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# SUFFICIENT CONDITIONS FOR THE STABILITY OF A SYSTEM OF DIFFERENCE EQUATIONS WITH FINITE DELAY

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Abstract. The stability properties of the zero solution of a special class of system of delay difference equations are considered. The fundamental matrix solution is used to convert the system of equations into an equivalent summation equation. A Krasnosel'skii's fixed point theorem is then used to obtain sufficient conditions for the zero solution of the system of delay difference equations to be asymptotically stable.

## 1. Introduction

The study of the stability of solutions for difference equations has gained the attention of many researchers in recent times, see for example [1] [2],[7],[9],[11] and the references cited therein.

In this paper we consider the system of difference equations

$$\Delta x(n) = A(n)x(n-\tau), \tag{1}$$

where $A(n) \in \mathbb{R}^{s\times s}$ is a nonsigular matrix and $\tau$ is a positive constant. We are mainly motivated by the work of Raffoul in [9] where he obtained sufficient conditions for the asymptotic stability of the zero solution of a scalar version of (1). In this paper however, we establish sufficient conditions for the zero solution of (1) to be asymptotically stable.

The first author proved the existence and uniqueness of periodic solutions for (1) in [12]. Throughout this paper $\Delta$ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n),\ n = 0, 1, 2, ...\}$. Also, we define the operator $E$ by $Ex(n) = x(n+1)$. For more on difference calculus we refer the reader to [8].

## 2. Preliminaries

In this section we obtain an equivalent summation equation for (1).

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**Lemma 1.** Suppose there exists a nonsingular $s \times s$ matrix $G(n)$ such that

$$\Delta x(n) = G(n)x(n) - \Delta_n \sum_{k=n-\tau}^{n-1} G(k)x(k) + \Big[A(n) - G(n-\tau)\Big]x(n-\tau). \tag{2}$$

Then equation (1) is equivalent to (2).

**Proof.** By taking the difference with respect to $n$ of the summation term in (2) we obtain

$$\Delta_n \sum_{k=n-\tau}^{n-1} G(k)x(k) = G(n)x(n) - G(n-\tau)x(n-\tau). \tag{3}$$

Substituting (3) into (2) gives the desired result. This completes the proof.

We now state in the following lemma one of the fundamental properties of the difference operator which will be used in the proof of our next lemma.

**Lemma 2.**[12] For functions $y(n)$ and $z(n)$ of a real variable $n$,

$$\Delta(y(n)z(n)) = Ey(n)\Delta z(n) + [\Delta y(n)]z(n). \tag{4}$$

In the rest of the paper we let $\Phi(n, n_0)$ denote the fundamental matrix solution of

$$\Delta x(n) = G(n)x(n). \tag{5}$$

**Lemma 3.** Suppose the hypotheses of Lemma 1 hold. Then $x(n)$ is a solution of (1) if and only if

$$\begin{aligned} x(n) \quad = \quad & -\sum_{k=n-\tau}^{n-1} G(k)x(k) + \Phi(n, n_0)\Big(x(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)x(k)\Big) \\ & + \sum_{u=n_0}^{n-1} \Phi(n, u+1)\Big[A(u)x(u-\tau) - G(u) \sum_{k=u-\tau}^{u-1} G(k)x(k) \\ & - G(u-\tau)x(u-\tau)\Big]. \end{aligned} \tag{6}$$

**Proof.** Rewrite equation (3) as

$$\begin{aligned} \Delta[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)] \quad = \quad & G(n)[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)] \\ & - G(n) \sum_{k=n-\tau}^{n-1} G(k)x(k) + A(n)x(n-\tau) \\ & - G(n-\tau)x(n-\tau). \end{aligned}$$

Since $\Phi(n, n_0)\Phi^{-1}(n, n_0) = I$, it follows from Lemma 2 that

$$\begin{aligned} 0 \quad &= \quad \Delta(\Phi(n, n_0)\Phi^{-1}(n, n_0)) = \Phi(n+1, n_0)\Delta\Phi^{-1}(n, n_0) + [\Delta\Phi(n, n_0)]\Phi^{-1}(n, n_0) \\ &= \quad \Phi(n+1, n_0)\Delta\Phi^{-1}(n, n_0) + [G(n)\Phi(n, n_0)]\Phi^{-1}(n, n_0) \\ &= \quad \Phi(n+1, n_0)\Delta\Phi^{-1}(n, n_0) + G(n). \end{aligned}$$

