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ON THE ADMISSIBILITY OF NONLINEAR SINGULAR SYSTEMS WITH TIME-VARYING DELAYS

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ABSTRACT. In this paper, using Jensen and Wirtinger inequalities and Lyapunov-Krasovskiĭ functionals (LKFs), we discuss the asymptotic admissibility, i.e., the concepts of asymptotic stability, regular and impulse-free, to a class of nonlinear singular systems of first order with two time-varying delays. Moreover, we give four examples with numerical simulations via MATLAB-Simulink to show applications of the obtained results. By this work, we extend some related results in the related literature.

1. Introduction

As it is known, time-delay systems are widely appeared in numerous mathematics and practical engineering systems such as nuclear reactors, long transmission lines in pneumatic systems, chemical process control, telecommunication, economic, aircraft, manufacturing systems, hydraulic systems and so on. Therefore, scientific works on time-delay differential systems have very important and effective roles in theoretically and practically researches fields. When the relevant literature is examined, it can be seen that many studies have been carried out on the characteristic features of such systems (see, for instance [6, 7, 12] and resources found in them).

A singular system is formulated as a mathematically combined set of differential and algebraic equations. Thus, singular systems are also called differential-algebraic systems. Singular systems are encountered in engineering systems (such as aerospace engineering, electrical circuit network and power systems), biological systems, network analysis. Due to this wide application areas, numerous books and articles have been published on linear and nonlinear singular systems, and numerous results related to the characteristic properties of these systems such as stability, instability, regular, impulse free and admissibility have been obtained (see, for example, [1-5, 13-18] and references therein).

Motivated by above discussions and [8-11], in this article, we consider the problem of asymptotic admissibility of nonlinear singular systems with two time-varying

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delays. By defining new LKFs, new asymptotic admissibility conditions, which guarantee the regularity, impulse-free and asymptotic stability, are established in terms of the matrix inequality technique. Numerical examples with their simulations are given to illustrate effectiveness of the used method.

We consider non-linear singular system with two variable delays:

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + \sum_{i=1}^{2} A_{d_i} x(t - d_i(t)) + F(t, x(t)), \\ x(t) =& \phi(t), t \in [-\tau_0, 0], \end{aligned} \tag{1}$$

where $t \in [-\tau_0, \infty), \tau_0$ is the constant delay, $x(t) \in \Re^n$, $\phi(t)$ is continuous initial function. $E, A, A_{d_i} \in \Re^{n\times n}, (i = 1, 2)$, are known real constant matrices with appropriate dimensions, the matrix $E \in \Re^{n\times n}$ is singular and $rankE = r \leq n, n \geq 1$. We also assume that $F \in C^1(\Re^+ \times \Re^n \to \Re^n), F(t, 0) = 0$, and

$$\|F(t, x_0) - F(t, y_0)\| \leq \|U(x_0 - y_0)\|, \forall t \in \Re^+, \forall x_0, y_0 \in \Re^n, \tag{2}$$

where U is a known constant matrix. $d_1(t), d_2(t) \in C^1(\Re^+, \Re^+)$ are variable delays and satifying

$$\begin{aligned} &0 \leq d_i(t) \leq \tau_i, \\ &\dot{d}_i(t) \leq \mu_i \leq 1, (i = 1, 2), \\ &\tau_0 = max\{\tau_1, \tau_2\}, \end{aligned} \tag{3}$$

where τ_0, τ_i and $\mu_i, (i = 1, 2)$, are positive constants.

We now give some basic information.

Definition 1.1 ([1]). If the conditions $det(sE - A) \neq 0$ and $deg(det(sE - A)) = rank(E)$ are satisfied, then the pair (E, A) is called regular and impulse-free, respectively.

Definition 1.2 ([15]). The singular system (1) is regular and impulse free if the pair (E, A) is regular and impulse free. In addition, the singular system (1) is admissible if the singular system (1) regular, impulse free and stable.

Lemma 1.1 (Jensen inequality [2]). Let $Z > 0$ be a matrix and $x : [a, b] \to \Re^n$ be an differentiable vector function. Then,

$$(b - a) \int_a^b x^T(s)Zx(s)ds \geq (\int_a^b x^T(s)ds)Z(\int_a^b x(s)ds).$$

Lemma 1.2 (Wirtinger inequality [6]). For any symmetric matrix $R \in \Re^{n\times n}$ and a continuously differentiable function $x : [a, b] \to \Re^n$ the following integral inequality holds:

$$\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{b - a}[x(b) - x(a)]^T R[x(b) - x(a)] + \frac{3}{b - a}\Omega^T R\Omega$$

Where

$$\Omega = x(a) + x(b) - (\frac{2}{b-a}) \int_a^b x(s)ds.$$

2. Main Results and Numerical Applications

Firstly, we show that the singular system (1) is asymptotically admissible.

