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SOME RESULTS ON VALUE DISTRIBUTION THEORY FOR MEROMORPHIC FUNCTION IN AN ANGULAR DOMAIN

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Abstract. In this paper, we establish analogous of Milloux inequality and Hayman's alternative for meromorphic functions in an angular domain. As an application of our results, we deduce some interesting analogous results for meromorphic function in an angular domain. And also we have given the applications of homogeneous differential polynomials to the Nevanlinna's theory of meromorphic functions in an angular domain and given some generalisations by these polynomials.

1. Introduction

The uniqueness theory of meromorphic functions is an interesting problem in the value distribution theory. In 1929, R. Nevanlinna proved that, if f and $\widehat{f}$ be two non-constant meromorphic functions in $\mathbb{C}$ and if they share five distinct values IM, then $f \equiv \widehat{f}$; if they share four distinct values CM, then f is a Mobius transformation of $\widehat{f}$. After this work, many authors proved several results on uniqueness of meromorphic functions concerning shared values in the complex plane. In 2004, J. H. Zheng (see [1]) extended the uniqueness of meromorphic functions dealing with five shared values in an angular domains of $\mathbb{C}$. Also in 2010, He Ping proved some important results on the uniqueness of meromorphic functions sharing values in an angular domain (see [2]-[35]). It is interesting to prove some important uniqueness results in the whole of the complex plane to an angular domain.

2. Basic Notations and Definitions

Nevanlinna theory in an angular domain will play a key role in the proof of theorems. Let $f(z)$ be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$,

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|sin\omega(\theta - a)d\theta,$$

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$$C_{\alpha,\beta}(r,f)=\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^{\omega}}-\frac{|b_n|^{\omega}}{r^{2\omega}}\right)sin\omega(\theta_n-a)d\theta,$$

where $\omega=\pi/(\beta-\alpha)$ and $b_n=|b_n|e^{i\theta_n}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appearing according to the multiplicities. $C_{\alpha,\beta}$ is called angular counting function of the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ and Nevanlinna's angular characteristic function is defined as follows

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Throughout, we denote by $R_{\alpha,\beta}(r,*)$ a quantity satisfying satisfying

$$R_{\alpha,\beta}(r,*)=O\{\log(rS_{\alpha,\beta}(r,*))\},\quad r\in E,$$

where E denotes a set of positive real numbers with finite linear measure.

Definition . *Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. Then function*

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f)$$

is called angular Nevanlinna characteristic of $f(z)$.

3. Some Lemmas

Lemma 3.1. [1] *Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, $a\in\mathbb{C}$*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}\left(r,f\right)+O(1).$$

and for an integer $p\geq 0$,

$$S_{\alpha,\beta}(r,f^{(p)})\leq 2pS_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f),$$

$$A_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right)=R_{\alpha,\beta}(r,f),$$

and $R_{\alpha,\beta}(r,f^{(p)})=R_{\alpha,\beta}(r,f)$.

Lemma 3.2. [1] *Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. Then for arbitrary q distinct $a_j\in\overline{\mathbb{C}}$ $(1\leq j\leq q)$, we have*

$$(q-2)S_{\alpha,\beta}(r,f)\leq\sum_{j=1}^{q}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+R_{\alpha,\beta}(r,f),$$

where the term $\overline{C}_{\alpha,\beta}(r,1/f-a_j)$ will be replaced by $\overline{C}_{\alpha,\beta}(r,f)$ when some $a_j=\infty$.

We use $\overline{C}^{k)}_{\alpha,\beta}(r,1/f-a_j)$ to denote the zeros of $f(z)-a$ in $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ whose multiplicities are no greater than k and are counted only once. Likewise, we use $\overline{C}^{(k+1}_{\alpha,\beta}(r,1/f-a_j)$ to denote the zeros of $f(z)-a$ in $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ whose multiplicities are greater than k and are counted only once.

4. **Main Results**

In the value distribution theory, it is very important to introduce and study the derivative of a given function. It is natural to ask whether can we establish the analogous of Milloux inequality and Hayman's alternative in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$.

By adopting the notaions of Nevanlinna functions in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, we proved the following theorems and establish an interesting and remarkable result of the Milloux inequality and Heyman's alternative in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$.

Theorem 4.1. *Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. Let*

$$\Theta(z)=\sum_{l=0}^{k}a_l\,f^{(l)}(z) \tag{4.1}$$

for any positive integer k.Where $a_0,a_1,a_2,a_3,.....,a_k$ are small functions of f. Then

$$A_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right)+B_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right)=R_{\alpha,\beta}(r,f) \tag{4.2}$$

and

$$S_{\alpha,\beta}\left(r,\Theta\right)\leq(k+1)S_{\alpha,\beta}\left(r,f\right)+R_{\alpha,\beta}(r,f). \tag{4.3}$$

Theorem 4.2. *Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ and $\Theta(z)$ be the function defined by (4.1). If $\Theta(z)$ is not a constant, then*

$$\begin{aligned}S_{\alpha,\beta}(r,f) \quad &< \quad \overline{C}_{\alpha,\beta}(r,f)+C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)\\ &\quad +R_{\alpha,\beta}(r,f)\end{aligned} \tag{4.4}$$

where $(a\neq 0,\infty)$ and $C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)$ counts only zeros of Θ' but not the repeated roots of $\Theta=a$ in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$.

Theorem 4.3. *Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, $\Theta=f^{(k)}$ and $C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)$ be defined as in Theorem 4.2. Then*

$$k\,C^{1}_{\alpha,\beta}(r,f)\quad\leq\quad\overline{C}^{(2}_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(r,f) \tag{4.5}$$

where $C^{1}_{\alpha,\beta}(r,f)$ counts the simple poles of $f(z)$ and $\overline{C}^{(2}_{\alpha,\beta}(r,f)$ counts the multiple poles of $f(z)$, not including multiplicity in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$.

