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DYNAMICAL BEHAVIOR AND SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS OF TWENTIETH ORDER

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Abstract. Most natural phenomena arising in nonlinear sciences can be often described using difference equations. This work aims to extract some new analytic solutions for some rational difference equations of twentieth order. We also investigate local and global stability, periodic behavior, oscillation, and boundedness of the constructed solutions. The solutions are obtained using the iteration method and the modulus operator. Moreover, the obtained results are confirmed with some numerical examples which have been plotted with the help of MATLAB software. The proposed approaches can be simply applied for other high-order difference equations.

1. Introduction

Difference equations usually model the evolution of a specific real life problem over the course of time. Several natural phenomena can be simply described on a discrete time. Therefore, difference equations play a significant role in mathematics. Many scholars have successfully used difference equations to investigate some biological, physical, economical, and engineering problems. For example, Elaydi [1] used difference equations to study various phenomena such as the drug in the blood system, the size of a population, the pricing of a certain commodity, the Fibonacci Sequence, the propagation of annual plants, the transmission of information, and others. Furthermore, difference equations have been well utilized to solve differential equations numerically. In other words, when we discretize a given differential equation, we obtain a corresponding difference equation. For instance, Euler method, which is used to solve a first order differential equations numerically, is the discretization of a first order differential equation.

A massive number of researchers have widely discussed the solutions, stability, boundedness and other properties of difference equations. We mention some of them. Sanbo and Elsayed [2] studied the local and global behavior, periodicity, boundedness and some solutions of a fifth order recursive equation. Alayachi et al. [3] discussed the stability, periodicity and the solutions of a sixth order recursive equation. Almatrafi and Alzubaidi [4] presented an extensive study about the equilibrium points, stability, periodic nature, and the exact solutions of an eighth

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order recursive relation. In [5], the authors explored the structures of the analytic solutions of a rational recursive equation. Furthermore, Elsayed [6] discussed the qualitative behaviors and the solutions of a nonlinear recursive equation. Ahmed et al. [7], introduced some new forms and dynamical analysis for the solutions of some nonlinear difference relations of fifteenth order. Finally, Kara and Yazlik [8] solved a $(k+l)$-order difference equation and presented the asymptotic approach of the obtained solutions of the equation when $k=3$, and $l=k$. More results about such equations can be simply obtained in refs. [9–21].
The essential purpose of this article is to discuss and present some qualitative behaviors such as the equilibrium points, local and global approaches, boundedness, and the analytic solutions of the nonlinear difference equations

$$u_{n+1} = \frac{u_{n-19}}{\pm 1 \pm \Pi_{i=0}^{4} u_{n-(4i+3)}}, \quad n = 0, 1, 2, ...,$$

where the initial data u_{-19}, u_{-18}, ..., u_0 are arbitrary non-zero real numbers. In addition, some 2D figures are depicted with the help of MATLAB to validate the constructed results.
In this article, we let $mod\,(k,4) = k - 4\left[\frac{k}{4}\right]$, where $[x]$ is defined to be the greatest integer less than or equal to the real number x.

2. The Difference Equation $u_{n+1} = \frac{u_{n-19}}{1+u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}$

This section is devoted to introduce some new forms of solutions to the following equation:

$$u_{n+1} = \frac{u_{n-19}}{1 + u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}, \quad n = 0, 1, 2, ..., \tag{1}$$

where the conditions u_{-j}, $j = 0, 1, 2, ..., 19$, are real numbers. In addition, the stability analysis and bounded solutions are extensively investigated.

Theorem 1. *Assume that* $\{u_n\}_{n=-19}^{\infty}$ *is a solution to Eq.* (1). *Then, for* $n = 0, 1, 2, ...$,

$$u_{20n-k} = a_k \prod_{i=0}^{n-1} \left(\frac{1 + (5i + \mu_k - 1)\,\Psi_k}{1 + (5i + \mu_k)\,\Psi_k} \right), \tag{2}$$

where $\Psi_k = \prod_{j=0}^{4} a_{mod(k,4)+4j}$, $\mu_k = 5 - \left[\frac{k}{4}\right]$ *and* $u_{-k} = a_k$, *with* $r\Psi_k \neq -1$ *such that* $r \in \{1, 2, 3, ...\}$, $k = 0, 1, 2, ..., 19$.