This implies that

$$\Delta\Phi^{-1}(n,n_0) = -\Phi^{-1}(n+1,n_0)G(n).$$

If $x(n)$ is a solution of (1) then

$$\begin{aligned}
&\Delta\Big[\Phi^{-1}(n,n_0)\Big(x(n)+\sum_{k=n-\tau}^{n-1}G(k)x(k)\Big)\Big]\\
= &\ \Phi^{-1}(n+1,n_0)\Delta\Big(x(n)+\sum_{k=n-\tau}^{n-1}G(k)x(k)\Big)\\
&+[\Delta\Phi^{-1}(n,n_0)][x(n)+\sum_{k=n-\tau}^{n-1}G(k)x(k)]\\
= &\ \Phi^{-1}(n+1,n_0)\Big[G(n)[x(n)+\sum_{k=n-\tau}^{n-1}G(k)x(k)]\\
&-G(n)\sum_{k=n-\tau}^{n-1}G(k)x(k)+A(n)x(n-\tau)\\
&-G(n-\tau)x(n-\tau)\Big]\\
&-[\Phi^{-1}(n+1,n_0)G(n)][x(n)+\sum_{k=n-\tau}^{n-1}G(k)x(k)]\\
= &\ \Phi^{-1}(n+1,n_0)\Big[A(n)x(n-\tau)-\ G(n)\sum_{k=n-\tau}^{n-1}G(k)x(k)\\
&-G(n-\tau)x(n-\tau)\Big].
\end{aligned}$$

Summing the above equation from $n_0$ to $n-1$ gives,

$$\begin{aligned}
x(n) = &-\sum_{k=n-\tau}^{n-1}G(k)x(k)+\Phi(n,n_0)\Big(x(n_0)+\sum_{k=n_0-\tau}^{n_0-1}G(k)x(k)\Big)\\
&+\sum_{u=n_0}^{n-1}\Phi(n,u+1)\Big[A(u)x(u-\tau)-\ G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k)\\
&-G(u-\tau)x(u-\tau)\Big].
\end{aligned} \tag{7}$$

This completes the proof.

Define

$$S = \{\varphi : \mathbb{Z} \to \mathbb{R}^s \mid \varphi(n) \to 0 \text{ as } n \to \infty\}.$$

Let

$$||\varphi|| = \max\{|\varphi(n)|,\ n \geq n_0\},$$

with $|.|$ denoting the infinity norm for $\varphi \in \mathbb{R}^s$. Then $(S, ||.||)$ is a Banach space. Also, if $A$ is an $s \times s$ real matrix, then we define the norm of $A$ by

$$||A|| = \sup_{n \in \mathbb{Z}} |A(n)|$$

where

$$|A(n)| = \max_{1 \le i \le s} \sum_{j=1}^{s} |a_{ij}|.$$

Let $\psi : [-\tau, n_0] \to \mathbb{R}$ be a given initial bounded sequence. Define mapping $H : S \to S$ by

$$(H\varphi)(n) = \psi(n) \text{ for } n \le n_0, \tag{8}$$

and

$$\begin{aligned}(H\varphi)(n) = & -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k) + \Phi(n, n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\ & + \sum_{u=n_0}^{n-1} \Phi(n, u+1)\Big[A(u)\varphi(u-\tau) - G(u) \sum_{k=u-\tau}^{u-1} G(k)\varphi(k) \\ & - G(u-\tau)\varphi(u-\tau)\Big], \quad n \ge n_0. \end{aligned} \tag{9}$$

Next we state Krasnosel'skii's fixed point theorem which is the main mathematical tool that we will use to obtain the stability results for equation (1). We refer the reader to [10] for the proof of Krasnosel'skii's fixed point theorem.

**Theorem 1.** [Krasnosel'skii] Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, ||.||)$. Suppose that $C$ and $B$ map $\mathbb{M}$ into $\mathbb{B}$ such that
(i) C is continuous and $C\mathbb{M}$ is contained in a compact set,
(ii) B is a contraction mapping.
(iii) $x, y \in \mathbb{M}$, implies $Cx + By \in \mathbb{M}$.
Then there exists $z \in \mathbb{M}$ with $z = Cz + Bz$.

