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FIXED POINT RESULTS FOR GENERALIZED (α, ψ) - GERAGHTY CONTRACTION IN B-RECTANGULAR METRIC SPACES

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Abstract. In this paper, we introduce generalized (α, ψ)-Geraghty Contraction mappings in b-rectangular metric spaces and study fixed point results for the mappings introduced. Our results extend and generalize related fixed point results in the existing literature. Finally, we provide an example in support of our main findings.

1. Introduction

Fixed point theory is one of the most important topic in Mathematics, specially in analysis. Due to its application in various disciplines like engineering, computer science, biological sciences, economics etc., many researchers took interest in fixed point theory and its application. It is well known that Banach Contraction Principle is the most important result in fixed point theory [13]. During the last many years this result was extended in different directions. Taking the key role of the notion of the metric in mathematics and hence in quantitative sciences, it has been extended and generalized in several distinct directions by many authors. One of the generalization of metric spaces was rectangular metric spaces which was introduced by Branciari [3]. In 1993, Czerwik [4] introduced and studied b-metric spaces, which is an interesting generalization of metric space. The concept of b-rectangular metric space was introduced as a generalization of metric, b-metric space and rectangular metric space by Geoge et al. [6]. Useful results on rectangular metric, b-metric and rectangular b-metric spaces can be seen in ([14]-[22]). Inspired and motivated by the works of Erhan [11] and Baiya and Kaewcharoen [2] the main purpose of this paper is to define generalized (α, ψ)-Geraghty contraction and establish new fixed point results for the class of map that satisfy generalized (α, ψ)-Geraghty contraction condition in the setting of b-rectangular metric spaces and prove the existence and uniqueness of fixed points for the maps introduced.

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2. Preliminaries

Notation: We need the following symbols and class of functions to prove certain results of this section:

- $\Re^+ = [0, \infty)$;
- $\mathbb{N}$ is the set of all natural numbers;
- $\Psi_1 = \{\psi = (0, \infty) \to (1, \infty)$, such that, ψ is nondecreasing, for each sequence $\{t_n\} \subset (0, \infty)$, $\psi(t_n) \to 1$ if and only if $t_n \to 0$ and there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim\limits_{t \longrightarrow 0} \frac{\psi(t)-1}{t^r} = l\}$;
- $\Psi_2 = \{\psi = (0, \infty) \to (1, \infty)$, such that, ψ is continuous and nondecreasing$\}$;
- $\Psi = \{\psi \colon \Re^+ \to \Re^+$, such that, ψ is continuous and non-decreasing $\}$;
- $\Theta = \{\theta \colon \Re^+ \to [0, 1)$, such that, $\theta(t_n) \to 1 \Rightarrow t_n \to 0$, as $n \to \infty\}$;
- $\Theta_s =\{\theta \colon \Re^+ \to [0, \frac{1}{s})$, such that, $\theta(t_n) \to \frac{1}{s} \Rightarrow t_n \to 0$ as $n \to \infty$ for $s \geq 1\}$.

Definition 1 [5] Let X be a nonempty set and $d : X \times X \to \Re^+$ be a function satisfying the following conditions:
(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Then the pair (X, d) is called a metric space.

Definition 2 [5] Let X be a nonempty set and $d : X \times X \to \Re^+$ be a function satisfying the following conditions:
(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Then the pair (X, d) is called a metric space.

Definition 3 [4] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \Re^+$ is said to be a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:
(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space.

Definition 4 [3] Let X be a nonempty set and let $d : X \times X \to \Re^+$ be a mapping such that for all $x, y \in X$and all distinct points $u, v \in X$, each distinct from x and y:
(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, y) \leq [d(x, u) + d(u, v) + d(v, y)]$(rectangular inequality).
The pair (X, d) is called a rectangular metric space.

Definition 5 [6] Let X be a nonempty set, $s \geq 1$ be a given real number, and $d : X \times X \to \Re^+$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:
a) $d(x, y) = 0$ if and only if $x = y$;
b) $d(x, y) = d(y, x)$;
c) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$(b-rectangular inequality).
The pair (X, d) is called a b-rectangular metric space.

Definition 6 [6] Let X be a b-rectangular metric space and $\{x_n\}$ be a sequence in X, we say that
a. $\{x_n\}$ is a b-rectangular converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$;
b. $\{x_n\}$ is a b-rectangular Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;
c. (X, d) is a b-rectangular complete if every b-rectangular Cauchy sequence in X is b-rectangular convergent.

Definition 7 [7] An operator $T : X \to X$ is called a Geraghty contraction if there exists a function $\theta \in \Theta$ which satisfies for all $x, y \in X$ the condition;

$$d(Tx, Ty) \leq \theta(d(x, y))d(x, y).$$

Theorem 1 [7] Let (X, d) be a complete metric space. If $T\colon X \to X$ is a Geraghty contractive mapping, then T has a unique fixed point.

Definition 8 [10] Let X be a nonempty set, $\alpha : X \times X \to \Re^+$ be a function. A mapping $T : X \to X$ is said to be α- admissible, if for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty)) \geq 1$.