Theorem 4.4. *Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. Then*

$$S_{\alpha,\beta}(r,f)\leq\left(2+\frac{1}{k}\right)C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+R_{\alpha,\beta}(r,f). \tag{4.6}$$

By replacing $\Theta = f^{(k)}(z)$ in the Theorem 4.2, we get the following result

Corollary 4.1. *Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$ and k is any positive integer. Then*

$$S_{\alpha,\beta}(r,f) \leq \overline{C}_{\alpha,\beta}(r,f) + C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-a}\right) \tag{4.7}$$

$$-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{f^{(k+1)}}\right) + R_{\alpha,\beta}(r,f). \tag{4.8}$$

By Theorem 4.2, we get the following Corollary

Corollary 4.2. *Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$ with only a finite number of zeros and poles. Then every function Θ as defined in (4.13) assumes every finite complex value, except possibly zero, infinitely often or else is identically constant in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$..*

By replacing $\Theta = f^{(k)}(z)$ in the Theorem 4.4, we get the following result

Corollary 4.3. *Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$. Then*

$$S_{\alpha,\beta}(r,f) \leq \left(2+\frac{1}{k}\right) C_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{k}\right) \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-a}\right) + R_{\alpha,\beta}(r,f).$$

By replacing the value of $F = \dfrac{f-\omega_1}{\omega_2}$, where ω_1 and ω_2 be complex numbers $\omega_2 \neq 0$ and $S_{\alpha,\beta}(r,F) = S_{\alpha,\beta}(r,f) + O(1)$ in Theorem 4.2.4. Then we get the following result.

Corollary 4.4. *(Hayman's Alternative in annuli.) Let $f(z)$ be a transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$. Then either f assumes every finite value infinitely often or $f^{(k)}$ assumes every finite value except possibly zero infinitely often in an angular $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$.*

5. Proof of the main results

Proof of the Theorem 4.1 :
First of all, we prove the Theorem 4.1 for the case $\Theta(z) = f^{(k)}$ using induction on the number k and then deduce the conclusion of the Theorem 4.1 for the general case.
By Lemma 3.1, we have

$$\begin{aligned} S_{\alpha,\beta}(r,f') &= S_{\alpha,\beta}\left(r,f\frac{f'}{f}\right) \leq S_{\alpha,\beta}(r,f) + S_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + O(1) \\ &= S_{\alpha,\beta}(r,f) + A_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + B_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + C_{\alpha,\beta}\left(r,\frac{f'}{f}\right) \\ &\leq S_{\alpha,\beta}(r,f) + \overline{C}_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f) \\ &\leq 2\,S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f). \end{aligned}$$

Hence the result is true for $k=1$.
Suppose that the theorem is true for $k=n$. Then by assumption, we have

$$A_{\alpha,\beta}\left(r,\frac{f^{(n)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(n)}}{f}\right)=\ R_{\alpha,\beta}(r,f) \tag{5.1}$$

and

$$S_{\alpha,\beta}(r,f^{(n)})\leq\ (n+1)\,S_{\alpha,\beta}(r,f)\ +\ R_{\alpha,\beta}(r,f). \tag{5.2}$$

Also we have,

$$\begin{aligned} A_{\alpha,\beta}\left(r,f^{(n+1)}\right)+B_{\alpha,\beta}\left(r,f^{(n+1)}\right) &= A_{\alpha,\beta}\left(r,f^{(n)}\right)+B_{\alpha,\beta}\left(r,f^{(n)}\right) \\ &\quad +A_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f^{(n)}}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f^{(n)}}\right) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} C_{\alpha,\beta}(r,f^{(n+1)}) &= C_{\alpha,\beta}(r,f^{(n)})+\overline{C}_{\alpha,\beta}(r,f^{(n)}) \\ &= C_{\alpha,\beta}(r,f^{(n)})+\overline{C}_{\alpha,\beta}(r,f) \\ &\leq C_{\alpha,\beta}(r,f^{(n)})+C_{\alpha,\beta}(r,f). \end{aligned} \tag{5.4}$$

By Lemma 3.1, we have

$$\begin{aligned} A_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f}\right) &\leq A_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f^{(n)}}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f^{(n)}}\right) \\ &\quad +A_{\alpha,\beta}\left(r,\frac{f^{(n)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(n)}}{f}\right) \\ &\leq R_{\alpha,\beta}(r,f^{(n)})+R_{\alpha,\beta}(r,f) \\ &\leq R_{\alpha,\beta}(r,f) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} S_{\alpha,\beta}(r,f^{(n+1)}) &= A_{\alpha,\beta}\left(r,f^{(n+1)}\right)+B_{\alpha,\beta}\left(r,f^{(n+1)}\right)+\ C_{\alpha,\beta}(r,f^{(n+1)}) \\ &\leq A_{\alpha,\beta}(r,f^{(n)})+B_{\alpha,\beta}(r,f^{(n)})+\ A_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f^{(n)}}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(n+1)}}{f^{(n)}}\right) \\ &\quad +C_{\alpha,\beta}(r,f^{(n)})+\ C_{\alpha,\beta}(r,f)+O(1) \\ &\leq S_{\alpha,\beta}(r,f^{(n)})+\ C_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f) \\ &\leq (n+1)\,S_{\alpha,\beta}(r,f)+S_{\alpha,\beta}(r,f)+\ R_{\alpha,\beta}(r,f) \\ &\leq (n+2)\,S_{\alpha,\beta}(r,f)+\ R_{\alpha,\beta}(r,f). \end{aligned} \tag{5.6}$$