Next we define $C, B : S \to S$ by

$$(B\varphi)(n) = -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k) + \Phi(n, n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big), \tag{10}$$

and

$$\begin{aligned}(C\varphi)(n) = & \sum_{u=n_0}^{n-1} \Phi(n, u+1)\Big[A(u)\varphi(u-\tau) - G(u) \sum_{k=u-\tau}^{u-1} G(k)\varphi(k) \\ & - G(u-\tau)\varphi(u-\tau)\Big]. \end{aligned} \tag{11}$$

It follows from (10) and (11) that $(H\varphi)(n) = (B\varphi)(n) + (C\varphi)(n)$.

## 3. Main Results

In this section we state and prove our main results.

**Theorem 2.** Suppose that

$$\Phi(n, n_0) \to 0 \text{ as } n \to \infty, \tag{12}$$

and there exists $\alpha \in (0, 1)$ such that

$$\tau||G|| + \sum_{u=n_0}^{n-1} |\Phi(n, u+1)| \Big[ ||A|| + \tau||G||^2 + ||G|| \Big] \leq \alpha. \tag{13}$$

Then the zero solution of (1) is asymptotically stable.

**Proof.** We first show that the mapping $H$ defined by (9) $\to 0$ as $n \to \infty$. To show that the first term on the right hand side of (9) goes to zero as $n \to \infty$, let $\varphi \in S$, then $\varphi(n) \to 0$ as $n \to \infty$. Thus by the continuity of norms we have $\| \varphi \| \to 0$ as $n \to \infty$.
Hence,

$$\begin{aligned} \Big| - \sum_{k=n-\tau}^{n-1} G(k)\varphi(k) \Big| &\leq \sum_{k=n-\tau}^{n-1} ||G||||\varphi|| \\ &\leq \tau||\varphi||||G|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

The second term on the on the right hand side of (9) goes to zero because of condition (12).
Now we show that the last term on the right hand side of (9) tends to zero as $n \to \infty$. Let $m > 0$ such that for $\varphi \in S$, $|\varphi(n)| < K$ for $K > 0$. Since $\varphi(n) \to 0$ as $n \to \infty$, for $\epsilon_1 > 0$, there exists an $n_1 > m$ such that $n \geq n_1$ implies $|\varphi(n)| < \epsilon_1$. By condition (12), there exists a $n_2 > n_1$ such that $n > n_2$ implies

$$|\Phi(n, n_1)| < \frac{\epsilon_1}{\alpha K}.$$

Thus for $n \geq n_2$, we have

$$\begin{aligned} &\Big| \sum_{u=n_0}^{n-1} \Phi(n, u+1) \Big[ A(u)\varphi(u-\tau) - G(u) \sum_{k=u-\tau}^{u-1} G(k)\varphi(k) \\ &- G(u-\tau)\varphi(u-\tau) \Big] \Big| \end{aligned}$$

$$
\begin{aligned}
\leq\ & \Big|\sum_{u=n_0}^{n_1-1}\Phi(n,u+1)\Big[A(u)\varphi(u-\tau)-\ G(u)\sum_{k=u-\tau}^{u-1}G(k)\varphi(k)\\
&-G(u-\tau)\varphi(u-\tau)\Big]\\
&+\sum_{u=n_1}^{n-1}\Phi(n,u+1)\Big[A(u)\varphi(u-\tau)-\ G(u)\sum_{k=u-\tau}^{u-1}G(k)\varphi(k)\\
&-G(u-\tau)\varphi(u-\tau)\Big]\Big|\\
\leq\ & K\sum_{u=n_0}^{n_1-1}|\Phi(n,u+1)|\Big[||A||+\tau||G||^2+||G||\Big]\\
&+\epsilon_1\sum_{u=n_1}^{n-1}|\Phi(n,u+1)|\Big[|A|+\tau||G||^2+||G||\Big]\\
\leq\ & K|\Phi(n,n_1)|\sum_{u=n_0}^{n_1-1}|\Phi(n_1,u+1)|\Big[|A|+\tau||G||^2+||G||\Big]+\epsilon_1\alpha\\
\leq\ & K|\Phi(n,n_1)|\alpha+\epsilon_1\alpha<\epsilon_1+\epsilon_1\alpha.
\end{aligned}
$$