Definition 9 Let X be a nonempty set, $\alpha : X \times X \to \Re^+$ be a functional. A mapping $T : X \to X$ is said to be α-orbital admissible, if for all $x \in X$, $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.

Definition 10 [9] Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to \Re^+$. We say that T is triangular α-orbital admissible if:
i. T is α-orbital admissible;
ii. for all $x, y \in X, \alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply that $\alpha(x, Ty) \geq 1$.

Lemma 1 [9] Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to \Re^+$. Suppose that T is a triangular α-orbital admissible mapping and assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

Theorem 2 [8] Let (X, d) be a complete rectangular metric space and $T : X \to X$. Suppose that there exist $\psi \in \Psi_1$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^\lambda,$$

where $R(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)\}$. Then T has a fixed point.

Theorem 3 [2] Let (X, d) be a complete rectangular metric space, $T : X \to X$ and $\alpha : X \times X \to \Re^+$. Suppose that the following conditions hold:
(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y).\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^\lambda,$$

where $R(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)\}$;
(ii) there exists $x_1 \in X$ such that $\alpha(x_1 Tx_1) \geq 1$;
(iii) T is a triangular α-orbital admissible mapping;
(iv) T is continuous.
Then T has a fixed point.

Theorem 4 [2] Let (X, d) be a complete rectangular metric space, $T : X \to X$ and $\alpha : X \times X \to \Re^+$. Suppose that the following conditions hold:

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y).\psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda},$$

where $R(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}$;
(ii) there exists $x_1 \in X$ such that $\alpha(x_1 Tx_1) \geq 1$;
(iii) T is a triangular α-orbital admissible mapping;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$.
Then T has a fixed point.

Theorem 5 [1] Let (X,d) be a complete rectangular metric space, $T : X \to X$ and $\alpha : X \times X \to \Re^+$. Suppose that the following conditions hold:
(i) there exist $\psi \in \Psi_1$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y).\psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda},$$

where $R(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\}$;
(ii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$, and $\alpha(x_1, T^2_{x_1}) \geq 1$;
(iii) T is a triangular α -orbital admissible mapping;
(iv) if $\{T^n_{x_1}\}$ is a sequence in X such that $\alpha(T^n_{x_1}, T^{n+1}_{x_1}) \geq 1$ for all n and $x_n \to x \in X$as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}_{x_1}\}$ of $\{T^n_{x_1}\}$ such that $\alpha(T^{n(k)}_{x_1}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$. Then T has a fixed point z in X and $\{T^n_{x_1}\}$ converges to z.

Theorem 6 [11] Let (X,d) be a complete b-rectangular metric space with a constant $s \geq 1$ and $\alpha : X \times X \to \Re^+$ and $\theta \in \Theta_s$ be two given functions. Let $T : X \to X$ be a continuous α-admissible mapping satisfying
$\alpha(x,y)d(Tx,Ty) \leq \theta(M(x,y))M(x,y)$,for all $x, y \in X$,
where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}$.
Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.
Then T has a fixed point.

Theorem 7 [11] Let (X,d) be a complete b-rectangular metric space with a constant $s \geq 1$ and $\alpha : X \times X \to \Re^+$ and $\theta \in \Theta_s$ be two given functions. Let $T : X \to X$ be an α-admissible mapping satisfying
$\alpha(x,y)d(Tx,Ty) \leq \theta(M(x,y))M(x,y)$,for all $x, y \in X$,
where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}$.
Suppose also that
(i) there exists $x_0 \in X$ such that$\alpha(x_0, Tx_0) \geq 1$and $\alpha(x_0, T^2x_0) \geq 1$.
(ii) for any sequence $\{x_n\} \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
(iii) for every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$.
Then T has a unique fixed point.

3. Main Results

In this section, we define generalized (α, ψ) - Geraghty contraction mappings in the setting of b-rectangular metric spaces and prove fixed point results for the mappings introduced.

Definition 11 Suppose that $\psi \in \Psi$, $\alpha : X \times X \to \Re^+$ and $\theta \in \Theta_s$. A self-mapping T on a b-rectangular metric space (X,d) is called generalized (α, ψ) -

Geraghty contraction if it satisfies for all $x, y \in X$ the following condition:

$$\alpha(x,y)\psi(s^2 d(Tx,Ty)) \leq \theta(M(x,y))\psi(M(x,y)), \tag{1}$$

where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx).d(y,Ty)}{1+d(Tx,Ty)}\}$.

Theorem 8 Let (X,d) be a complete b-rectangular metric space, $T : X \to X$ and $\alpha : X \times X \to \Re^+$. Suppose that the following conditions hold:

(1) T is a triangular α- orbital admissible mapping;
(2) T is generalized (α,ψ)-Geraghty contraction mapping;
(3) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(4) for every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$;
(5) T is continuous.

Then T has a unique fixed point.