Proof. The results are true when $n = 0$. We then assume that the results hold when $n-1$, as follows:

$$u_{20n-20-k} = a_k \prod_{i=0}^{n-2} \left(\frac{1 + (5i + \mu_k - 1)\,\Psi_k}{1 + (5i + \mu_k)\,\Psi_k} \right). \tag{3}$$

Next, from Eq. (1) and Eq. (3), one obtains

$$\begin{aligned}
u_{20n-19} &= \frac{u_{20n-39}}{1+u_{20n-23}u_{20n-27}u_{20n-31}u_{20n-35}u_{20n-39}} \\
&= \frac{a_{19}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{19}-1)P_{19}}{1+(5i+M_{19})P_{19}}\right)}{1+\prod_{j=0}^{4}\left(a_{4j+3}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{4j+3}-1)P_{4j+3}}{1+(5i+M_{4j+3})P_{4j+3}}\right)\right)} \\
&= \frac{a_{19}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_3a_7a_{11}a_{15}a_{19}}{1+(5i+1)a_3a_7a_{11}a_{15}a_{19}}\right)}{1+a_3a_7a_{11}a_{15}a_{19}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_3a_7a_{11}a_{15}a_{19}}{1+(5i+5)a_3a_7a_{11}a_{15}a_{19}}\right)} = a_{19}\prod_{i=0}^{n-1}\left(\frac{1+(5i)a_3a_7a_{11}a_{15}a_{19}}{1+(5i+1)\,a_3a_7a_{11}a_{15}a_{19}}\right).
\end{aligned}$$

Moreover, using Eq. (1) and Eq. (3) gives

$$\begin{aligned}
u_{20n-18} &= \frac{u_{20n-38}}{1+u_{20n-22}u_{20n-26}u_{20n-30}u_{20n-34}u_{20n-38}} \\
&= \frac{a_{18}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{18}-1)P_{18}}{1+(5i+M_{18})P_{18}}\right)}{1+\prod_{j=0}^{4}\left(a_{4j+2}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{4j+2}-1)P_{4j+2}}{1+(5i+M_{4j+2})P_{4j+2}}\right)\right)} \\
&= \frac{a_{18}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_2a_6a_{10}a_{14}a_{18}}{1+(5i+1)a_2a_6a_{10}a_{14}a_{18}}\right)}{1+a_2a_6a_{10}a_{14}a_{18}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_2a_6a_{10}a_{14}a_{18}}{1+(5i+5)a_2a_6a_{10}a_{14}a_{18}}\right)} = a_{18}\prod_{i=0}^{n-1}\left(\frac{1+(5i)a_2a_6a_{10}a_{14}a_{18}}{1+(5i+1)\,a_2a_6a_{10}a_{14}a_{18}}\right).
\end{aligned}$$

Further, using Eq. (1) and Eq. (3), we have

$$\begin{aligned}
u_{20n-17} &= \frac{u_{20n-37}}{1+u_{20n-21}u_{20n-25}u_{20n-29}u_{20n-33}u_{20n-37}} \\
&= \frac{a_{17}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{17}-1)P_{17}}{1+(5i+M_{17})P_{17}}\right)}{1+\prod_{j=0}^{4}\left(a_{4j+1}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{4j+1}-1)P_{4j+1}}{1+(5i+M_{4j+1})P_{4j+1}}\right)\right)} \\
&= \frac{a_{17}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_1a_5a_9a_{13}a_{17}}{1+(5i+1)a_1a_5a_9a_{13}a_{17}}\right)}{1+a_1a_5a_9a_{13}a_{17}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_1a_5a_9a_{13}a_{17}}{1+(5i+5)a_1a_5a_9a_{13}a_{17}}\right)} = a_{17}\prod_{i=0}^{n-1}\left(\frac{1+(5i)a_1a_5a_9a_{13}a_{17}}{1+(5i+1)\,a_1a_5a_9a_{13}a_{17}}\right).
\end{aligned}$$

