$P(x) = G\mathcal{J}(x) * C(x)$$

*Proof.* We get the following generator function from the first column of matrix  $C(x)$

$$g_{C(x)}(t) = \frac{2}{i} \left( \frac{1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1}}{1-t} \right).$$

From the Riordan matrix definition, we write  $f_{C(x)}(t) = \frac{t}{1-t}$

Then we write matrix  $C(x)$  as follow,

$$C(x) = (g_{C(x)}(t), f_{C(x)}(t)) = \left( \frac{2(1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}{i(1-t)}, \frac{t}{1-t} \right).$$

By using the definition of infinite Pascal matrix and the infinite  $d$ -Gaussian Jacobsthal polynomials matrix, we obtain the Riordan representation as follows

$$P = \left( \frac{1}{1-t}, \frac{t}{1-t} \right), G\mathcal{J}(x) = \left( \frac{\frac{i}{2}}{1 - p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1}}, t \right).$$

Finally, we write  $C^*(x)$  and  $G\mathcal{J}(x)$  matrices instead of the desired equation by using the definition of Riordan Group matrix multiplication. Thus, the proof is completed.  $\square$

Now, we present other factorization of the Pascal matrix including the  $d$ -Gaussian Jacobsthal Polynomials matrix. For that, let's define an infinite  $D^*(x)$  as follows

$$\begin{aligned}
 D^*(x) &= \begin{bmatrix} \frac{2}{i} & 0 & 0 & \cdots \\ \frac{2(1-p_1(x))}{i} & \frac{2}{i} & 0 & \vdots \\ \frac{2(1-2p_1(x)-p_2(x))}{i} & \frac{2(2-p_1(x))}{i} & \frac{2}{i} & \vdots \\ \frac{2(1-3p_1(x)-3p_2(x)-p_3(x))}{i} & \frac{2(3-2p_1(x)-p_2(x))}{i} & \frac{2(3-p_1(x))}{i} & \vdots \\ \vdots & \vdots & \frac{2(6-3p_1(x)-p_2(x))}{i} & \ddots \\ d_1(x) & d_3(x) & \cdots & \ddots \\ d_2(x) & d_4(x) & \cdots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\
 &= \frac{2}{i} \begin{bmatrix} 1 & 0 & 0 & \cdots \\ (1-p_1(x)) & 1 & 0 & \vdots \\ (1-2p_1(x)-p_2(x)) & (2-p_1(x)) & 1 & \vdots \\ (1-3p_1(x)-3p_2(x)-p_3(x)) & (3-2p_1(x)-p_2(x)) & (3-p_1(x)) & \vdots \\ \vdots & \vdots & (6-3p_1(x)-p_2(x)) & \ddots \\ d_1(x) & d_3(x) & \cdots & \ddots \\ d_2(x) & d_4(x) & \cdots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{9}
 \end{aligned}$$

where  $d_1(x) = \frac{2(1-dp_1(x)-\frac{d(d-1)}{2!}p_2(x)-\cdots-p_d(x))}{i}$ ,  $d_2(x) = \frac{2(d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-p_{d-1}(x))}{i}$ ,  $d_3(x) = \frac{2(1-(d+1)p_1(x)-\frac{d(d+1)}{2!}p_2(x)-\cdots-p_d(x))}{i}$  and  $d_4(x) = \frac{2((d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-p_d(x))}{i}$ .

From the infinite  $d$ -Gaussian Jacobsthal polynomials matrix and the infinite  $D(x)$  matrix as in (9), we introduce the second factorization of the infinite Pascal matrix with the following theorem.

**Theorem 3.3.** *The factorization of the infinite Pascal matrix is as follows,*

$$P(x) = G\mathcal{J}(x) * D(x).$$

*Proof.* The proof is like that of Theorem 3.1.2 □

Now, we can find the inverse of  $d$ -Gaussian Jacobsthal polynomials matrix the using the definition of reverse element Riordan group in [20].