Proof. By (3) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define iterative sequence $\{x_n\}$ as $x_{n+1} = Tx_n$, for $n \in \mathbb{N} \cup \{0\}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Since $Tx_{n_0} = x_{n_0+1} = x_{n_0}$ the point x_{n_0} forms a fixed point of T that completes the proof. From now on we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By condition (3), we have $\alpha(x_0, Tx_0) \geq 1$. Using Lemma 1, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2}$$

From (1) and (2), for all $n \in \mathbb{N} \cup \{0\}$, we have the following.

$$\begin{aligned}
\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\
&\leq \alpha(x_{n-1}, x_n)\psi(s^2 d(Tx_{n-1}, Tx_n)) \\
&\leq \theta(M(x_{n-1}, x_n))\psi(M(x_{n-1}, x_n)) \\
&< \frac{1}{s}\psi(M(x_{n-1}, x_n)).
\end{aligned}$$

So, we obtain

$$\psi(d(x_n, x_{n+1})) < \frac{1}{s}\psi(M(x_{n-1}, x_n)). \tag{3}$$

Where

$$\begin{aligned}
M(x_{n-1}, x_n) &= max\Bigg\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\
&\quad \frac{d(x_{n-1}, Tx_{n-1}).d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, Tx_{n-1}).d(x_n, Tx_n)}{1 + d(Tx_{n-1}, Tx_n)} \Bigg\} \\
&= max\Bigg\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\
&\quad \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} \Bigg\} \\
&= max\left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\}.
\end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then by (3) we get,
$\psi(d(x_n, x_{n+1})) < \frac{1}{s}\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1}))$ which is a contradiction.
Hence $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Using (3), we have
$\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n))$.
Since ψ is non-decreasing, we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$. Hence the sequence

$\{d(x_n, x_{n+1})\}$ is decreasing. So, $\{d(x_n, x_{n+1})\}$ converges to a non-negative real number. Thus there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \text{ and } d(x_n, x_{n+1}) \geq r. \tag{4}$$

We prove that $r = 0$. Suppose that $r > 0$. Since ψ is non-decreasing and by using (3) and (4), we obtain that,

$$\begin{aligned}
\frac{1}{s}\psi(r) \leq \psi(r) \leq \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\
&\leq \alpha(x_n, x_{n+1})\psi(s^2 d(Tx_n, Tx_{n+1})) \\
&\leq \theta(M(x_n, x_{n+1}))\psi(M(x_n, x_{n+1})) \\
&= \theta(d(x_n, x_{n+1}))\psi(d(x_n, x_{n+1})) \\
&< \frac{1}{s}\psi(d(x_n, x_{n+1})).
\end{aligned}$$

By applying limit as $n \to \infty$ in the above inequality and by the property of θ, we get
$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = \frac{1}{s}$, implies $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ which is a contradiction. Hence we have $r = 0$ and therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5}$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$ or $x_n \neq x_{n+p}$ for all $n, p \in \mathbb{N}$. Assume on the contrary there exist $n, p \in \mathbb{N}$ such that $x_n = x_{n+p}$. Since $d(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N} \cup \{0\}$. Without loss of generality, we may assume that $p > 1$. Using (1) and (2), we obtain that

$$\begin{aligned}
\psi(d(x_n, x_{n+1})) &= \psi(d(x_{n+p}, x_{n+p+1})) = \psi(d(Tx_{n+p-1}, Tx_{n+p})) \\
&\leq \alpha(x_{n+p-1}, x_{n+p})\psi(S^2 d(Tx_{n+p-1}, Tx_{n+p})) \\
&\leq \theta(M(x_{n+p-1}, x_{n+p}))\psi(M(x_{n+p-1}, x_{n+p})) \\
&< \frac{1}{s}\psi(M(x_{n+p-1}, x_{n+p})) \leq \psi(M(x_{n+p-1}, x_{n+p})).
\end{aligned}$$

So, we get

$$\psi(d(x_n, x_{n+1})) = \psi(d(x_{n+p}, x_{n+p+1})) < \psi(M(x_{n+p-1}, x_{n+p})). \tag{6}$$

Where

$$\begin{aligned}
M(x_{n+p-1}, x_{n+p}) &= max\Big\{ d(x_{n+p-1}, x_{n+p}), d(x_{n+p-1}, Tx_{n+p-1}), d(x_{n+p}, Tx_{n+p}, \\
&\quad \frac{d(x_{n+p-1}, Tx_{n+p-1}).d(x_{n+p}, Tx_{n+p})}{1 + d(x_{n+p-1}, x_{n+p})}, \frac{d(x_{n+p-1}, Tx_{n+p-1}).d(x_{n+p}, Tx_{n+p})}{1 + d(Tx_{n+p-1}, Tx_{n+p})} \Big\} \\
&= max\Big\{ d(x_{n+p-1}, x_{n+p}), d(x_{n+p-1}, x_{n+p}), d(x_{n+p}, x_{n+p+1}, \\
&\quad \frac{d(x_{n+p-1}, x_{n+p}).d(x_{n+p}, x_{n+p+1})}{1 + d(x_{n+p-1}, x_{n+p})}, \frac{d(x_{n+p-1}, x_{n+p}).d(x_{n+p}, x_{n+p+1})}{1 + d(x_{n+p}, x_{n+p+1})} \Big\} \\
&= max\{d(x_{n+p-1}, x_{n+p}), d(x_{n+p}, x_{n+p+1})\}.
\end{aligned}$$