Hence the result is true for all positive integer k.
In the following, we consider the general case.
By above case, it is obvious that

$$A_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right)+B_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right)\ \leq\ \sum_{l=0}^{k}A_{\alpha,\beta}\left(R,\frac{a_l f^{(l)}}{f}\right)+\sum_{l=0}^{k}B_{\alpha,\beta}\left(R,\frac{a_l f^{(l)}}{f}\right)\ +\ log(k+1)$$

$$\begin{aligned} &\leq \sum_{l=0}^{k}\left[A_{\alpha,\beta}\left(r,a_l\right)+B_{\alpha,\beta}\left(r,a_l\right)+A_{\alpha,\beta}\left(r,\frac{f^{(l)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(l)}}{f}\right)\right]+log(k+1) \\ &\leq R_{\alpha,\beta}(r,f). \end{aligned} \tag{5.7}$$

Thus, we have

$$\begin{aligned} A_{\alpha,\beta}\left(r,\Theta\right)+B_{\alpha,\beta}\left(r,\Theta\right) &\leq A_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right)+B_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right)+A_{\alpha,\beta}\left(r,f\right)+B_{\alpha,\beta}\left(r,f\right) \\ &\leq A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f). \end{aligned} \tag{5.8}$$

On the other hand, we have

$$C_{\alpha,\beta}\left(r,\Theta\right)\leq C_{\alpha,\beta}\left(r,f^{(k)}\right)\leq C_{\alpha,\beta}\left(r,f\right)+k\,\overline{C}_{\alpha,\beta}\left(r,f\right). \tag{5.9}$$

Therefore, by (5.8) and (5.9), we have

$$\begin{aligned} S_{\alpha,\beta}\left(r,\Theta\right) &= A_{\alpha,\beta}\left(r,\Theta\right)+B_{\alpha,\beta}\left(r,\Theta\right)+C_{\alpha,\beta}\left(r,\Theta\right) \\ &\leq A_{\alpha,\beta}\left(r,f\right)+B_{\alpha,\beta}\left(r,f\right)+C_{\alpha,\beta}\left(r,f\right)+k\,\overline{C}_{\alpha,\beta}\left(r,f\right)+R_{\alpha,\beta}(r,f) \\ &\leq S_{\alpha,\beta}\left(r,f\right)+k\,\overline{C}_{\alpha,\beta}\left(r,f\right)+R_{\alpha,\beta}(R,f) \\ &\leq (k+1)S_{\alpha,\beta}\left(r,f\right)+R_{\alpha,\beta}(r,f) \\ S_{\alpha,\beta}\left(r,\Theta\right) &\leq (k+1)S_{\alpha,\beta}\left(r,f\right)+R_{\alpha,\beta}(r,f) \end{aligned}$$

which completes the proof of Theorem 4.1.

Proof of the Theorem 4.2:
By Lemma 3.1, we have

$$\begin{aligned} &A_{\alpha,\beta}\left(r,\Theta\right)+B_{\alpha,\beta}\left(r,\Theta\right)+A_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right)+B_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right) \\ &+A_{\alpha,\beta}\left(R,\frac{1}{\Theta-a}\right)+B_{\alpha,\beta}\left(R,\frac{1}{\Theta-a}\right) \\ &\leq 2\,S_{\alpha,\beta}\left(r,\Theta\right)-C^{(1)}_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,\Theta). \end{aligned} \tag{5.10}$$

By Lemma 3.1, we have

$$\begin{aligned} &2\,S_{\alpha,\beta}\left(r,\Theta\right)-C^{(1)}_{\alpha,\beta}(r,f) \\ =\ &A_{\alpha,\beta}\left(r,\Theta\right)+B_{\alpha,\beta}\left(r,\Theta\right)+A_{\alpha,\beta}\left(r,a,\Theta\right)+B_{\alpha,\beta}\left(r,a,\Theta\right)+C_{\alpha,\beta}\left(r,\Theta\right)+C_{\alpha,\beta}\left(r,a,\Theta\right) \\ &-\left[2\,C_{\alpha,\beta}\left(r,\Theta\right)-C_{\alpha,\beta}\left(r,\Theta'\right)+C_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)\right] \\ =\ &A_{\alpha,\beta}\left(r,\Theta\right)+B_{\alpha,\beta}\left(r,\Theta\right)+A_{\alpha,\beta}\left(r,a,\Theta\right)+B_{\alpha,\beta}\left(r,a,\Theta\right)+C_{\alpha,\beta}\left(r,a,\Theta\right) \\ &-C_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+C_{\alpha,\beta}\left(r,\Theta'\right)-C_{\alpha,\beta}\left(r,\Theta\right). \end{aligned} \tag{5.11}$$

It is obvious that

$$C_{\alpha,\beta}\left(r,\Theta'\right)-C_{\alpha,\beta}\left(r,\Theta\right)\quad\leq\quad\overline{C}_{\alpha,\beta}\left(r,f\right) \tag{5.12}$$

and

$$C_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)=\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right). \tag{5.13}$$

Hence it follows from (5.10), (5.11), (5.12) and (5.13) that

$$A_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right)+B_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right)\leq \overline{C}_{\alpha,\beta}\left(r,f\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C^{0}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(R,\Theta). \tag{5.14}$$

From (4.3), we have

$$R_{\alpha,\beta}(r,\Theta)=R_{\alpha,\beta}(r,f).$$

By Lemma 3.1, we have

$$\begin{aligned} S_{\alpha,\beta}\left(r,f\right) &= A_{\alpha,\beta}\left(r,\frac{1}{f}\right)+B_{\alpha,\beta}\left(r,\frac{1}{f}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+O(1) \\ &\leq A_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right)+B_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right)+A_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right)+B_{\alpha,\beta}\left(r,\frac{\Theta}{f}\right) \\ &\quad +C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+O(1) \\ &\leq A_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right)+B_{\alpha,\beta}\left(r,\frac{1}{\Theta}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+R_{\alpha,\beta}(r,f). \end{aligned} \tag{5.15}$$

From (5.14) and (5.15), we have

$$\begin{aligned} S_{\alpha,\beta}\left(r,f\right) &\leq \overline{C}_{\alpha,\beta}\left(r,f\right)+C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right) \\ &\quad -C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(R,f) \end{aligned}$$

which completes the Proof of Theorem 4.2.