A. Assumption

(B1) The system matrix E is singular, $rankE = r \leq n$, there exist positive definite symmetric matrices $P, R_i, Z, Q_i, (i = 1, 2)$, and a known constant matrix U of appropriate dimension such that the matrix inequality holds:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & \Xi_{26} \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} \\ * & * & * & \Xi_{44} & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0, \tag{4}$$

with

$$\Xi_{11} = A^T P + PA + \epsilon U^T U + \sum_{i=1}^{2} \tau_i A^T Z A - \sum_{i=1}^{2} \tau_i^{-1} E^T Z E + \sum_{i=1}^{2} (R_i + Q_i),$$

$$\Xi_{12} = PA_{d_1} + \sum_{i=1}^{2} \tau_i A^T Z A_{d_1}, \Xi_{13} = PA_{d_2} + \sum_{i=1}^{2} \tau_i A^T Z A_{d_2}, \Xi_{14} = \tau_1^{-1} E^T Z E,$$

$$\Xi_{15} = \tau_2^{-1} E^T Z E, \Xi_{16} = P + \sum_{i=1}^{2} \tau_i A^T Z, \Xi_{22} = \sum_{i=1}^{2} \tau_i A_{d_1}^T Z A_{d_1} - (1 - \mu_1) R_1,$$

$$\Xi_{23} = \sum_{i=1}^{2} \tau_i A_{d_1}^T Z A_{d_2}, \Xi_{26} = \sum_{i=1}^{2} \tau_i A_{d_1}^T Z, \Xi_{33} = \sum_{i=1}^{2} \tau_i A_{d_2}^T Z A_{d_2} - (1 - \mu_2) R_2,$$

$$\Xi_{36} = \sum_{i=1}^{2} \tau_i A_{d_2}^T Z, \Xi_{44} = -\tau_1^{-1} E^T Z E - Q_1, \Xi_{55} = -\tau_2^{-1} E^T Z E - Q_2,$$

$$\Xi_{66} = \sum_{i=1}^{2} \tau_i Z - \epsilon I,$$

where ϵ is a positive constant and I is $n \times n-$ identity matrix.

Theorem 2.1. If the conditions (3) and (B1) are satisfied, then the singular system (1) is asymptotically admissible.

proof. First, we show that the system (1) is regular and impulse-free. Hence, we choose two regular matrices $M, N \in \Re^{n \times n}$ such that

$$\overline{E} = MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

and define

$$\begin{aligned}\overline{A} =& MAN = \begin{bmatrix} \overline{A}_1 & \overline{A}_2 \\ \overline{A}_3 & \overline{A}_4 \end{bmatrix}, \overline{A}_{d_i} = MA_{d_i}N = \begin{bmatrix} \overline{A}_{d_{i1}} & \overline{A}_{d_{i2}} \\ \overline{A}_{d_{i3}} & \overline{A}_{d_{i4}} \end{bmatrix},\\ \overline{P} =& M^{-T}PN = \begin{bmatrix} \overline{P}_1 & \overline{P}_2 \\ \overline{P}_3 & \overline{P}_4 \end{bmatrix}, \overline{Z} = M^{-T}ZN = \begin{bmatrix} \overline{Z}_1 & \overline{Z}_2 \\ \overline{Z}_3 & \overline{Z}_4 \end{bmatrix}.\end{aligned} \tag{5}$$

Using (4), (5) and some simple calculations, we have $A_4^T P_4 + P_4 A_4 < 0$. Thus, $\overline{A}_4$ is regular. Because of this reason, the pair (E, A) is regular and impulse-free (see [15]). Next, in view of Definition 1.2, we conclude that the system (1) is regular and impulse-free.

We now show that the system (1) is asymptotically stable. We define an LKF as follows:

$$V(t,x) = \sum_{i=1}^{4} V_i(t,x), \tag{6}$$

where

$$\begin{aligned}V_1(t,x) =& x^T(t)E^T Px(t),\\ V_2(t,x) =& \sum_{i=1}^{2} \int_{-\tau_i}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha)E^T ZE\dot{x}(\alpha)d\alpha d\beta,\\ V_3(t,x) =& \sum_{i=1}^{2} \int_{t-d_i(t)}^{t} x^T(\alpha)R_i x(\alpha)d\alpha,\\ V_4(t,x) =& \sum_{i=1}^{2} \int_{t-\tau_i}^{t} x^T(\alpha)Q_i x(\alpha)d\alpha.\end{aligned}$$