If $M(x_{n+p-1},x_{n+p})=d(x_{n+p},x_{n+p+1})$, then from (6), we obtain that

$$\psi(d(x_n,x_{n+1})) = \psi(d(x_{n+p},x_{n+p+1}))<\psi(d(x_{n+p},x_{n+p+1})),$$

which is a contradiction. Hence $M(x_{n+p-1},x_{n+p})=d(x_{n+p-1},x_{n+p})$.
By (6) we obtain that

$$\psi(d(x_n,x_{n+1})) = \psi(d(x_{n+p},x_{n+p+1}))<\psi(d(x_{n+p-1},x_{n+p})). \tag{7}$$

From (7) we get,
$d(x_n,x_{n+1})<d(x_{n+p-1},x_{n+p})$.
By using (1) we have,

$$\begin{aligned}\psi(d(x_{n+p-1},x_{n+p})) &= \psi(d(Tx_{n+p-2},Tx_{n+p-1}))\\ &\le \alpha(x_{n+p-2},x_{n+p-1})\psi(s^2d(Tx_{n+p-2},Tx_{n+p-1}))\\ &\le \theta(M(x_{n+p-2},x_{n+p-1}))\psi(M(x_{n+p-2},x_{n+p-1}))\\ &< \frac{1}{s}\psi(M(x_{n+p-2},x_{n+p-1}))\\ &\le \psi(M(x_{n+p-2},x_{n+p-1}))=\psi(d(x_{n+p-2},x_{n+p-1})).\end{aligned}$$

Since ψ is non-decreasing, we have
$d(x_{n+p-1},x_{n+p})<d(x_{n+p-2},x_{n+p-1})$.
By continuing this process, we obtain the following inequality

$$d(x_n,x_{n+1})<d(x_{n+p-1},x_{n+p})<d(x_{n+p-2},x_{n+p-1})<\ldots<d(x_n,x_{n+1}),$$

which is a contradiction. Hence $x_n\neq x_m$ for all $n\neq m$. We now prove that $\{d(x_n,x_{n+2})\}$ is bounded. Since $\{d(x_n,x_{n+1})\}$ is bounded, there exists $B>0$ such that $d(x_n,x_{n+1})\le B$ for all $n\in\mathbb{N}\cup\{0\}$.
If $d(x_n,x_{n+2})>B$ for all $n\in\mathbb{N}\cup\{0\}$, then from

$$\begin{aligned}M(x_{n-1},x_{n+1}) &= max\Big\{d(x_{n-1},x_{n+1}),d(x_{n-1},Tx_{n-1}),d(x_{n+1},Tx_{n+1}),\\ &\qquad \frac{d(x_{n-1},Tx_{n-1})d(x_{n+1},Tx_{n+1})}{1+d(x_{n-1},x_{n+1})},\frac{d(x_{n-1},Tx_{n-1})d(x_{n+1},Tx_{n+1})}{1+d(Tx_{n-1},Tx_{n+1})}\Big\}\\ &= max\Big\{d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2}),\\ &\qquad \frac{d(x_{n-1},x_n)d(x_{n+1},x_{n+2})}{1+d(x_{n-1},x_{n+1})},\frac{d(x_{n-1},x_n)d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+2})}\Big\}\\ &= d(x_{n-1},x_{n+1})\end{aligned}$$

and Lemma 1, we obtain that

$$\begin{aligned}\psi(d(x_n,x_{n+2})) &= \psi(d(Tx_{n-1},Tx_{n+1}))\\ &\le \alpha(x_{n-1},x_{n+1})\psi(s^2d(Tx_{n-1},Tx_{n+1}))\\ &\le \theta(M(x_{n-1},x_{n+1}))\psi(M(x_{n-1},x_{n+1}))\\ &< \frac{1}{s}\psi(M(x_{n-1},x_{n+1}))\\ &\le \psi(M(x_{n-1},x_{n+1}))=\psi(d(x_{n-1},x_{n+1})).\end{aligned} \tag{8}$$