Hence, $(H\varphi)\to 0$ as $n\to\infty$ and so $H$ maps the set $S$ into itself.
We next show that $B$ is a contraction. Let $\varphi,\eta\in S$, then we have from (13) that

$$
\begin{aligned}
||(B\varphi)-(B\eta)|| &\leq \tau||G||||\varphi-\eta||\\
&\leq \mu||\varphi-\eta||,\ \text{ for some }\ \mu\in(0,1).
\end{aligned}
$$

Next we prove that the map $C$ is compact. To this end we let $\{\varphi^l\}$ be a sequence in $S$ such that

$$\lim_{l\to\infty}||\varphi^l-\varphi||=0.$$

Since $S$ is closed, we have that $\varphi\in S$. Then by the definition of $C$

$$||C(\varphi^l)-C(\varphi)||=\max_{n\in\mathbb{Z}}|(C\varphi^l)(n)-(C\varphi)(n)|.$$

Thus, for $\varphi\in S$, we have that

$$
\begin{aligned}
|(C\varphi^l)(n)-(C\varphi)(n)| \leq\ & \sum_{u=n_0}^{n-1}|\Phi(n,u+1)|\Big[||A||\Big(|\varphi^l(u-\tau)-\varphi(u-\tau)|\Big)\\
&+||G||\sum_{k=u-\tau}^{u-1}||G||\Big(|\varphi^l(k)-\varphi(k)|\Big)\\
&+||G||\Big(|\varphi^l(u-\tau)-\varphi(u-\tau)|\Big)\Big].
\end{aligned}
$$

The continuity of $\varphi$ along with the Lebesgue dominated convergence theorem imply that

$$\lim_{l\to\infty}|(C\varphi^l)(n)-(C\varphi)(n)|=0,\ \text{ for }\ n\in\mathbb{Z}.$$

This shows that $C$ is continuous. To show that $CS$ is precompact, let $\varphi^l$ be a sequence in $S$. Then for each $n\in\mathbb{Z}$, $\varphi^l$ is a bounded sequence. This shows that $\{\varphi^l\}$ has a convergent subsequence $\{\varphi^{l_k}\}$ in $S$. Since $C$ is continuous, we know that $\{C\varphi^l\}$ has a convergent subsequence in $CS$. This means $CS$ is precompact. This completes the proof for compactness.

We finally show that the zero solution of (1) is stable. To this end let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$\delta||\Phi||(1+\tau||G||)+\alpha\epsilon<\epsilon.$$

Let $\psi(n)$ be any given initial function such that $|\psi(n)|<\delta$. Define

$$\mathbb{M}=\{\varphi\in S:||\varphi||<\epsilon\}.$$

Let $\varphi,\eta\in\mathbb{M}$, then

$$\begin{aligned}
||(B\eta)+(C\varphi)|| \leq\ & \Big|\sum_{k=n-\tau}^{n-1}G(k)\eta(k)\Big|+\Big|\Phi(n,n_0)\Big(\psi(n_0)+\sum_{k=n_0-\tau}^{n_0-1}G(k)\psi(k)\Big)\Big| \\
& +\Big|\sum_{u=n_0}^{n-1}\Phi(n,u+1)\Big[A(u)\varphi(u-\tau)-\ G(u)\sum_{k=u-\tau}^{u-1}G(k)\varphi(k) \\
& -G(u-\tau)\varphi(u-\tau)\Big]\Big| \\
\leq\ & \delta||\Phi||\Big(1+\tau||G||\Big)+\Big\{\tau||G||+\sum_{u=n_0}^{n-1}|\Phi(n,u+1)|\Big[||A|| \\
& +\tau||G||^2+||G||\Big]\Big\} \\
\leq\ & \delta||\Phi||\Big(1+\tau||G||\Big)+\epsilon\alpha \\
\leq\ & \epsilon.
\end{aligned}$$

It follows from the above work that all the conditions of the Krasnoselskii's fixed point theorem are satisfied. Thus, there exists a fixed point $z\in\mathbb{M}$ such that $z=Bz+Cz$. This completes the proof.

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