It can be easily seen that the LKF defined by (6) is positive definite. In the light of Newton-Leibnitz formula and Jensen inequality, i.e., Lemma 1.1, taking the derivative of the LKF $V(t,x)$ all along the solutions of the system (1), we have:

$$\dot{V}(t,x) = \sum_{i=1}^{4} \dot{V}_i(t,x), \tag{7}$$

where

$$\begin{aligned}\dot{V}_1(t,x) =& x^T(t)[A^T P + PA]x(t) + \sum_{i=1}^{2} x^T(t)PA_{d_i}x(t-d_i(t))\\ &+ \sum_{i=1}^{2} x^T(t-d_i(t))A_{d_i}^T Px(t) + F^T(t,x(t))Px(t)\\ &+ x^T(t)PF(t,x(t)),\end{aligned} \tag{8}$$

$$\begin{aligned}
\dot{V}_2(t,x) = &\sum_{i=1}^{2}\tau_i x^T(t)A^T ZAx(t) + \sum_{i=1}^{2}\tau_i x^T(t)A^T ZF(t,x(t)) \\
&+ (\sum_{i=1}^{2}\tau_i)x^T(t)A^T Z(\sum_{i=1}^{2}A_{d_i}x(t-d_i(t))) \\
&+ (\sum_{i=1}^{2}\tau_i)(\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T)ZAx(t) \\
&+ (\sum_{i=1}^{2}\tau_i)(\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T)Z(\sum_{i=1}^{2}A_{d_i}x(t-d_i(t))) \\
&+ (\sum_{i=1}^{2}\tau_i)(\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T)ZF(t,x(t)) \\
&+ \sum_{i=1}^{2}\tau_i F^T(t,x(t))ZAx(t) \\
&+ (\sum_{i=1}^{2}\tau_i)F^T(t,x(t))Z(\sum_{i=1}^{2}A_{d_i}x(t-d_i(t))) \\
&+ (\sum_{i=1}^{2}\tau_i)F^T(t,x(t))ZF(t,x(t)) \\
&- \sum_{i=1}^{2}\tau_i^{-1}x^T(t)E^T ZEx(t) + \sum_{i=1}^{2}\tau_i^{-1}x^T(t)E^T ZEx(t-\tau_i) \\
&+ \sum_{i=1}^{2}\tau_i^{-1}x^T(t-\tau_i)E^T ZEx(t) \\
&- \sum_{i=1}^{2}\tau_i^{-1}x^T(t-\tau_i)E^T ZEx(t-\tau_i), \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\dot{V}_3(t,x) &= \sum_{i=1}^{2}x^T(t)R_i x(t) - \sum_{i=1}^{2}x^T(t-d_i(t))R_i x(t-d_i(t))(1-\dot{d}_i(t)) \\
&\leq \sum_{i=1}^{2}x^T(t)R_i x(t) - \sum_{i=1}^{2}x^T(t-d_i(t))R_i x(t-d_i(t))(1-\mu_i), \qquad (10) \\
\dot{V}_4(t,x) &= \sum_{i=1}^{2}x^T(t)Q_i x(t) - \sum_{i=1}^{2}x^T(t-\tau_i)Q_i x(t-\tau_i). \qquad (11)
\end{aligned}$$

Consider the function $F(t,x(t))$ with $\epsilon > 0$. Then, we get

$$0 \leq -\epsilon F^T(t,x(t))F(t,x(t)) + \epsilon x^T(t)U^T Ux(t). \qquad (12)$$

Combining the estimates (7)-(12), we obtain

$$
\begin{aligned}
\dot{V}(t,x) \leq & x^T(t)[A^TP + PA + \epsilon U^TU + \sum_{i=1}^{2}\tau_i A^TZA - \sum_{i=1}^{2}\tau_i^{-1}E^TZE \\
& + \sum_{i=1}^{2}(R_i + Q_i)]x(t) + \sum_{i=1}^{2}x^T(t-d_i(t))[A_{d_i}^TP + A_{d_i}^T(\sum_{i=1}^{2}\tau_i) \\
& \times ZA]x(t) + \sum_{i=1}^{2}x^T(t)\tau_i^{-1}E^TZEx(t-\tau_i) + \sum_{i=1}^{2}x^T(t)[PA_{d_i} \\
& + (\sum_{i=1}^{2}\tau_i)A^TZA_{d_i}]x(t-d_i(t)) + (\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T)(\sum_{i=1}^{2}\tau_i) \\
& \times Z(\sum_{i=1}^{2}A_{d_i}x(t-d_i(t))) - \sum_{i=1}^{2}x^T(t-d_i(t))R_ix(t-d_i(t))(1-\mu_i) \\
& + \sum_{i=1}^{2}x^T(t-\tau_i)\tau_i^{-1}E^TZEx(t) + \sum_{i=1}^{2}x^T(t-\tau_i)[-\tau_i^{-1}E^TZE - Q_i] \\
& \times x(t-\tau_i) + x^T(t)[P + \sum_{i=1}^{2}\tau_i A^TZ]F(t,x(t)) + (\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T) \\
& \times (\sum_{i=1}^{2}\tau_i)ZF(t,x(t)) + F^T(t,x(t))[P + \sum_{i=1}^{2}\tau_i ZA]x(t) + \sum_{i=1}^{2}F^T(t,x(t)) \\
& \times (\sum_{i=1}^{2}\tau_i)ZA_{d_i}x(t-d_i(t)) + \sum_{i=1}^{2}F^T(t,x(t))\tau_iF(t,x(t)).
\end{aligned}
$$