From (8) we get,
$d(x_n, x_{n+2}) < d(x_{n-1}, x_{n+1})$.
This implies that $\{d(x_n, x_{n+2})\}$ is decreasing and bounded.
If $d(x_n, x_{n+2}) \leq B$ for some $n \in \mathbb{N} \cup \{0\}$, then from

$$\begin{aligned}
M(x_n, x_{n+2}) &= max\Big\{ d(x_n, x_{n+2}), d(x_n, Tx_n), d(x_{n+2}, Tx_{n+2}), \\
&\quad \frac{d(x_n, Tx_n)d(x_{n+2}, Tx_{n+2})}{1 + d(x_n, x_{n+2})}, \frac{d(x_n, Tx_n)d(x_{n+2}, Tx_{n+2})}{1 + d(Tx_n, Tx_{n+2})} \Big\} \\
&= max\Big\{ d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3}), \\
&\quad \frac{d(x_n, x_{n+1})d(x_{n+2}, x_{n+3})}{1 + d(x_n, x_{n+2})}, \frac{d(x_n, x_{n+1})d(x_{n+2}, x_{n+3})}{1 + d(x_{n+1}, x_{n+3})} \Big\} \\
&= B
\end{aligned}$$

and Lemma 1, we obtain that

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+3})) &= \psi(d(Tx_n, Tx_{n+2})) \\
&\leq \alpha(x_n, x_{n+2})\psi(s^2 d(Tx_n, Tx_{n+2})) \\
&\leq \theta(M(x_n, x_{n+2}))\psi(M(x_n, x_{n+2})) \\
&< \frac{1}{s}\psi(M(x_n, x_{n+2})) \leq \psi(M(x_n, x_{n+2})) = \psi(B).
\end{aligned}$$

Therefore, $d(x_{n+1}, x_{n+3}) < B$. This implies that $\{d(x_n, x_{n+2})\}$ is bounded. We next prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. Suppose that $\lim_{n\to\infty} d(x_n, x_{n+2}) \neq 0$. So there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+2}) = a$ for some $a > 0$. Using (1) and Lemma 1 we have,

$$\begin{aligned}
\psi(d(x_{n_k+1}, x_{n_k+3})) &= \psi(d(Tx_{n_k}, Tx_{n_k+2})) \\
&\leq \alpha(x_{n_k}, x_{n_k+2})\psi(s^2 d(Tx_{n_k}, Tx_{n_k+2})) \\
&\leq \theta(M(x_{n_k}, x_{n_k+2}))\psi(M(x_{n_k}, x_{n_{k+2}})) \\
&= \theta(d(x_{n_k}, x_{n_k+2}))\psi(d(x_{n_k}, x_{n_k+2})) \\
&< \frac{1}{s}\psi(d(x_{n_k}, x_{n_k+2})).
\end{aligned}$$

Letting $k \to \infty$ in above inequality we obtain that, $\lim_{k\to\infty} \theta(d(x_{n_k}, x_{n_k+2})) = \frac{1}{s}$ this is $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+2})) = 0$, which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{9}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{10}$$

This implies that

$$d(x_{m_k}, x_{n_k-1}) < \frac{\epsilon}{s}. \tag{11}$$

By applying the b-rectangular inequality, using (10) and (11) we get that

$$\begin{aligned}\epsilon &\leq d(x_{m_k},x_{n_k})\\ &\leq sd(x_{m_k},x_{n_k-1})+sd(x_{n_k-1},x_{n_k-2})+sd(x_{_k-2},x_{n_k})\\ &< s.\frac{\epsilon}{s}+sd(x_{n_k-1},x_{n_k-2})+sd(x_{_k-2},x_{n_k})\\ &= \epsilon+sd(x_{n_k-1},x_{n_k-2})+sd(x_{n_k-2},x_{n_k}).\end{aligned}$$

Letting $k\to\infty$ in above inequality, using (5) and (9) we get that

$$\lim_{k\to\infty} d(x_{n_k},x_{m_k})=\epsilon. \tag{12}$$

By using (5) and (12) we obtain that

$$\lim_{k\to\infty} M(x_{n_k},x_{m_k})=\epsilon. \tag{13}$$

By (12) and (13), there exists a positive integer k_0 such that $d(x_{n_{k+1}},x_{m_{k+1}})>0$ and $M(x_{n_k},x_{m_k})>0$, for all $k\geq k_0$.
By Lemma 1 and using (1), for all $n_k>m_k>k>k_0$ we get that

$$\begin{aligned}\psi(d(x_{n_k+1},x_{m_k+1})) &= \psi(d(Tx_{n_k},Tx_{m_k}))\\ &\leq \alpha(x_{n_k},x_{m_k})\psi(s^2d(Tx_{n_k},Tx_{m_k}))\\ &\leq \theta(M(x_{n_k},x_{m_k}))\psi(M(x_{n_k},x_{m_k}))\\ &< \frac{1}{s}\psi(M(x_{n_k},x_{m_k})).\end{aligned}$$