We also use Eq. (1) and Eq. (3) to have

$$\begin{aligned}
u_{20n-16} &= \frac{u_{20n-36}}{1+u_{20n-20}u_{20n-24}u_{20n-28}u_{20n-32}u_{20n-36}} \\
&= \frac{a_{16}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{16}-1)P_{16}}{1+(5i+M_{16})P_{16}}\right)}{1+\prod_{j=0}^{4}\left(a_{4j}\prod_{i=0}^{n-2}\left(\frac{1+(5i+M_{4j}-1)P_{4j}}{1+(5i+M_{4j})P_{4j}}\right)\right)} \\
&= \frac{a_{16}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_0a_4a_8a_{12}a_{16}}{1+(5i+1)a_0a_4a_8a_{12}a_{16}}\right)}{1+a_0a_4a_8a_{12}a_{16}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_0a_4a_8a_{12}a_{16}}{1+(5i+5)a_0a_4a_8a_{12}a_{16}}\right)} = a_{16}\prod_{i=0}^{n-1}\left(\frac{1+(5i)a_0a_4a_8a_{12}a_{16}}{1+(5i+1)\,a_0a_4a_8a_{12}a_{16}}\right).
\end{aligned}$$

Similarly, we can straightforwardly extract other formulas.

Theorem 2. *Assume that the initial values $u_{-19}, u_{-18}, ..., u_0 \in [0, \infty)$, then every solution of Eq.* (1) *is bounded.*

Proof. Suppose that $\{u_n\}_{n=-19}^{\infty}$ is a solution to Eq. (1). Then, from Eq. (1), we have

$$0 \leq u_{n+1} = \frac{u_{n-19}}{1 + u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}} \leq u_{n-19} \quad \text{for all} \quad n \geq 0.$$

Hence, the sequence $\{u_{20n-i}\}_{n=0}^{\infty}$, $i = 0, 1, ..., 19$ is decreasing and bounded from above by $\mu = \max\{u_{-19}, u_{-18}, ..., u_0\}$.

Theorem 3. *Equation* (1) *has only one equilibrium point which is* $\overline{u} = 0$.

Proof. Using Eq. (1), we have

$$\overline{u} = \frac{\overline{u}}{1 + \overline{u}^5},$$

which is

$$\overline{u} + \overline{u}^6 = \overline{u}.$$

Therefore, $\overline{u} = 0$.

Theorem 4. *Let* $u_{-19}, u_{-18}, ..., u_0 \in [0, \infty)$, *then the equilibrium point* $\overline{u} = 0$ *of Eq.* (1) *is locally stable.*

Proof. Suppose that $\epsilon > 0$, and assume that $\{u_n\}_{n=-19}^{\infty}$ is a solution to Eq. (1) with

$$\sum_{j=0}^{19} |u_{-j}| < \epsilon.$$

It suffices to show that $|u_1| < \epsilon$. That is

$$0 < u_1 = \frac{u_{-19}}{1 + u_{-3}u_{-7}u_{-11}u_{-15}u_{-19}} \leq u_{-19} < \epsilon.$$

This completes the proof.

Theorem 5. *Let* $u_{-19}, u_{-18}, ..., u_0 \in [0, \infty)$, *then the equilibrium point* $\overline{u} = 0$ *of Eq.* (1) *is globally asymptotically stable.*

Proof. In Theorem 4, we showed that the fixed point $\overline{u} = 0$ is locally stable. Suppose that $\{u_n\}_{n=-19}^{\infty}$ is a positive solution to Eq. (1). Next, it is sufficient to prove that $\lim_{n\to\infty} u_n = \overline{u} = 0$. Theorem 2 leads to $u_{n+1} < u_{n-19}$ for all $n \geq 0$. Hence, the sequences $\{u_{20n-i}\}_{n=0}^{\infty}$, $i = 0, 1, ..., 19$ are decreasing and bounded which imply that the sequences $\{u_{20n-i}\}_{n=0}^{\infty}, i = 0, 1, ..., 19$ converge to limit (say $L_i \geq 0$). Thus,

$$L_{19} = \frac{L_{19}}{1 + L_3L_7L_{11}L_{15}L_{19}}, L_{18} = \frac{L_{18}}{1 + L_2L_6L_{10}L_{14}L_{18}}, ..., L_0 = \frac{L_0}{1 + L_0L_4L_8L_{12}L_{16}},$$

which imply that $L_0 = L_1 = ... = L_{19} = 0$.

3. The Difference Equation $u_{n+1} = \frac{u_{n-19}}{1 - u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}$

In this part, we give new forms of exact solutions for the following equation:

$$u_{n+1} = \frac{u_{n-19}}{1 - u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}, \qquad n = 0, 1, 2, ..., \tag{4}$$

where $u_{-j},\ j = 0, 1, 2, ..., 19$, are real numbers.