Proof of the Theorem 4.3:
We first define the function

$$g=\frac{\left(f^{(k+1)}\right)^{k+1}}{\left(a-f^{(k)}\right)^{k+2}}=\frac{(\Theta')^{k+1}}{(a-\Theta)^{k+2}}. \tag{5.16}$$

Suppose f has a simple pole at z_0, in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq \beta\}$ i,e $f(z)=b(z-z_0)^{-1}+O(1)$ for some $b\neq 0$. Then differentiating k times,

$$f^{(k)}(z)=\frac{(-1)^k a\,k!}{(z-z_0)^{k+1}}\left(1+O((z-z_0)^{k+1})\right).$$

Differentiating again and then substituting it into g, we find that

$$g=\frac{(-1)^k(k+1)^{k+1}}{a\,k!}\left(1+O((z-z_0)^{k+1})\right).$$

Thus, at a simple pole of f, $g\neq 0,\infty$, in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ but g' has a zero of order at least k in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. Now we apply First Lemma 3.1 to $\frac{g'}{g}$, assuming g to be non constant, giving

$$\begin{aligned} A_{\alpha,\beta}\left(r,\frac{g'}{g}\right)+B_{\alpha,\beta}\left(r,\frac{g'}{g}\right) &- A_{\alpha,\beta}\left(r,\frac{g}{g'}\right)+B_{\alpha,\beta}\left(r,\frac{g}{g'}\right)+O(1)\\ &= C_{\alpha,\beta}\left(r,\frac{g}{g'}\right)-C_{\alpha,\beta}\left(r,\frac{g'}{g}\right)\\ &= C_{\alpha,\beta}\left(r,g\right)+C_{\alpha,\beta}\left(r,\frac{1}{g'}\right)-C_{\alpha,\beta}\left(r,g'\right)-C_{\alpha,\beta}\left(r,\frac{1}{g}\right)\\ &= C_{\alpha,\beta}\left(r,\frac{1}{g'}\right)-C_{\alpha,\beta}\left(r,\frac{1}{g}\right)-\overline{C}_{\alpha,\beta}\left(r,g\right)\\ &= C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{g'}\right)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{g}\right)-\overline{C}_{\alpha,\beta}\left(r,g\right). \qquad (5.17) \end{aligned}$$

Thus using (5.17), Lemma 1 and the property that $A_{\alpha,\beta}\left(r,\frac{g}{g'}\right)$ is non negative, we have

$$\begin{aligned} k\,C^{1}_{\alpha,\beta}(r,f) &\le C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{g'}\right)\le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{g}\right)\\ &\quad+\overline{C}_{\alpha,\beta}\left(r,g\right)+A_{\alpha,\beta}\left(r,\frac{g'}{g}\right)+B_{\alpha,\beta}\left(r,\frac{g'}{g}\right)+O(1)\\ &\le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{g}\right)+\overline{C}_{\alpha,\beta}\left(r,g\right)+R_{\alpha,\beta}(r,g). \qquad (5.18) \end{aligned}$$

By (5.18) and zeros and poles of g can only occur at multiple poles of f, a-points of Θ or zeros of Θ' which are not a-points of Θ in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\le argz\le\beta\}$ and so

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{g}\right)+\overline{C}_{\alpha,\beta}\left(r,g\right)\le \overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+\overline{C}^{(2}_{\alpha,\beta}\left(r,f\right)+C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right).$$

Hence by (5.15), we have

$$k\,C^{1}_{\alpha,\beta}(r,f)\le \overline{C}^{(2}_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(r,f).$$

Proof of the Theorem 4.4 :
We start by noting that in $N_0(R,f)$, multiple poles are counted at least twice in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\le argz\le\beta\}$ and then apply (4.4)

$$\begin{aligned} &C^{1}_{\alpha,\beta}(r,f)+2\,\overline{C}^{(2}_{\alpha,\beta}(r,f)\le S_{\alpha,\beta}(r,f)\\ &\quad\le \overline{C}_{\alpha,\beta}\left(r,f\right)+C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(r,f). \qquad (5.19) \end{aligned}$$

Since $\overline{C}_{\alpha,\beta}(r,f)=C^{1}_{\alpha,\beta}(r,f)+\overline{C}^{(2}_{\alpha,\beta}(r,f)$, hence by (5.19), we get

$$\overline{C}^{(2}_{\alpha,\beta}(r,f)\le C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(r,f). \qquad (5.20)$$

By (5.20) and (4.5), we get

$$
\begin{aligned}
k\,C^{1}_{\alpha,\beta}(r,f) &\leq C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)\\
&\quad+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(r,f)\\
k\,N^{1}_{\alpha,\beta}(r,f) &\leq C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+2\,\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+R_{\alpha,\beta}(r,f). \qquad (5.21)
\end{aligned}
$$

By (5.20) and (5.21), we can write

$$
\begin{aligned}
\overline{C}_{\alpha,\beta}(r,f) &= C^{1}_{\alpha,\beta}(r,f)+\overline{C}^{(2}_{\alpha,\beta}(r,f)\\
&\leq \frac{1}{k}C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\frac{2}{k}\,\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+N_0\left(R,\frac{1}{f}\right)\\
&\quad+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(r,f)
\end{aligned}
$$

$$
\begin{aligned}
\overline{C}_{\alpha,\beta}(r,f) &\leq \left(1+\frac{1}{k}\right)C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\left(1+\frac{2}{k}\right)\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)\\
&\quad-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)+R_{\alpha,\beta}(r,f). \qquad (5.22)
\end{aligned}
$$

Since $C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{\Theta'}\right)\geq 0$, we substitute this and (5.22) into (4.4), we get

$$
S_{\alpha,\beta}(r,f)\leq\left(2+\frac{1}{k}\right)C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\overline{C}_{\alpha,\beta}\left(r,\frac{1}{\Theta-a}\right)+R_{\alpha,\beta}(R,f).
$$

6. On the deficiencies of differential polynomials for meromorphic functions in an angular domain

We shall concerned with meromorphic functions P which are polynomials in the meromorphic function $f(z)$ and derivatives of $f(z)$ with coefficients of the form $a(z)$ in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$.