Letting $k\to\infty$ in above inequality, by (12), (13), the property of θ and the continuity of ψ we obtain that,
$\lim\limits_{k\to\infty}\theta(M(x_{n_k},x_{m_k}))=\frac{1}{s}$ it follows that $\lim\limits_{k\to\infty} M(x_{n_k},x_{m_k}))=0$, which is a contradiction.
Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete b-rectangular metric space, it follows that $\{x_n\}$ converges to $x\in X$. Since T is continuous, we have
$x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}Tx_n=T(\lim\limits_{n\to\infty}x_n)=Tx.$
Therefore, x is a fixed point of T. Next we show uniqueness of fixed point.
Assume that T has two distinct fixed points, say $x,y\in X$, such that $x\neq y$, or $d(x,y)>0$. Using (1) and the fact that $\alpha(x,y)\geq 1$ gives

$$\begin{aligned}\psi(d(x,y)) &= \psi(d(Tx,Ty))\\ &\leq \alpha(x,y)\psi(s^2d(Tx,Ty))\\ &\leq \theta(M(x,y))\psi(M(x,y))\\ &< \frac{1}{s}\psi(M(x,y))\leq\psi(M(x,y))\end{aligned}$$

where

$$\begin{aligned}M(x,y) &= max\Bigg\{d(x,y),d(x,Tx),d(y,Ty),\\ &\qquad \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)},\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\Bigg\}\\ &= max\{d(x,y),0\}=d(x,y).\end{aligned}$$

This implies, $\psi(d(x,y)) < \psi(d(x,y))$, which is a contradiction. Hence, $d(x,y) = 0$ or $x = y$. This completes the proof of the uniqueness of the fixed point. $\square$

Theorem 9 Let (X,d) be a complete b-rectangular metric spaces, $T : X \to X$ and $\alpha : X \times X \to \Re^+$. Suppose that the following conditions hold:

(1) T is a triangular α- orbital admissible mappings;
(2) T is generalized (α,ψ)-Geraghty contraction mappings;
(3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(4) For every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$;
(5) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.
Then T has a unique fixed point.

Proof. As in the proof of Theorem 8, we can construct the sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$, $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ is Cauchy. Since X is complete there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. By (5), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$. We can suppose that $x_{n_k} \neq Tx$ for all $k \in \mathbb{N} \cup \{0\}$. Using (1), we obtain that

$$\begin{aligned}
\psi(sd(x_{n_k+1}, Tx)) &= \psi(sd(Tx_{n_k}, Tx)) \\
&\leq \alpha(x_{n_k}, x)\psi(s^2 d(Tx_{n_k}, Tx)) \\
&\leq \theta(M(x_{n_k}, x))\psi(M(x_{n_k}, x)) \\
&< \frac{1}{s}\psi(M(x_{n_k}, x)) \leq \psi(M(x_{n_k}, x)).
\end{aligned}$$

By taking the limit as $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} \psi(sd(x_{n_k+1}, Tx)) \leq \lim_{k\to\infty} \theta(M(x_{n_k}, x))\psi(M(x_{n_k}, x)) \leq \frac{1}{s} \lim_{k\to\infty} \psi(M(x_{n_k}, x)). \quad (14)$$

Where

$$\begin{aligned}
M(x_{n_k}, x) &= max\left\{ d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}), d(x, Tx), \right. \\
&\quad \left. \frac{d(x_{n_k}, Tx_{n_k})d(x, Tx)}{1 + d(x_{n_k}, x)}, \frac{d(x_{n_k}, Tx_{n_k})d(x, Tx)}{1 + d(Tx_{n_k}, Tx)} \right\} \\
&= max\left\{ d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx), \right. \\
&\quad \left. \frac{d(x_{n_k}, x_{n_k+1})d(x, Tx)}{1 + d(x_{n_k}, x)}, \frac{d(x_{n_k}, x_{n_k+1})d(x, Tx)}{1 + d(x_{n_k+1}, Tx)} \right\} \to d(x, Tx), \text{ as } k \to \infty.
\end{aligned}$$

Now, We prove that $x = Tx$. Suppose that $x \neq Tx$. Then,
$d(x, Tx) \leq sd(x, x_{n_k}) + sd(x_{n_k}, x_{n_k+1}) + sd(x_{n_k+1}, Tx)$.
It follows that

$$d(x, Tx) \leq \lim_{k\to\infty} sd(x_{n_k+1}, Tx). \quad (15)$$

By using ψ, θ, (13), (14) and (15), we obtain that

$$\begin{aligned}\psi(d(x,Tx)) &\leq \lim_{k\to\infty}\psi(sd(x_{n_k+1},Tx))\\ &\leq \lim_{k\to\infty}[\theta(M(x_{n_k},x))\psi(M(x_{n_k},x))]\\ &\leq \frac{1}{s}\psi(d(x,Tx)) \leq \psi(d(x,Tx))\end{aligned}$$

thus, $\lim_{k\to\infty}\theta(M(x_{n_k},x)) = \frac{1}{s}$, this implies that $\lim_{k\to\infty}M(x_{n_k},x) = d(x,Tx) = 0$, which is a contradiction. Hence $x = Tx$ and so, x is a fixed point of T. The proof of uniqueness is identical to the proof of Theorem 8. □

Corollary 1

Let (X,d) be a complete rectangular b- metric spaces, $T : X \to X$, $\alpha : X \times X \to \Re^+$ and $\theta \in \Theta_s$. Suppose that the following conditions hold:

(1) $\alpha(x,y)d(Tx,Ty) \leq \theta(M(x,y))M(x,y)$,

where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx).d(y,Ty)}{1+d(Tx,Ty)}\}$;

(2) T is a triangular α- orbital admissible mapping;

(3) There exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;

(4) For every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$;

(5) Either T is continuous or $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a unique fixed point.

proof. The result follows by taking $\psi(t) = t$ in Theorem 8 (or Theorem 9).