**Corollary 3.1.** *The inverse of  $d$ -Gaussian Jacobsthal polynomial is given by the following.*

$$G\mathcal{J}^{-1}(x) = \left( \frac{1 - p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}{2/i}, t \right).$$

**3.2. The Infinite  $d$ -Gaussian Jacobsthal-Lucas Polynomials Matrix.**

The  $d$ -Gaussian Fibonacci matrix polynomials is denoted by

$$G\mathcal{J}\mathcal{L}(x) = [G\mathcal{J}\mathcal{L}_{P_1, P_2, \dots, P_{d+1}, i, j}(x)]$$

and defined as follows

$$\begin{aligned}
 G\mathcal{JL}(x) &= \begin{bmatrix} \binom{2-\frac{i}{2}}{0} & 0 & 0 & \cdots \\ p_1(x)\binom{2-\frac{i}{2}}{p_1(x)\binom{2-\frac{i}{2}}{2-\frac{i}{2}}+p_2(x)\binom{2-\frac{i}{2}}{2-\frac{i}{2}}} & \binom{2-\frac{i}{2}}{p_1(x)\binom{2-\frac{i}{2}}{2-\frac{i}{2}}} & 0 & \cdots \\ l_1(x) & l_2(x) & p_1(x)\binom{2-\frac{i}{2}}{2-\frac{i}{2}} & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\
 &= (g_{G\mathcal{JL}(x)}(t), f_{G\mathcal{JL}(x)}(t)),
 \end{aligned}$$

where

$$l_1(x) = p_1^3(x) \binom{2-\frac{i}{2}}{2-\frac{i}{2}} + 2p_1^2(x) + p_1(x)p_2(x)\binom{2-\frac{i}{2}}{2-\frac{i}{2}} + p_3(x) \binom{2-\frac{i}{2}}{2-\frac{i}{2}}$$

and

$$l_2(x) = p_1^2(x) \binom{2-\frac{i}{2}}{2-\frac{i}{2}} + p_2(x) \binom{2-\frac{i}{2}}{2-\frac{i}{2}}.$$

This  $d$ - Gaussian Jacobsthal-Lucas polynomial matrix can also be written as,

$$G\mathcal{JL}(x) = \begin{bmatrix} GJL_0(x) & 0 & 0 & \cdots \\ GJL_1(x) & GJL_0(x) & 0 & \cdots \\ GJL_2(x) & GJL_1(x) & GJL_0(x) & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Note that  $G\mathcal{JL}(x)$  is a Riordan matrix.

**Theorem 3.4.** *The first column of  $G\mathcal{JL}(x)$  matrix is,*

$$\left( \binom{2-\frac{i}{2}}{2-\frac{i}{2}}, p_1(x) \binom{2-\frac{i}{2}}{2-\frac{i}{2}}, p_1^2(x) \binom{2-\frac{i}{2}}{2-\frac{i}{2}} + p_2(x) \binom{2-\frac{i}{2}}{2-\frac{i}{2}}, l_1(x), \dots \right)^T.$$

According to the Riordan group theory, the generator function of the first column is as follows.

$$g_{G\mathcal{JL}(x)}(t) = \sum_{n=0}^{\infty} G\mathcal{JL}_{P_1, P_2, \dots, P_{d+1}, i, j}(x) t^n = \frac{\binom{2-\frac{i}{2}}{2-\frac{i}{2}}}{(1-p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1})}.$$

*Proof.* The proof is done analogously to that of Theorem 3.1 □

Then we write  $G\mathcal{JL}(x)$  as following.

$$G\mathcal{JL}(x) = (g_{G\mathcal{JL}(x)}(t), f_{G\mathcal{JL}(x)}(t)) = \left( \frac{\binom{2-\frac{i}{2}}{2-\frac{i}{2}}}{1-p_1(x)t - p_2(x)t^2 - \dots - p_{d+1}(x)t^{d+1}}, t \right).$$