Let

$$
F_k=a\,(f)^{t_0}[f^1]^{t_1}[f^2]^{t_2}.......[f^m]^{t_m}
$$

and

$$
P=\sum_{k=1}^{N}F_k
$$

where $f^{(1)},f^{(2)},.......,f^{(m)}$ are the successive derivatives of f in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ and $t_0,t_1,...,t_m$ are non negative integers.

Definition . *If $t_0+t_1+...+t_m$ for a fixed positive integer in every term of P, then P is called a homogeneous differential polynomial in $f(z)$ of degree n in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$.*

Lemma 6.1. [12] *Let $f(z)$ be a meromorphic function on the $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ and $a_j\in\overline{\mathbb{C}}$ $(j=1,2,...,q)$ be q distinct complex numbers. Then we have*

$$\begin{aligned}&\sum_{j=1}^{q}A_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{q}B_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)\\=\;&A_{\alpha,\beta}\left(r,\sum_{j=1}^{q}\frac{1}{f-a_j}\right)+B_{\alpha,\beta}\left(r,\sum_{j=1}^{q}\frac{1}{f-a_j}\right)+O(1).\end{aligned}$$

We introduce some lemmas which are important and used later.

Lemma 6.2. *If P is a homogeneous differential polynomial in f in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ of degree $n\geq 1$, then*

$$A_{\alpha,\beta}\left(r,\frac{P}{f^n}\right)+B_{\alpha,\beta}\left(r,\frac{P}{f^n}\right)=R_{\alpha,\beta}(r,f) \tag{6.1}$$

Proof. We know that

$$A_{\alpha,\beta}\left(r,\frac{f^{(i)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(i)}}{f}\right)=R_{\alpha,\beta}(r,f)$$

for $i=1,2,3,......$

By definition, P is the sum of finite number of terms of the type

$$F_k=a\,(f)^{t_0}[f^1]^{t_1}[ff^2]^{t_2}.......[f^m]^{t_m}$$

Where $t_0+t_1+...+t_m$ are non-negative integers satisfying

$$\sum_{i=0}^{m}t_i=n$$

Then

$$\frac{F_k}{f^n}=a\left(\frac{f^{(1)}}{f}\right)^{t_1}\left(\frac{f^{(2)}}{f}\right)^{t_2}...\left(\frac{f^{(1)}}{f}\right)^{t_m}.$$

So,

$$\begin{aligned}&A_{\alpha,\beta}\left(r,\frac{F^{(k)}}{f^n}\right)+B_{\alpha,\beta}\left(r,\frac{F^{(k)}}{f^n}\right)\\\leq\;&A_{\alpha,\beta}\,(r,a)+B_{\alpha,\beta}\,(r,a)+\sum_{i=0}^{m}t_i\;A_{\alpha,\beta}\left(r,\frac{f^{(i)}}{f}\right)+\sum_{i=0}^{m}t_i\;B_{\alpha,\beta}\left(r,\frac{f^{(i)}}{f}\right)\\\leq\;&R_{\alpha,\beta}(r,f).\end{aligned}$$

Thus,

$$\begin{aligned}&A_{\alpha,\beta}\left(r,\frac{P}{f^n}\right)+B_{\alpha,\beta}\left(r,\frac{P}{f^n}\right)\\=\;&A_{\alpha,\beta}\left(r,\sum_k\frac{F^{(k)}}{f^n}\right)+B_{\alpha,\beta}\left(r,\sum_k\frac{F^{(k)}}{f^n}\right)\\\leq\;&\sum_k A_{\alpha,\beta}\left(r,\frac{F^{(k)}}{f^n}\right)+\sum_k B_{\alpha,\beta}\left(r,\frac{F^{(k)}}{f^n}\right)+O(1)\\\leq\;&R_{\alpha,\beta}(r,f).\end{aligned}$$

Which proves lemma 6.2. □

Lemma 6.3. *Let P be a homogeneous differential polynomial in f of degree n in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ and suppose that P is a homogeneous polynomial of degree n in $ff^{(1)},f^{(2)},.......,f^{(m)}$ with coefficients of the form $a(z)$ in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. If P is not a constant and $a_1,a_2,.....,a_q$ are distinct elements of $\mathbb{C}$ where q is any positive integer, then*

$$n\sum_{i=1}^{q}A_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+n\sum_{i=1}^{q}B_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)\leq S_{\alpha,\beta}(r,P)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f) \tag{6.2}$$

or

$$n\,q\,S_{\alpha,\beta}(r,f)\leq S_{\alpha,\beta}(r,P)+n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f) \tag{6.3}$$

Proof. We may assume that $q\geq 2$.