Corollary 2 Let (X,d) be a complete rectangular metric spaces, $T : X \to X$ and $\psi \in \Psi$, $\alpha : X \times X \to \Re^+$, and $\theta \in \Theta$. Suppose that the following conditions hold:

(1) $\alpha(x,y)\psi(d(Tx,Ty)) \leq \theta(M(x,y))\psi(M(x,y))$,

where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx).d(y,Ty)}{1+d(Tx,Ty)}\}$;

(2) T is a triangular α- orbital admissible mapping;

(3) There exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;

(4) For every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$;

(5) Either T is continuous or $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a unique fixed point.

proof. The result follows by taking $s = 1$ in Theorem 8 (or Theorem 9).

Corollary 3 Let (X,d) be a complete rectangular metric spaces, $T : X \to X$ and $\psi \in \Psi$, $\alpha : X \times X \to \Re^+$, and $\theta \in \Theta$. Suppose that the following conditions hold:

(1) $\alpha(x,y)\psi(d(Tx,Ty)) \leq \theta(d(x,y))\psi(d(x,y))$;

(2) T is a triangular α- orbital admissible mapping;

(3) There exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;

(4) For every pair x and y of fixed points of T, $\alpha(x,y) \geq 1$;

(5) Either T is continuous or $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a unique fixed point.

proof. The result follows by taking $s = 1$ and $M(x, y) = d(x, y)$ in Theorem 8 (or Theorem 9).

Corollary 4 Let (X, d) be a complete rectangular metric spaces, $T : X \to X$, $\alpha : X \times X \to \Re^+$ and $\theta \in \Theta$. Suppose that the following conditions hold:

(1) $\alpha(x, y)d(Tx, Ty) \leq \theta(M(x, y))M(x, y)$,

where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx).d(y,Ty)}{1+d(Tx,Ty)}\}$;

(2) T is a triangular α- orbital admissible mapping;

(3) There exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(4) For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$;

(5) Either T is continuous or $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.
Then T has a unique fixed point.

proof. The result follows by taking $s = 1$ and $\psi(t) = t$ in Theorem 8 (or Theorem 9).

We now present an example for supporting our main result.

Example 1 Let $X = \{1, 2, 3, 4\}$. Defined $d : X \times X \to \Re^+$ as
$d(x, x) = 0$ for all $x \in X$
$d(1, 2) = d(2, 1) = 20$
$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = 2$
$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 4.$
Therefore (X, d) is complete rectangular b- metric spaces with $s = 2$ but (X, d) is not a metric and rectangular metric space because it lacks the triangle and rectangle inequality respectively as follows:
$20 = d(1, 2) \geq d(1, 3) + d(3, 2) = 2 + 2 = 4$
$20 = d(1, 2) \geq d(1, 3) + d(3, 4) + d(4, 2) = 2 + 4 + 4 = 10.$
Let $T : X \to X$ be the mapping defined by

$$T(x) = \begin{cases} 2 & \text{if } x \neq 4 \\ 3 & \text{if } x = 4. \end{cases}$$

Define $\alpha : X \times X \to \Re^+$, $\psi : \Re^+ \to \Re^+$ and $\theta : \Re^+ \to [0, \frac{1}{2})$ as

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in X\backslash\{4\} \\ \frac{1}{16} & otherwise. \end{cases};$$

$\psi(t) = \frac{t}{3}$ and $\theta(t) = \frac{1}{5}$.
We next illustrate that all conditions in Theorem 8 and Theorem 9 are holds. Taking $x_0 = 1$, we have $\alpha(x_0, Tx_0) = \alpha(1, T1) = \alpha(1, 2) = 1 \geq 1$.

We next prove that T is an α-orbital admissible.
Let $x \in X$ such that $\alpha(x, Tx) \geq 1$. Therefore $x, Tx \in X\backslash\{4\}$ and then $x \in \{1, 2, 3\}$. By definition of α, we obtain that
$\alpha(T1, T^21) = \alpha(2, 2) = 1 \geq 1,$
$\alpha(T2, T^22) = \alpha(2, 2) = 1 \geq 1,$
$\alpha(T3, T^23) = \alpha(2, 2) = 1 \geq 1.$
It follows that T is an α-orbital admissible. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$. By the definition of α, we have $x, y, Ty \in X\backslash\{4\}$. This yields