From the last inequality, we can write that

$$\dot{V}(t,x) \leq \xi^T(t)\Xi\xi(t),$$

where

$$\xi^T(t) = [\ x^T(t) \quad x^T(t-d_1(t)) \quad x^T(t-d_2(t)), x^T(t-\tau_1) \quad x^T(t-\tau_2) \quad F^T(t,x(t))\ ]$$

and Ξ is defined by (4). Since $\Xi < 0$, we arrive that $\dot{V}(t,x) < 0$. Thus, we conclude that the singular system (1) is asymptotically stable. As a result, since the singular system (1) is asymptotically stable, impulse free and regular, then it is asymptotically admissible.

Example 2.1. We consider nonlinear the following singular system with variable delay, which is a special case of the system (1):

$$
\begin{aligned}
\frac{d}{dt}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right) = & \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
& + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t - \frac{1+sint}{20}) \\ x_2(t - \frac{1+sint}{20}) \end{bmatrix}
\end{aligned}
$$

$$+\begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix},$$

where

$$E=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A=\begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}, A_d=\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F(t,x(t))=\begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix}$$

and

$$\epsilon = 0.85, 0 \le d(t) = \frac{1+sint}{20} \le 0.1 = \tau, \dot{d}(t) = \frac{cost}{20} \le 0.05 = \mu.$$

It can be easily seen that the above singular system is regular and impulse free.

Let

$$P=\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, Q=\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, R=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, Z=\begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, U=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Next, the eigenvalues of the matrix Ξ satisfy $\lambda_{max}(\Xi) \le -0.1536$. As a result, since all the conditions of the Theorem 2.1 are satisfied, we conclude that the above system is asymptotically admissible.

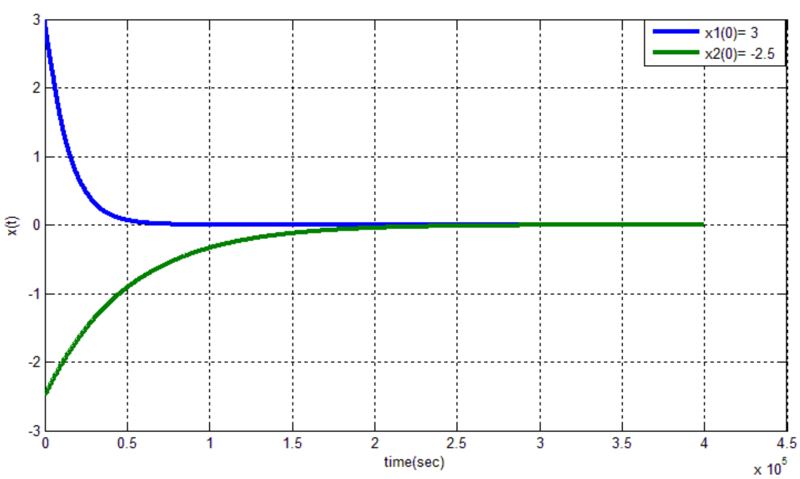

FIGURE 1. The simulation of Example 2.1 for $\tau = 0.1$.

Example 2.2. We consider the following nonlinear singular system with a variable delay, which is a special case of the system (1):

$$\frac{d}{dt}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}\right) = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.3 \end{bmatrix}\begin{bmatrix} x_1(t-\frac{1+sint}{20}) \\ x_2(t-\frac{1+sint}{20}) \\ x_3(t-\frac{1+sint}{20}) \end{bmatrix} + \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \\ x_3(t)e^{-x_3^2(t)} \end{bmatrix},$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.3 \end{bmatrix},$$

$$F(t, x(t)) = \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \\ x_3(t)e^{-x_3^2(t)} \end{bmatrix}$$

and

$$\epsilon = 0.85, 0 \le d(t) = \frac{1+sint}{20} \le 0.1 = \tau, \dot{d}(t) = \frac{cost}{20} \le 0.05 = \mu.$$

Hence, it can be easily seen that the above singular system is regular and impulse free.
Let

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, Q = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, U = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}.$$

Next, the eigenvalues of the matrix Ξ satisfy $\lambda_{max}(\Xi) \le -0.0686$. As a result, since all the conditions of the Theorem 2.1 are satisfied, then it is concluded that the system given in Example 2.2 is asymptotically admissible.