If this Gaussian Jacobsthal-Lucas polynomial matrix  $G\mathcal{JL}(x)$  is finite, then the matrix is

$$GJ\mathcal{L}_f(x) = \begin{bmatrix} GJL_0(x) & 0 & 0 & \cdots \\ GJL_1(x) & GJL_0(x) & 0 & \cdots \\ GJL_2(x) & GJL_1(x) & GJL_0(x) & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ GJL_n(x) & GJL_{n-1}(x) & \cdots & \ddots \end{bmatrix}$$

and

$$\det GJ\mathcal{L}_f(x) = |GJ\mathcal{L}_f(x)| = (GjL_0(x))^n = \left(2 - \frac{i}{2}\right)^n$$

Now we give two factorization of Pascal matrix including the  $d$ - Gaussian Jacobsthal-Lucas polynomials matrix. For that, let's define an infinite  $C^*(x)$  as follows.

$$C^*(x) = \begin{bmatrix} \frac{2}{4-i} & 0 & 0 & \cdots \\ \frac{2(1-p_1(x))}{4-i} & \frac{2}{4-i} & 0 & \cdots \\ \frac{2(1-p_1(x)-p_2(x))}{4-i} & \frac{2(2-p_1(x))}{4-i} & \frac{2}{4-i} & \cdots \\ \frac{2(1-p_1(x)-p_2(x)-p_3(x))}{4-i} & \frac{2(3-2p_1(x)-p_2(x))}{4-i} & \frac{2(3-p_1(x))}{4-i} & \ddots \\ \vdots & \vdots & \frac{2(6-2p_1(x)-p_2(x))}{4-i} & \cdots \\ k_1(x) & k_3(x) & \cdots & \cdots \\ k_2(x) & k_4(x) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{10}$$

where  $k_1(x) = \frac{2(1-p_1(x)-p_2(x)-\cdots-p_d(x))}{4-i}$ ,  $k_2(x) = \frac{2(d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-p_{d-1}(x))}{4-i}$ ,  $k_3(x) = \frac{2(1-p_1(x)-p_2(x)-\cdots-p_{d+1}(x))}{4-i}$  and  $k_4(x) = \frac{2((d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-p_d(x))}{4-i}$ .

By using the infinite  $d$ -Gaussian Jacobsthal-Lucas matrix and the infinite  $C^*(x)$  matrix as in(10), we introduce the first factorization of the infinite Pascal matrix with the following theorem

**Theorem 3.5.** *The factorization of the infinite Pascal matrix is as follows  $P(x) = GJ\mathcal{L}(x) * C^*(x)$ .*

*Proof.* We get the following generator function from the first column of matrix  $C^*(x)$ ,

$$g_{C^*(x)}(t) = \left(\frac{2}{4-i}\right) \left(\frac{1-p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}{1-t}\right).$$

According to the Riordan matrix definition, we write  $f_{C^*(x)}(t) = \frac{t}{1-t}$ .

Then we write matrix  $C^*(x)$  as follow,

$$C^*(x) = (g_{C^*(x)}(t), f_{C^*(x)}(t)) = \left(\frac{2(1-p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}{(4-i)(1-t)}, \frac{t}{1-t}\right).$$

From the definition of infinite Pascal matrix and the infinite  $d$ -Gaussian Jacobsthal-Lucas polynomials matrix, we have the following Riordan representation

$$P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right), GJ\mathcal{L}(x) = \left(\frac{(2 - \frac{i}{2})}{1-p_1(x)t - p_2(x)t^2 - \cdots - p_{d+1}(x)t^{d+1}}, t\right).$$

Finally, we write  $C^*(x)$  and  $G\mathcal{J}\mathcal{L}(x)$  matrices instead of the desired equation by using the definition of Riordan Group matrix multiplication. Thus, the proof is completed.  $\square$