$\alpha(1,2) \geq 1$ and $\alpha(2,T2) \geq 1$ implies $\alpha(1,T2) \geq 1$,
$\alpha(1,3) \geq 1$ and $\alpha(3,T3) \geq 1$ implies $\alpha(1,T3) \geq 1$,
$\alpha(2,3) \geq 1$ and $\alpha(3,T3) \geq 1$ implies $\alpha(2,T3) \geq 1$,
$\alpha(2,1) \geq 1$ and $\alpha(1,T1) \geq 1$ implies $\alpha(2,T1) \geq 1$,
$\alpha(3,1) \geq 1$ and $\alpha(1,T1) \geq 1$ implies $\alpha(3,T1) \geq 1$,
$\alpha(3,2) \geq 1$ and $\alpha(2,T2) \geq 1$ implies $\alpha(3,T2) \geq 1$.
This implies that T is triangular α-orbital admissible. Let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$. By the definition of α, for each $n \in N$, we get that $x_n \in X \backslash \{4\} = \{1,2,3\}$. We obtain that $x \in \{1,2,3\}$. Thus we have $\alpha(x_n, x) \geq 1$ for each $n \in N$. We next prove T is generalized (α, ψ)-Geraghty contraction mapping. So we consider the following cases:
Case(i) For $x, y \in X \backslash \{4\}$,we have
$\alpha(x,y)\psi(s^2 d(Tx,Ty)) = \psi(4d(2,2)) = 0 \leq \theta(M(x,y))\psi(M(x,y))$.
Case(ii) For $x, y \in \{1,4\}$

$$\begin{aligned} M(1,4) &= max\{d(1,4), d(1,T1), d(4,T4), \frac{d(1,T1)d(4,T4)}{1+d(1,4)}, \frac{d(1,T1)d(4,T4)}{1+d(T1,T4)}\} \\ &= max\{4, 20, 4, 16, \frac{80}{3}\} = \frac{80}{3}. \end{aligned}$$

This implies that
$\alpha(1,4)\psi(s^2 d(T1,T4)) = \frac{1}{16}\psi(4d(2,3)) = \frac{1}{6} \leq \theta(M(1,4))\psi(M(1,4)) = \frac{80}{45}$.
Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that
$\alpha(4,1)\psi(s^2 d(T4,T1)) \leq \theta(M(4,1))\psi(M(4,1))$.
Case(iii) For $x, y \in \{2,4\}$

$$\begin{aligned} M(2,4) &= max\{d(2,4), d(2,T2), d(4,T4), \frac{d(2,T2)d(4,T4)}{1+d(2,4)}, \frac{d(2,T2)d(4,T4)}{1+d(T2,T4)}\} \\ &= max\{4, 0, 4, 0, 0\} = 4. \end{aligned}$$

This implies that
$\alpha(2,4)\psi(s^2 d(T2,T4)) = \frac{1}{16}\psi(4d(2,3)) = \frac{1}{6} \leq \theta(M(2,4))\psi(M(2,4)) = \frac{4}{15}$.
Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that
$\alpha(4,2)\psi(s^2 d(T4,T2)) \leq \theta(M(4,2))\psi(M(4,2))$.
Case(iv) For $x, y \in \{3,4\}$

$$\begin{aligned} M(3,4) &= max\{d(3,4), d(3,T3), d(4,T4), \frac{d(3,T3)d(4,T4)}{1+d(3,4)}, \frac{d(3,T3)d(4,T4)}{1+d(T3,T4)}\} \\ &= max\{4, 2, 4, \frac{8}{5}, \frac{8}{3}\} = 4. \end{aligned}$$

This implies that
$\alpha(3,4)\psi(s^2 d(T3,T4)) = \frac{1}{16}\psi(4d(2,3)) = \frac{1}{6} \leq \theta(M(3,4))\psi(M(3,4)) = \frac{4}{15}$.
Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that
$\alpha(4,3)\psi(s^2 d(T4,T3)) \leq \theta(M(4,3))\psi(M(4,3))$.
Case(v) For $x = y = 4$

$$\begin{aligned} M(4,4) &= max\{d(4,4), d(4,T4), d(4,T4), \frac{d(4,T4)d(4,T4)}{1+d(4,4)}, \frac{d(4,T4)d(4,T4)}{1+d(T4,T4)}\} \\ &= max\{0, 4, 4, 16\} = 16. \end{aligned}$$

This implies that
$\alpha(4,4)\psi(s^2 d(T4,T4)) = \frac{1}{16}\psi(4d(3,3)) = 0 \leq \theta(M(4,4))\psi(M(4,4)) = \frac{16}{15}$.
Finally let $x, y \in F(T)$. Clearly $x = y = 2$, therefore, by the definition of α, we have $\alpha(x,y) = \alpha(2,2) = 1 \geq 1$. Hence all assumptions in Theorem 8 and Theorem 9 are satisfied and thus T has a unique fixed point which is $x = 2$.

4. Conclusion

The development of the field of fixed point theory depends on the generalization of the Banach Contraction principle on complete metric spaces. This generalization or extension comes up by either introducing new types of contractions or by working on a more general structured space such as rectangular b-metric spaces. In this article, we have proven some fixed point theorems for generalized (α,ψ)-Geraghty Contraction mappings in b-rectangular metric spaces and hence our results generalize many existing results in the literature.

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