As for the next theorem, it includes new admissibility criteria for the given singular system (1). Here, we show that the system (1) is asymptotically admissible. For this aim, we use an LKF and the Wirtinger-based integral inequality.

B. Assumption

(B2) E is a singular matrix, $rankE = r \le n$, $P, R_i, S_i, Z, Q_i, (i = 1, 2)$, are positive definite symmetric matrices and U is a known constant matrix with suitable

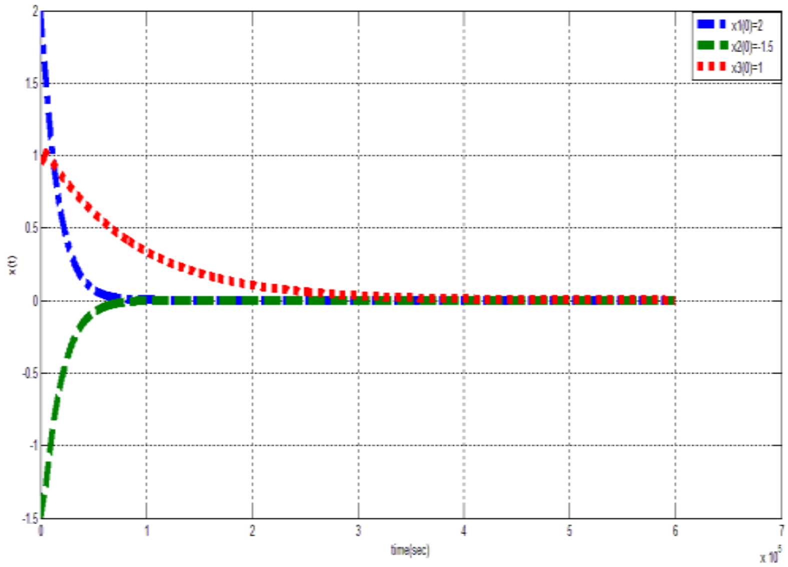

FIGURE 2. The simulation of Example 2.2 for $\tau = 0.1$.

dimension such that the matrix inequality holds:

$$\overline{\Xi} = \begin{bmatrix} \overline{\Xi}_{11} & \overline{\Xi}_{12} & \overline{\Xi}_{13} & \overline{\Xi}_{14} & \overline{\Xi}_{15} & \overline{\Xi}_{16} & \overline{\Xi}_{17} & \overline{\Xi}_{18} \\ * & \overline{\Xi}_{22} & \overline{\Xi}_{23} & 0 & 0 & \overline{\Xi}_{26} & 0 & 0 \\ * & * & \overline{\Xi}_{33} & 0 & 0 & \overline{\Xi}_{36} & 0 & 0 \\ * & * & * & \overline{\Xi}_{44} & 0 & 0 & \overline{\Xi}_{47} & 0 \\ * & * & * & * & \overline{\Xi}_{55} & 0 & 0 & \overline{\Xi}_{58} \\ * & * & * & * & * & \overline{\Xi}_{66} & 0 & 0 \\ * & * & * & * & * & * & \overline{\Xi}_{77} & 0 \\ * & * & * & * & * & * & * & \overline{\Xi}_{88} \end{bmatrix} < 0 \tag{13}$$

where

$$\begin{aligned} \overline{\Xi}_{11} =& A^T P + PA + \epsilon U^T U + \sum_{i=1}^{2} \tau_i A^T Z A - \sum_{i=1}^{2} \frac{4E^T Z E}{\tau_i} \\ &+ \sum_{i=1}^{2} (R_i + Q_i + \tau_i S_i), \\ \overline{\Xi}_{12} =& PA_{d_1} + \sum_{i=1}^{2} \tau_i A^T Z A_{d_1}, \overline{\Xi}_{13} = PA_{d_2} + \sum_{i=1}^{2} \tau_i A^T Z A_{d_2}, \\ \overline{\Xi}_{14} =& \frac{-2E^T Z E}{\tau_1}, \overline{\Xi}_{15} = \frac{-2E^T Z E}{\tau_2}, \overline{\Xi}_{16} = P + \sum_{i=1}^{2} \tau_i A^T Z, \end{aligned}$$