Let

$$F(z)=\sum_{i=1}^{q}\frac{1}{(f(z)-a)^n}$$

By Lemma 6.1, we have

$$\begin{aligned}
& A_{\alpha,\beta}(r,P)+B_{\alpha,\beta}(r,P)+B_{\alpha,\beta}(r,P)+B_{\alpha,\beta}(r,P)+O(1)\\
\geq\ & \sum_{i=1}^{q}A_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)^n+\sum_{i=1}^{q}B_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)^n\\
=\ & n\sum_{i=1}^{q}A_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+n\sum_{i=1}^{q}B_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right) \qquad (6.4)
\end{aligned}$$

Thus,

$$\begin{aligned}
& n\sum_{i=1}^{q}A_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+n\sum_{i=1}^{q}B_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)\\
\leq\ & A_{\alpha,\beta}(r,F)+B_{\alpha,\beta}(r,F)+O(1)\\
\leq\ & A_{\alpha,\beta}(r,PF)+B_{\alpha,\beta}(r,PF)+A_{\alpha,\beta}\left(r,\frac{1}{P}\right)+B_{\alpha,\beta}\left(r,\frac{1}{P}\right)+O(1)\\
\leq\ & A_{\alpha,\beta}(r,\frac{1}{P})+B_{\alpha,\beta}(r,\frac{1}{P})+\sum_{i=1}^{q}A_{\alpha,\beta}\left(r,\frac{P}{f-a_i}\right)^n+\sum_{i=1}^{q}B_{\alpha,\beta}\left(r,\frac{P}{f-a_i}\right)^n
\end{aligned} \tag{6.5}$$

Now for $1\leq i\leq q$, P is a homogeneous differential polynomial of degree n in $f-a_i$ in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, since the successive derivative of $f-a_i$ are precisely those of f and so by Lemma 6.1, we have

$$A_{\alpha,\beta}\left(r,\frac{P}{f-a_i}\right)^n+B_{\alpha,\beta}\left(r,\frac{P}{f-a_i}\right)^n=R_{\alpha,\beta}(r,W)$$

for $i = 1, 2, 3, ..., q$.

Hence from (6.5), we have

$$n\sum_{i=1}^{q} A_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+n\sum_{i=1}^{q} B_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right) \leq A_{\alpha,\beta}\left(r,\frac{1}{P}\right)+B_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f)$$

So,

$$\begin{aligned} & n\sum_{i=1}^{q} A_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+n\sum_{i=1}^{q} B_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P}\right) \\ \leq\ & A_{\alpha,\beta}\left(r,\frac{1}{P}\right)+B_{\alpha,\beta}\left(r,\frac{1}{P}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f) \end{aligned}$$

$$\begin{aligned} & n\sum_{i=1}^{q} A_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+n\sum_{i=1}^{q} B_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right) \\ \leq\ & S_{\alpha,\beta}\left(r,\frac{1}{P}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f) \\ \leq\ & S_{\alpha,\beta}(r,P)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,W) \end{aligned} \tag{6.6}$$

Which proves (6.2).

Next by (6.6), we prove (6.3)

$$\begin{aligned} & n\sum_{i=1}^{q} A_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+n\sum_{i=1}^{q} B_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+n\sum_{i=1}^{q} C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right) \\ \leq\ & S_{\alpha,\beta}\left(r,P\right)+n\sum_{i=1}^{q} C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f) \end{aligned}$$

$$n\,q\,S_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right) \leq S_{\alpha,\beta}\left(r,\frac{1}{P}\right)+n\sum_{i=1}^{q} C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f)$$

$$n\,q\,S_{\alpha,\beta}\left(r,f\right) \leq S_{\alpha,\beta}\left(r,\frac{1}{P}\right)+n\sum_{i=1}^{q} C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f).$$

Which proves (6.3). □

Theorem 6.1. *Let P be a homogeneous differential polynomial in f of degree n in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ and $a\neq b$. If f is a non constant meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, then we have the following inequality*

$$S_{\alpha,\beta}(r,f)\leq C_{\alpha,\beta}(r,P)+C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f-b}\right)-C_{\alpha,\beta}(r,f)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+C_{\alpha,\beta}(r,f) \tag{6.7}$$

Proof. Since $a\neq b$, we have

$$\frac{1}{f-b}=\left(\frac{P}{f-b}-\frac{P}{f-a}\right)\left(\frac{f-a}{P}\right)\frac{1}{b-a}$$

By Lemma 3.1, we have

$$\begin{aligned}
& A_{\alpha,\beta}\left(r, \frac{1}{f-b}\right) + B_{\alpha,\beta}\left(r, \frac{1}{f-b}\right) \\
\leq\ & A_{\alpha,\beta}\left(r, \frac{P}{f-b}\right) + B_{\alpha,\beta}\left(r, \frac{P}{f-b}\right) + A_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) + B_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) \\
& + A_{\alpha,\beta}\left(r, \frac{f-a}{P}\right) + B_{\alpha,\beta}\left(r, \frac{f-a}{P}\right) + O(1) \\
\leq\ & A_{\alpha,\beta}\left(r, \frac{P}{f-b}\right) + B_{\alpha,\beta}\left(r, \frac{P}{f-b}\right) + A_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) + B_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) \\
& + A_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) + B_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) + C_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) - C_{\alpha,\beta}\left(r, \frac{f-a}{P}\right) + O(1) \\
\leq\ & C_{\alpha,\beta}(r, P) + C_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) - C_{\alpha,\beta}\left(r, \frac{1}{P}\right) - C_{\alpha,\beta}(r, f) + R_{\alpha,\beta}(r, f). \qquad (6.8)
\end{aligned}$$

Where

$$A_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) + B_{\alpha,\beta}\left(r, \frac{P}{f-b}\right) \leq R_{\alpha,\beta}(r, f)$$

and

$$C_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) - C_{\alpha,\beta}\left(r, \frac{f-a}{P}\right) = C_{\alpha,\beta}(r, P) + C_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) - C_{\alpha,\beta}\left(r, \frac{1}{P}\right) - C_{\alpha,\beta}(r, f)$$