Now we give other factorization of Pascal matrix including the  $d$ - Gaussian Jacobsthal- Lucas polynomials matrix. For that, let's define an infinite  $D^*(x)$  as follows

$$\begin{aligned}
 D^*(x) &= \begin{bmatrix} \frac{2}{4-i} & 0 & 0 & \cdots \\ \frac{2(1-p_1(x))}{4-i} & \frac{2}{4-i} & 0 & \cdots \\ \frac{2(1-2p_1(x)-p_2(x))}{4-i} & \frac{2(2-p_1(x))}{4-i} & \frac{2}{4-i} & \cdots \\ \frac{2(1-3p_1(x)-3p_2(x)-p_3(x))}{4-i} & \frac{2(3-2p_1(x)-p_2(x))}{4-i} & \frac{2(3-p_1(x))}{4-i} & \ddots \\ \cdots & \vdots & \frac{2(6-2p_1(x)-p_2(x))}{4-i} & \cdots \\ l_1(x) & l_3(x) & \cdots & \cdots \\ l_2(x) & l_4(x) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\
 &= \left(\frac{2}{4-i}\right) \begin{bmatrix} 1 & 0 & 0 & \cdots \\ (1-p_1(x)) & 1 & 0 & \cdots \\ (1-2p_1(x)-p_2(x)) & (2-p_1(x)) & 1 & \cdots \\ (1-3p_1(x)-3p_2(x)-p_3(x)) & (3-2p_1(x)-p_2(x)) & (3-p_1(x)) & \ddots \\ \vdots & \vdots & (6-2p_1(x)-p_2(x)) & \cdots \\ l_1(x) & l_3(x) & \cdots & \cdots \\ l_2(x) & l_4(x) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{11}
 \end{aligned}$$

where  $l_1(x) = \frac{2(1-dp_1(x)-\frac{d(d-1)}{2!}p_2(x)-\cdots-p_d(x))}{4-i}$ ,  $l_2(x) = \frac{2(d-(d-1)p_1(x)-(d-2)p_2(x)-\cdots-p_{d-1}(x))}{4-i}$ ,  
 $l_3(x) = \frac{2(1-(d+1)p_1(x)-\frac{d(d-1)}{2!}p_2(x)-\cdots-p_d(x))}{4-i}$  and  $l_4(x) = \frac{2((d+1)-dp_1(x)-(d-1)p_2(x)-\cdots-p_d(x))}{4-i}$ .

From the infinite  $d$ -Gaussian Jacobsthal- Lucas polynomials matrix and the infinite  $D^*(x)$  matrix as in (11), we present the second factorization of the infinite Pascal matrix with the following theorem

**Theorem 3.6.** *The factorization of the infinite Pascal matrix is as follows*

$$P(x) = G\mathcal{J}\mathcal{L}(x) * D^*(x).$$

*Proof.* The proof is similar to that of Theorem 3.2  $\square$

Now, we can find the inverse of  $d$ -Gaussian Jacobsthal- Lucas polynomials matrix by using from the definition of reverse element Riordan group in [20].

**Corollary 3.2.** *The inverse of  $d$ -Gaussian Jacobsthal-Lucas polynomial is given by the following.*

$$G\mathcal{J}\mathcal{L}^{-1}(x) = \left(\frac{1-p_1(x)t-p_2(x)t^2-\cdots-p_{d+1}(x)t^{d+1}}{(2-\frac{i}{2})}, t\right).$$

## 4. CONCLUSIONS

New generalized Gaussian Jacobsthal polynomials and Gaussian Jacobsthal-Lucas polynomials have been introduced and studied. We gave the matrix representations of  $d$ -Gaussian Jacobsthal and  $d$ -Gaussian Jacobsthal-Lucas polynomials. Also, we introduced these matrices as binary representations according to the Riordan group matrix representation. Using the Riordan method, we found the factorizations of the Pascal matrix involving these polynomials. Also, we gave the inverse of matrices of these polynomials.

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