Theorem 6. *Let $\{u_n\}_{n=-19}^{\infty}$ be a solution to Eq. (4). Then, for $n = 0, 1, 2, ...$*

$$u_{20n-k} = a_k \prod_{i=0}^{n-1} \left(\frac{-1 + (5i + \mu_k - 1)\,\Psi_k}{-1 + (5i + \mu_k)\,\Psi_k} \right), \tag{5}$$

where $\Psi_k = \prod_{j=0}^{4} a_{mod(k,4)+4j}$, $\mu_k = 5 - \left[\frac{k}{4}\right]$ and $u_{-k} = a_k$, with $r\Psi_k \neq 1$ such that $r \in \{1, 2, 3, ...\}$, $k = 0, 1, 2, ..., 19$.

Proof. The solutions are true at $n = 0$. We now assume that they are true at $n-1$. This gives

$$u_{20n-20-k} = a_k \prod_{i=0}^{n-2} \left(\frac{-1 + (5i + \mu_k - 1)\,\Psi_k}{-1 + (5i + \mu_k)\,\Psi_k} \right). \tag{6}$$

From Eq. (4) and Eq. (6), we have

$$\begin{aligned} u_{20n-19} &= \frac{u_{20n-39}}{1 - u_{20n-23}u_{20n-27}u_{20n-31}u_{20n-35}u_{20n-39}} \\ &= \frac{a_{19}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{19}-1)P_{19}}{-1+(5i+M_{19})P_{19}}\right)}{1 - \prod_{j=0}^{4}\left(a_{4j+3}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{4j+3}-1)P_{4j+3}}{-1+(5i+M_{4j+3})P_{4j+3}}\right)\right)} \\ &= \frac{a_{19}\prod_{i=0}^{n-2}\left(\frac{-1+(5i)a_3a_7a_{11}a_{15}a_{19}}{-1+(5i+1)a_3a_7a_{11}a_{15}a_{19}}\right)}{1 - a_3a_7a_{11}a_{15}a_{19}\prod_{i=0}^{n-2}\left(\frac{-1+(5i)a_3a_7a_{11}a_{15}a_{19}}{-1+(5i+5)a_3a_7a_{11}a_{15}a_{19}}\right)} = a_{19}\prod_{i=0}^{n-1}\left(\frac{-1+(5i)a_3a_7a_{11}a_{15}a_{19}}{-1+(5i+1)\,a_3a_7a_{11}a_{15}a_{19}}\right). \end{aligned}$$

Moreover, using Eq. (4) and Eq. (6) gives

$$\begin{aligned} u_{20n-18} &= \frac{u_{20n-38}}{1 - u_{20n-22}u_{20n-26}u_{20n-30}u_{20n-34}u_{20n-38}} \\ &= \frac{a_{18}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{18}-1)P_{18}}{-1+(5i+M_{18})P_{18}}\right)}{1 - \prod_{j=0}^{4}\left(a_{4j+2}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{4j+2}-1)P_{4j+2}}{-1+(5i+M_{4j+2})P_{4j+2}}\right)\right)} \\ &= \frac{a_{18}\prod_{i=0}^{n-2}\left(\frac{-1+(5i)a_2a_6a_{10}a_{14}a_{18}}{-1+(5i+1)a_2a_6a_{10}a_{14}a_{18}}\right)}{1 - a_2a_6a_{10}a_{14}a_{18}\prod_{i=0}^{n-2}\left(\frac{-1+(5i)a_2a_6a_{10}a_{14}a_{18}}{-1+(5i+5)a_2a_6a_{10}a_{14}a_{18}}\right)} = a_{18}\prod_{i=0}^{n-1}\left(\frac{-1+(5i)a_2a_6a_{10}a_{14}a_{18}}{-1+(5i+1)\,a_2a_6a_{10}a_{14}a_{18}}\right). \end{aligned}$$