$$\begin{aligned}
\overline{\Xi}_{17} =&\frac{6E^TZE}{\tau_1^2}, \overline{\Xi}_{18} = \frac{6E^TZE}{\tau_2^2},\\
\overline{\Xi}_{22} =&\sum_{i=1}^{2}\tau_i A_{d_1}^T Z A_{d_1} - (1-\mu_1)R_1, \overline{\Xi}_{23} = \sum_{i=1}^{2}\tau_i A_{d_1}^T Z A_{d_2},\\
\overline{\Xi}_{26} =&\sum_{i=1}^{2}\tau_i A_{d_1}^T Z, \overline{\Xi}_{33} = \sum_{i=1}^{2}\tau_i A_{d_2}^T Z A_{d_2} - (1-\mu_2)R_2,\\
\overline{\Xi}_{36} =&\sum_{i=1}^{2}\tau_i A_{d_2}^T Z, \overline{\Xi}_{44} = -Q_1 - \frac{4E^TZE}{\tau_1}, \overline{\Xi}_{47} = \frac{6E^TZE}{\tau_1^2},\\
\overline{\Xi}_{55} =& - Q_2 - \frac{4E^TZE}{\tau_2}, \overline{\Xi}_{58} = \frac{6E^TZE}{\tau_2^2}, \overline{\Xi}_{66} = \sum_{i=1}^{2}\tau_i Z - \epsilon I,\\
\overline{\Xi}_{77} =& - \frac{12E^TZE}{\tau_1^3} - \frac{S_1}{\tau_1}, \overline{\Xi}_{88} = -\frac{12E^TZE}{\tau_2^3} - \frac{S_2}{\tau_2},
\end{aligned}$$

where ϵ is a positive constant and I is $n \times n$ - identity matrix.

Theorem 2.2. If the conditions (3) and (B2) are satisfied, then the system (1) is asymptotically admissible.

proof. Define the LKF:

$$V(t,x) = \sum_{i=1}^{5} V_i(t,x), \tag{14}$$

where

$$\begin{aligned}
V_1(t,x) =& x^T(t)E^TPx(t),\\
V_2(t,x) =& \sum_{i=1}^{2}\int_{t-d_i(t)}^{t} x^T(\alpha)R_i x(\alpha)d\alpha,\\
V_3(t,x) =& \sum_{i=1}^{2}\int_{-\tau_i}^{0}\int_{t+\beta}^{t} \dot{x}^T(\alpha)E^TZE\dot{x}(\alpha)d\alpha d\beta,\\
V_4(t,x) =& \sum_{i=1}^{2}\int_{t-\tau_i}^{t} x^T(\alpha)Q_i x(\alpha)d\alpha,\\
V_5(t,x) =& \sum_{i=1}^{2}\int_{-\tau_i}^{0}\int_{t+\beta}^{t} x^T(\alpha)S_i x(\alpha)d\alpha d\beta.
\end{aligned}$$

It can be easily seen that the LKF defined by (14) is positive definite. In the light of the Newton-Leibnitz formula and Wirtinger-based integral inequality, i.e., Lemma 1.2, taking the derivative of the LKF $V(t,x)$ along the system (1), we have

$$\dot{V}(t,x) = \sum_{i=1}^{5} \dot{V}_i(t,x), \tag{15}$$

where

$$\begin{aligned}\dot{V}_1(t,x) =& x^T(t)[A^TP+PA]x(t)+\sum_{i=1}^{2}x^T(t)PA_{d_i}x(t-d_i(t))\\&+\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^TPx(t)+F^T(t,x(t))Px(t)\\&+x^T(t)PF(t,x(t)),\end{aligned}\tag{16}$$

$$\begin{aligned}\dot{V}_2(t,x) =&\sum_{i=1}^{2}x^T(t)R_ix(t)-\sum_{i=1}^{2}x^T(t-d_i(t))R_ix(t-d_i(t))(1-\dot{d}_i(t))\\\leq&\sum_{i=1}^{2}x^T(t)R_ix(t)-\sum_{i=1}^{2}x^T(t-d_i(t))R_ix(t-d_i(t))(1-\mu_i),\end{aligned}\tag{17}$$