If we add the term $C_{\alpha,\beta}\left(r, \frac{1}{f-b}\right)$ on both sides of the inequality (6.8), we get

$$S_{\alpha,\beta}(r, f) \leq C_{\alpha,\beta}(r, P) + C_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) + C_{\alpha,\beta}\left(r, \frac{1}{f-b}\right) - C_{\alpha,\beta}\left(r, \frac{1}{P}\right) - C_{\alpha,\beta}(r, f) + R_{\alpha,\beta}(r, f) \qquad (6.9)$$

If we restrict $P = f'(z)$, the inequality (6.9) becomes

$$S_{\alpha,\beta}(r, f) \leq \overline{C}_{\alpha,\beta}(r, f) + C_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) + C_{\alpha,\beta}\left(r, \frac{1}{f-b}\right) - C_{\alpha,\beta}\left(r, \frac{1}{f'}\right) - C_{\alpha,\beta}(r, f) + R_{\alpha,\beta}(r, f).$$

□

Theorem 6.2. *Let P be a homogeneous differential polynomial in f of degree n and $b \neq 0$ in an angular domain $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \leq argz \leq \beta\}$. If f is a non constant algebroid function in an angular domain $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \leq argz \leq \beta\}$, then we have the following inequality*

$$S_{\alpha,\beta}(r, f) \leq \overline{C}_{\alpha,\beta}(r, f) + C_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) + C_{\alpha,\beta}\left(r, \frac{1}{P-b}\right) - C^{(0)}_{\alpha,\beta}\left(r, \frac{1}{P'}\right) + R_{\alpha,\beta}(r, f) \qquad (6.10)$$

Proof. Since $b \neq 0$, we have

$$\frac{1}{W-a} = \left(\frac{P}{W-a} - \frac{P'}{W-a}\,\frac{P-b}{P'}\right)\frac{1}{b}$$

By Lemma 3.1, we have

$$\begin{aligned}
& A_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) + B_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) \\
\leq\ & A_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) + B_{\alpha,\beta}\left(r, \frac{P}{f-a}\right) + A_{\alpha,\beta}\left(r, \frac{P'}{f-a}\right) + B_{\alpha,\beta}\left(r, \frac{P'}{f-a}\right) \\
& + A_{\alpha,\beta}\left(r, \frac{P-b}{P'}\right) + B_{\alpha,\beta}\left(r, \frac{P-b}{P'}\right) + O(1) \\
\leq\ & C_{\alpha,\beta}\left(r, \frac{P'}{P-b}\right) + C_{\alpha,\beta}\left(r, \frac{P-b}{P'}\right) + S(r, W)
\end{aligned}$$

Where

$$A_{\alpha,\beta}\left(r,\frac{P}{f-a}\right)+B_{\alpha,\beta}\left(r,\frac{P}{f-a}\right)+A_{\alpha,\beta}\left(r,\frac{P'}{f-a}\right)+B_{\alpha,\beta}\left(r,\frac{P'}{f-a}\right)$$
$$+A_{\alpha,\beta}\left(r,\frac{P-b}{P'}\right)+B_{\alpha,\beta}\left(r,\frac{P-b}{P'}\right)\leq R_{\alpha,\beta}(r,f)$$

Therefore,

$$\begin{aligned}
&A_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+B_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)\\
\leq\;& C_{\alpha,\beta}(r,P')+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)-C_{\alpha,\beta}(r,P)+R_{\alpha,\beta}(r,f)\\
\leq\;& \overline{C}_{\alpha,\beta}(r,f)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f)\\
\leq\;& \overline{C}_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f)
\end{aligned}\tag{6.11}$$

Thus,

$$\begin{aligned}
&C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)\\
&=\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-\left[\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)\right]\\
&=\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{P'}\right)
\end{aligned}$$

where

$$C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{P'}\right)=\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)$$

If we add the term $N\left(r,\frac{1}{W-b}\right)$ on both sides of the inequality (6.11) we get

$$S_{\alpha,\beta}(r,f)\leq\overline{C}_{\alpha,\beta}(r,f)+C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f)$$

If we restrict $P=W^{(k)}(z)$, the inequality (6.10) becomes

$$S_{\alpha,\beta}(r,f)\leq\overline{C}_{\alpha,\beta}(r,f)+C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-b}\right)-C^{(0)}_{\alpha,\beta}\left(r,\frac{1}{f^{(k+1)}}\right)+R_{\alpha,\beta}(r,f)$$

□

Which is one of the Milloux result.

Theorem 6.3. *Let P be a homogeneous differential polynomial in f of degree n in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. If f is a non constant meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, then*

$$S_{\alpha,\beta}(r,f)\leq C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-c}\right)-C^{1}_{\alpha,\beta}(r,P)+R_{\alpha,\beta}(r,f)\tag{6.12}$$

where

$$C^{1}_{\alpha,\beta}(r,P)=2\,C_{\alpha,\beta}(r,P)-C_{\alpha,\beta}(r,P')+C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)$$

are non negative.

Proof. It is easy to write

$$\frac{1}{f-a}=\frac{1}{P}\,\frac{P}{f-a}$$

By Lemma 3.1, we have

$$\begin{aligned}
& A_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+B_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)\\
\leq\ & A_{\alpha,\beta}\left(r,\frac{1}{P}\right)+B_{\alpha,\beta}\left(r,\frac{1}{P}\right)+\ A_{\alpha,\beta}\left(r,\frac{P}{f-a}\right)+B_{\alpha,\beta}\left(r,\frac{P}{f-a}\right)\\
\leq\ & A_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f)\\
\leq\ & S_{\alpha,\beta}\left(r,P\right)-\ C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,f) \qquad (6.13)
\end{aligned}$$

By Lemma 3.2, we have

$$S_{\alpha,\beta}\left(r,P\right)\leq C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-c}\right)-C^{1}_{\alpha,\beta}\left(r,P\right)+R_{\alpha,\beta}(r,f)$$