Eq. (4) and Eq. (6) also lead to

$$\begin{aligned} u_{20n-17} &= \frac{u_{20n-37}}{1 - u_{20n-21}u_{20n-25}u_{20n-29}u_{20n-33}u_{20n-37}} \\ &= \frac{a_{17}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{17}-1)P_{17}}{-1+(5i+M_{17})P_{17}}\right)}{1 - \prod_{j=0}^{4}\left(a_{4j+1}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{4j+1}-1)P_{4j+1}}{-1+(5i+M_{4j+1})P_{4j+1}}\right)\right)} \\ &= \frac{a_{17}\prod_{i=0}^{n-2}\left(\frac{1+(5i)a_1a_5a_9a_{13}a_{17}}{1+(5i+1)a_1a_5a_9a_{13}a_{17}}\right)}{1 + a_1a_5a_9a_{13}a_{17}\prod_{i=0}^{n-2}\left(\frac{-1+(5i)a_1a_5a_9a_{13}a_{17}}{-1+(5i+5)a_1a_5a_9a_{13}a_{17}}\right)} = a_{17}\prod_{i=0}^{n-1}\left(\frac{-1+(5i)a_1a_5a_9a_{13}a_{17}}{-1+(5i+1)\,a_1a_5a_9a_{13}a_{17}}\right). \end{aligned}$$

In addition, using Eq. (4) and Eq. (6) yields

$$\begin{aligned} u_{20n-16} &= \frac{u_{20n-36}}{1-u_{20n-20}u_{20n-24}u_{20n-28}u_{20n-32}u_{20n-36}} \\ &= \frac{a_{16}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{16}-1)P_{16}}{-1+(5i+M_{16})P_{16}}\right)}{1-\prod_{j=0}^{4}\left(a_{4j}\prod_{i=0}^{n-2}\left(\frac{-1+(5i+M_{4j}-1)P_{4j}}{-1+(5i+M_{4j})P_{4j}}\right)\right)} \\ &= \frac{a_{16}\prod_{i=0}^{n-2}\left(\frac{-1+(5i)a_0a_4a_8a_{12}a_{16}}{-1+(5i+1)a_0a_4a_8a_{12}a_{16}}\right)}{1-a_0a_4a_8a_{12}a_{16}\prod_{i=0}^{n-2}\left(\frac{-1+(5i)a_0a_4a_8a_{12}a_{16}}{-1+(5i+5)a_0a_4a_8a_{12}a_{16}}\right)} = a_{16}\prod_{i=0}^{n-1}\left(\frac{-1+(5i)a_0a_4a_8a_{12}a_{16}}{-1+(5i+1)\,a_0a_4a_8a_{12}a_{16}}\right). \end{aligned}$$

In a similar way, one can prove the remaining relations.

Theorem 7. *Equation* (4) *has a unique fixed point* $\overline{u}=0$, *which is non-hyperbolic.*

Proof. Using Eq. (4), we obtain

$$\overline{u}=\frac{\overline{u}}{1-\overline{u}^5},$$

which can be easily rearranged as follows:

$$\overline{u}-\overline{u}^6=\overline{u},$$

or, $\overline{u}^6=0$. As a result, the unique fixed point of Eq. (4) is $\overline{u}=0$. Next, we define a function

$$g(x,y,z,u,v)=\frac{x}{1-xyzuv},$$

on I^5 where I is a subset of R such that $0\in I$ and $g(I^5)\subseteq I$. Obviously, g is continuously differentiable on I^5. Therefore, we have

$$\begin{aligned} g_x(x,y,z,u,v) &= \frac{1}{(1-xyzuv)^2}, \qquad g_y(x,y,z,u,v)=\frac{x^2zuv}{(1-xyzuv)^2}, \\ g_z(x,y,z,u,v) &= \frac{x^2yuv}{(1-uyzuv)^2}, \qquad g_u(x,y,z,u,v)=\frac{x^2yzv}{(1-xyzuv)^2}, \\ g_v(x,y,z,u,v) &= \frac{x^2yzu}{(1-xyzuv)^2}. \end{aligned}$$

Hence,

$$g_x(\overline{u},\overline{u},\overline{u},\overline{u},\overline{u})=1,\ g_y(\overline{u},\overline{u},\overline{u},\overline{u},\overline{u})=g_z(\overline{u},\overline{u},\overline{u},\overline{u},\overline{u})=g_u(\overline{u},\overline{u},\overline{u},\overline{u},\overline{u})=g_v(\overline{u},\overline{u},\overline{u},\overline{u},\overline{u})=0.$$

Consequently, the linearized equation of Eq. (4) about the obtained equilibrium point is

$$z_{n+1}=z_{n-19},$$

whose characteristic equation is $\lambda^{20}-1=0$. This gives

$$|\lambda_i|=1, \quad i=1,2,...,20.$$

Hence, the equilibrium point is non-hyperbolic.