$$\begin{aligned}\dot{V}_3(t,x)\leq&\sum_{i=1}^{2}\tau_ix^T(t)A^TZAx(t)+(\sum_{i=1}^{2}\tau_i)x^T(t)A^TZ(\sum_{i=1}^{2}A_{d_i}x(t-d_i(t)))\\&+\sum_{i=1}^{2}\tau_ix^T(t)A^TZF(t,x(t))+\sum_{i=1}^{2}\tau_iF^T(t,x(t))ZAx(t)\\&+(\sum_{i=1}^{2}\tau_i)(\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T)ZAx(t)\\&+(\sum_{i=1}^{2}\tau_i)(\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T)Z(\sum_{i=1}^{2}A_{d_i}x(t-d_i(t)))\\&+(\sum_{i=1}^{2}\tau_i)(\sum_{i=1}^{2}x^T(t-d_i(t))A_{d_i}^T)ZF(t,x(t))\\&+(\sum_{i=1}^{2}\tau_i)F^T(t,x(t))Z(\sum_{i=1}^{2}A_{d_i}x(t-d_i(t)))\\&+(\sum_{i=1}^{2}\tau_i)F^T(t,x(t))ZF(t,x(t))\\&-\sum_{i=1}^{2}\{x^T(t)\frac{4E^TZE}{\tau_i}x(t)+x^T(t)\frac{2E^TZE}{\tau_i}x(t-\tau_i)\\&+x^T(t-\tau_i)\frac{2E^TZE}{\tau_i}x(t)+x^T(t-\tau_i)\frac{4E^TZE}{\tau_i}x(t-\tau_i)\\&-x^T(t)\frac{6E^TZE}{\tau_i^2}(\int_{t-\tau_i}^{t}x(s)ds)-x^T(t-\tau_i)\frac{6E^TZE}{\tau_i^2}(\int_{t-\tau_i}^{t}x(s)ds)\\&-(\int_{t-\tau_i}^{t}x(s)ds)^T\frac{6E^TZE}{\tau_i^2}x(t)-(\int_{t-\tau_i}^{t}x(s)ds)^T\frac{6E^TZE}{\tau_i^2}x(t-\tau_i)\end{aligned}$$

$$+(\int_{t-\tau_i}^{t} x(s)ds)^T \frac{12E^TZE}{\tau_i^3}(\int_{t-\tau_i}^{t} x(s)ds)\}, \tag{18}$$

$$\dot{V}_4(t,x) = \sum_{i=1}^{2} x^T(t)Q_ix(t) - \sum_{i=1}^{2} x^T(t-\tau_i)Q_ix(t-\tau_i), \tag{19}$$

$$\dot{V}_5(t,x) \leq \sum_{i=1}^{2} \tau_i x^T(t)S_ix(t) - \sum_{i=1}^{2} \tau_i^{-1}(\int_{t-\tau_i}^{t} x(s)ds)^T S_i(\int_{t-\tau_i}^{t} x(s)ds). \tag{20}$$

Let $\epsilon > 0$. Then, we have

$$0 \leq -\epsilon F^T(t,x(t))F(t,x(t)) + \epsilon x^T(t)U^TUx(t). \tag{21}$$

Combining the estimates (15)-(21), we obtain the following inequality:

$$\dot{V}(t,x) \leq \xi^T(t)\overline{\Xi}\xi(t),$$

where

$$\begin{aligned}\xi^T(t) = [\; & x^T(t) \quad x^T(t-d_1(t)) \quad x^T(t-d_2(t)) \quad x^T(t-\tau_1) \\ & x^T(t-\tau_2) \quad F^T(t,x(t)) \quad (\textstyle\int_{t-\tau_1}^{t} x(s)ds)^T \quad (\textstyle\int_{t-\tau_2}^{t} x(s)ds)^T \;]\end{aligned}$$

and $\overline{\Xi}$ is defined by (13). Since $\overline{\Xi} < 0$, then we arrive that $\dot{V}(t,x) < 0$. Hence, we conclude that the system (1) is asymptotically stable. As a result, the system (1) is asymptotically admissible, which is regular, impulse free and asymptotically stable.

Example 2.3. We consider nonlinear singular system with variable delay, which is a special case of the system (1):

$$\begin{aligned}\frac{d}{dt}\left(\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) &= \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t-\frac{1+sint}{20}) \\ x_2(t-\frac{1+sint}{20}) \end{bmatrix} \\ &+ \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix}, t \geq \frac{1+sint}{20},\end{aligned}$$

where

$$E = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, F(t,x(t)) = \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix}$$

and

$$\epsilon = 0.6, 0 \leq d(t) = \frac{1+sint}{20} \leq 0.1 = \tau, \dot{d}(t) = \frac{cost}{20} \leq 0.05 = \mu.$$

It can be easily seen that the singular system of Example 2.3 is regular and impulse free.

Let

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, Q = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, Z = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}.$$

Next, the eigenvalues of the matrix $\overline{\Xi}$ satisfy $\lambda_{max}(\overline{\Xi}) \leq -0.1256$. Finally, since all the conditions of Theorem 2.2 are satisfied, then it is concluded that the system of Example 2.3 is asymptotically admissible.