If we use Lemma 3.2 in the equality (6.13), we have

$$\begin{aligned}
S_{\alpha,\beta}\left(r,f\right)\ \leq\ & C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-c}\right)\\
& -C^{1}_{\alpha,\beta}\left(r,P\right)-C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+R_{\alpha,\beta}(r,W)
\end{aligned}$$

or

$$S_{\alpha,\beta}\left(r,f\right)\ \leq\ C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-c}\right)-C^{1}_{\alpha,\beta}\left(r,P\right)+R_{\alpha,\beta}(r,f)$$

If we restrict $P=f^{(k)}(z)$, the inequality (6.12) becomes

$$S_{\alpha,\beta}\left(r,f\right)\ \leq\ C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-b}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-c}\right)-C^{(1)}_{\alpha,\beta}\left(r,f^{(k)}\right)+R_{\alpha,\beta}(r,f).$$

□

Theorem 6.4. *Let P be a homogeneous differential polynomial in f of degree n in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. If f is a non constant meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, then*

$$nq\,S_{\alpha,\beta}\left(r,f\right)\leq\overline{C}_{\alpha,\beta}\left(r,f\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f). \qquad (6.14)$$

Proof. By Lemma 3.2, we have

$$S_{\alpha,\beta}\left(r,P\right)\leq C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+C_{\alpha,\beta}\left(r,P\right)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C^{1}_{\alpha,\beta}\left(r,P\right)+R_{\alpha,\beta}(r,P)$$

where

$$C^{1}_{\alpha,\beta}\left(r,P\right)=2\,C_{\alpha,\beta}\left(r,P\right)-\ C_{\alpha,\beta}\left(r,P'\right)+\ C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)$$

$$\begin{aligned} S_{\alpha,\beta}(r,P) &\leq C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+C_{\alpha,\beta}(r,P)+C_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right) \\ &\quad -2\,C_{\alpha,\beta}(r,P)+C_{\alpha,\beta}(r,P')+R_{\alpha,\beta}(r,P) \\ &\leq \overline{C}_{\alpha,\beta}(r,P)+C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,P) \end{aligned}$$

Therefore,

$$S_{\alpha,\beta}(r,P)\leq \overline{C}_{\alpha,\beta}(r,P)+C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-N_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,P)$$

On the other hand, it is easy to write $\overline{C}_{\alpha,\beta}(r,P)\leq \overline{C}_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f)$.

If we use the inequality (6.7), we can write

$$\begin{aligned} n\,q\,S_{\alpha,\beta}(r,f) &\leq \overline{C}_{\alpha,\beta}(r,P)+C_{\alpha,\beta}\left(r,\frac{1}{P}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right) \\ &\quad +n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f) \end{aligned}$$

or

$$n\,q\,S_{\alpha,\beta}(r,f)\leq \overline{C}_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b}\right)+n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f)$$

If $n=1$ and $q=1$ the inequality (6.14) gives the inequality (6.10). That is, the inequality (6.14) is the generalization of the inequality (6.10). □

Theorem 6.5. *Let P be a homogeneous differential polynomial in f of degree n in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$ and $s=1,2,3,...$, if f is a non constant meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$. Then*

$$\begin{aligned} (s-1)n\,q\,S_{\alpha,\beta}(r,f) &\leq \overline{C}_{\alpha,\beta}(r,f)+(s-1)\,n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+\sum_{j=1}^{s}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b_j}\right) \\ &\quad -C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f) \end{aligned} \tag{6.15}$$

If $s=3,4,5,...$, then we have

$$\begin{aligned} (s-2)n\,q\,S_{\alpha,\beta}(r,f) &\leq (s-2)\,n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right)+\sum_{j=1}^{s}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b_j}\right) \\ &\quad -C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f) \end{aligned} \tag{6.16}$$

Proof. Lemma 3.2 in terms of the homogeneous differential polynomial P in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leq argz\leq\beta\}$, then we have

$$(s-1)S_{\alpha,\beta}(r,f)\leq \overline{C}_{\alpha,\beta}(r,f)+\sum_{j=1}^{s}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b_j}\right)-C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)+R_{\alpha,\beta}(r,f) \tag{6.17}$$

and

$$(s-2)S_{\alpha,\beta}(r,f)\leq \sum_{j=1}^{s}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b_j}\right)-C^{(1)}_{\alpha,\beta}(r,P)+R_{\alpha,\beta}(r,f) \tag{6.18}$$

where

$$C_{\alpha,\beta}(1)\,(r,P) = 2\,C_{\alpha,\beta}\,(r,P) - \,C_{\alpha,\beta}\,(r,P') + \,C_{\alpha,\beta}\left(r,\frac{1}{P'}\right)$$

and non negative.

If we use the inequality (6.17) and (6.18) in the equality (6.7), we get

$$\begin{aligned}(s-1)n\,q\,S_{\alpha,\beta}\,(r,f) \quad &\leq \quad \overline{C}_{\alpha,\beta}\,(r,f) + \,(s-1)\,n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right) + \sum_{j=1}^{s}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b_j}\right)\\ &\quad - C_{\alpha,\beta}\left(r,\frac{1}{P'}\right) + R_{\alpha,\beta}(r,f)\end{aligned}$$

and

$$\begin{aligned}(s-2)n\,q\,S_{\alpha,\beta}\,(r,f) \quad &\leq \quad (s-2)\,n\sum_{i=1}^{q}C_{\alpha,\beta}\left(r,\frac{1}{f-a_i}\right) + \sum_{j=1}^{s}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{P-b_j}\right)\\ &\quad - C_{\alpha,\beta}\left(r,\frac{1}{P'}\right) + R_{\alpha,\beta}(r,f).\end{aligned}$$

□

7. Applications

We can use Milloux inequality and Hayman's alternative in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$ to prove results related to uniqueness and sharing of two meromorphic functions in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leq argz \leq \beta\}$.

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