4. The Difference Equation $u_{n+1} = \dfrac{u_{n-19}}{-1 + u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}$

This section presents some new theorems and solutions for the following equation:

$$u_{n+1} = \frac{u_{n-19}}{-1 + u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}, \qquad n = 0, 1, 2, ..., \tag{7}$$

where the initial data u_{-j}, $j = 0, 1, 2, ..., 19$ are real. The oscillation and the periodic nature are also investigated.

Theorem 8. *Every solution of Eq.* (7) *is periodic with period* 40.

Proof. Equation (7) leads to

$$u_{n+40} = \frac{u_{n+20}}{-1 + \prod_{i=0}^{4} u_{n+20+4i}}. \tag{8}$$

Since

$$-1 + \prod_{i=0}^{4} u_{n+20+4i} = -1 + \prod_{i=0}^{3} u_{n+20+4i} \frac{u_{n+16}}{-1 + \prod_{i=0}^{4} u_{n+16+4i}} = \frac{1}{-1 + \prod_{i=0}^{4} u_{n+16+4i}},$$

where

$$-1 + \prod_{i=0}^{4} u_{n+16+4i} = \frac{1}{-1 + \prod_{i=0}^{4} u_{n+12+4i}}.$$

Then,

$$-1 + \prod_{i=0}^{4} u_{n+20+4i} = -1 + \prod_{i=0}^{4} u_{n+12+4i}.$$

Similarly,

$$-1 + \prod_{i=0}^{4} u_{n+12+4i} = -1 + \prod_{i=0}^{4} u_{n+4+4i} \text{ and } -1 + \prod_{i=0}^{4} u_{n+4+4i} = \frac{1}{-1 + \prod_{i=0}^{4} u_{n+4i}},$$

Therefore, from Eq. (7) and Ed. (8), we have

$$u_{n+40} = \frac{\left(\frac{u_n}{-1+\prod_{i=0}^{4} u_{n+4i}}\right)}{\left(\frac{1}{-1+\prod_{i=0}^{4} u_{n+4i}}\right)} = u_n \ , n = 0, 1, 2, ... \ .$$

Theorem 9. *The periodic solution of Eq.* (7) *takes the form*

$$u_{40n-k} = \frac{a_{k-(q(k))(20)^{q(k)}}}{(-1 + \Psi_{k-20})^{\alpha(k)q(k)}}, \qquad k = 0, 1, ..., 39 \textit{ and } n = 1, 2, ...$$

Here, $u_{-j} = a_j$, $\Psi_j = \prod_{i=0}^{4} a_{mod(j,4)+4i}$, where $\Psi_j \neq 1$, $j = 0, 1, 2, ..., 19$, $\Psi_{-l} = 0, l = 1, 2, ..., 20$, $q(k) = \frac{1}{2}((-1)^{[\frac{k}{20}]+1} + 1)$ and $\alpha(k) = (-1)^{[\frac{k}{4}]+1}$.

Proof. The definition of $q(k)$ gives

$$q(0) = q(1) = ... = q(19) = 0 \text{ and } q(20) = q(21) = ... = q(39) = 1.$$

Further,

$$\alpha(i + 8r) = -1, \ \alpha(i + 4 + 8r) = 1, \quad i = 0, 1, 2, 3 \text{ and } r = 0, 1, 2, 3.$$