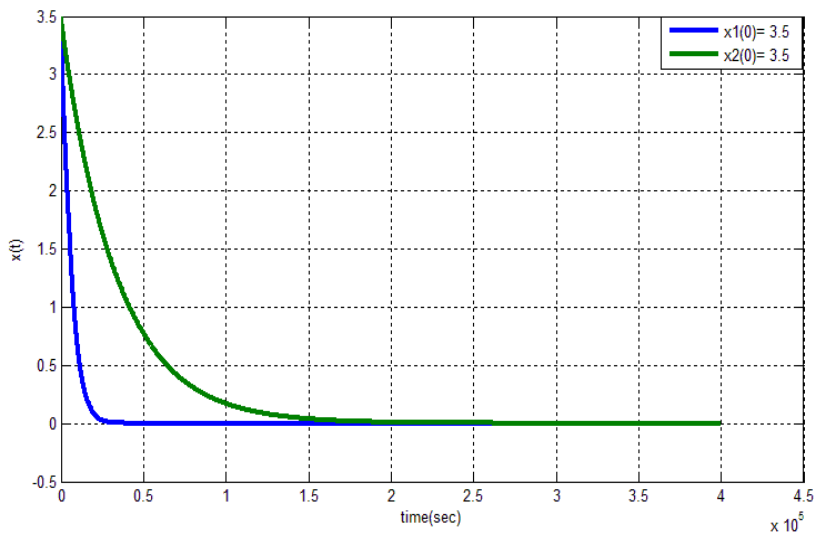

FIGURE 3. The simulation of Example 2.3 for $\tau = 0.1$.

Example 2.4. We consider nonlinear singular system with variable delay, which is a special case of the system (1) and defined by

$$\frac{d}{dt}\left(\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \right) = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$
$$+ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(t - \frac{1+sint}{20}) \\ x_2(t - \frac{1+sint}{20}) \\ x_3(t - \frac{1+sint}{20}) \end{bmatrix}$$
$$+ \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \\ x_3(t)e^{-x_3^2(t)} \end{bmatrix}, t \geq \frac{1+sint}{20},$$

where

$$E = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -4 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.5 \end{bmatrix},$$

$$F(t, x(t)) = \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \\ x_3(t)e^{-x_3^2(t)} \end{bmatrix}$$

and

$$\epsilon = 0.6, 0 \leq d(t) = \frac{1 + sint}{20} \leq 0.1 = \tau, \dot{d}(t) = \frac{cost}{20} \leq 0.05 = \mu.$$

It can be easily seen that the singular system of Example 2.4 is regular and impulse free.

Let

$$P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}, Q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.03 \end{bmatrix}, S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Next, the eigenvalues of the matrix $\overline{\Xi}$ satisfy $\lambda_{max}(\overline{\Xi}) \leq -0.1256$. At the end, since all the conditions of Theorem 2.2 are satisfied, then it is concluded that the system of Example 2.4 is asymptotically admissible.

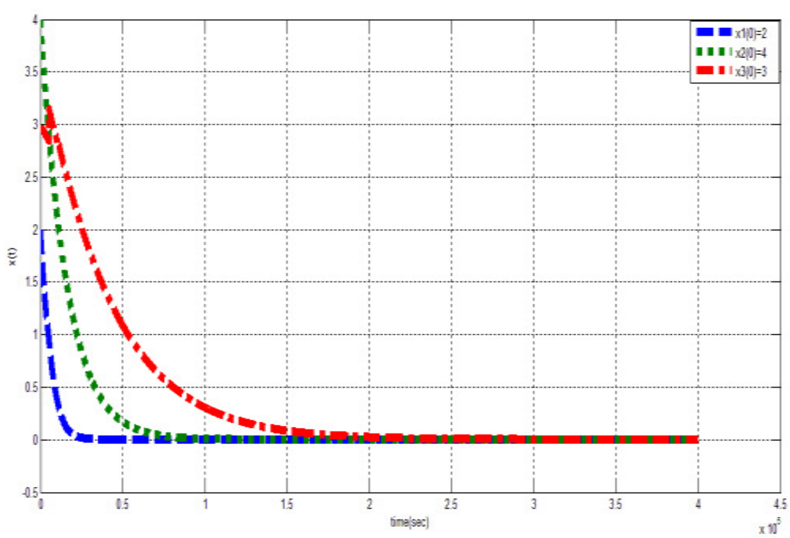

FIGURE 4. The simulation of Example 2.4 for $\tau = 0.1$.

3.Conclusion

In this paper, the asymptotic admissibility for a kind of nonlinear delay singular systems has been discussed. Firstly, we have proved that the given system is regular and impulse-free. Then, we proved the asymptotic stability of the given system with the help of an LKF and some well-known inequalities, which are Jensen inequality, Wirtinger-based integral inequality and matrix inequality. We have also proved that the given system is asymptotically admissible since the given system is asymptotically stable, impulse-free and regular. Consequently, we gave four examples together with simulations of their solutions. This paper extends and improves some results that can be found in the relevant literature.

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