Therefore,

$$
\begin{aligned}
u_1 &= \frac{u_{-19}}{-1+u_{-3}u_{-7}u_{-11}u_{-15}u_{-19}} = \frac{a_{19}}{-1+a_3a_7a_{11}a_{15}a_{19}},\\
u_2 &= \frac{u_{-18}}{-1+u_{-2}u_{-6}u_{-10}u_{-14}u_{-18}} = \frac{a_{18}}{-1+a_2a_6a_{10}a_{14}a_{18}},\\
u_3 &= \frac{u_{-17}}{-1+u_{-1}u_{-5}u_{-9}u_{-13}u_{-17}} = \frac{a_{17}}{-1+a_1a_5a_9a_{13}a_{17}},\\
u_4 &= \frac{u_{-16}}{-1+u_0u_{-4}u_{-8}u_{-12}u_{-16}} = \frac{a_{16}}{-1+a_0a_4a_8a_{12}a_{16}},\\
u_5 &= \frac{u_{-15}}{-1+u_1u_{-3}u_{-7}u_{-11}u_{-15}} = a_{15}(-1+a_3a_7a_{11}a_{15}a_{19}),\\
u_6 &= \frac{u_{-14}}{-1+u_2u_{-2}u_{-6}u_{-10}u_{-14}} = a_{14}(-1+a_2a_6a_{10}a_{14}a_{18}),\\
u_7 &= \frac{u_{-13}}{-1+u_3u_{-1}u_{-5}u_{-9}u_{-13}} = a_{13}(-1+a_1a_5a_9a_{13}a_{17}),\\
u_8 &= \frac{u_{-12}}{-1+u_4u_0u_{-4}u_{-8}u_{-12}} = a_{12}(-1+a_0a_4a_8a_{12}a_{16}),\\
u_9 &= \frac{u_{-11}}{-1+u_5u_1u_{-3}u_{-7}u_{-11}} = \frac{a_{11}}{-1+a_3a_7a_{11}a_{15}a_{19}},\\
u_{10} &= \frac{u_{-10}}{-1+u_6u_2u_{-2}u_{-6}u_{-10}} = \frac{a_{10}}{-1+a_2a_6a_{10}a_{14}a_{18}},\\
u_{11} &= \frac{u_{-9}}{-1+u_7u_3u_{-1}u_{-5}u_{-9}} = \frac{a_9}{-1+a_1a_5a_9a_{13}a_{17}},\\
u_{12} &= \frac{u_{-8}}{-1+u_8u_4u_0u_{-4}u_{-8}} = \frac{a_8}{-1+a_0a_4a_8a_{12}a_{16}},\\
u_{13} &= \frac{u_{-7}}{-1+u_9u_5u_1u_{-3}u_{-7}} = a_7(-1+a_3a_7a_{11}a_{15}a_{19}),\\
u_{14} &= \frac{u_{-6}}{-1+u_{10}u_6u_2u_{-2}u_{-6}} = a_6(-1+a_2a_6a_{10}a_{14}a_{18}),\\
u_{15} &= \frac{u_{-5}}{-1+u_{11}u_7u_3u_{-1}u_{-5}} = a_5(-1+a_1a_5a_9a_{13}a_{17}),\\
u_{16} &= \frac{u_{-4}}{-1+u_{12}u_8u_4u_0u_{-4}} = a_4(-1+a_0a_4a_8a_{12}a_{16}),\\
u_{17} &= \frac{u_{-3}}{-1+u_{13}u_9u_5u_1u_{-3}} = \frac{a_3}{-1+a_3a_7a_{11}a_{15}a_{19}},\\
u_{18} &= \frac{u_{-2}}{-1+u_{14}u_{10}u_6u_2u_{-2}} = \frac{a_2}{-1+a_2a_6a_{10}a_{14}a_{18}},\\
u_{19} &= \frac{u_{-1}}{-1+u_{15}u_{11}u_7u_3u_{-1}} = \frac{a_1}{-1+a_1a_5a_9a_{13}a_{17}},\\
u_{20} &= \frac{u_0}{-1+u_{16}u_{12}u_8u_4u_0} = \frac{a_0}{-1+a_0a_4a_8a_{12}a_{16}},\\
u_{21} &= \frac{u_1}{-1+u_{17}u_{13}u_9u_5u_1} = a_{19}, \qquad u_{22} = \frac{u_2}{-1+u_{18}u_{14}u_{10}u_6u_2} = a_{18},\\
u_{23} &= \frac{u_3}{-1+u_{19}u_{15}u_{11}u_7u_3} = a_{17}, \qquad u_{24} = \frac{u_4}{-1+u_{20}u_{16}u_{12}u_8u_4} = a_{16},\\
u_{25} &= \frac{u_5}{-1+u_{21}u_{17}u_{13}u_9u_5} = a_{15}, \qquad u_{26} = \frac{u_6}{-1+u_{22}u_{18}u_{14}u_{10}u_6} = a_{14},
\end{aligned}
$$

$$u_{27} = \frac{u_7}{-1 + u_{23}u_{19}u_{15}u_{11}u_7} = a_{13}, \qquad u_{28} = \frac{u_8}{-1 + u_{24}u_{20}u_{16}u_{12}u_8} = a_{12},$$
$$u_{29} = \frac{u_9}{-1 + u_{25}u_{21}u_{17}u_{13}u_9} = a_{11}, \qquad u_{30} = \frac{u_{10}}{-1 + u_{26}u_{22}u_{18}u_{14}u_{10}} = a_{10},$$
$$u_{31} = \frac{u_{11}}{-1 + u_{27}u_{23}u_{19}u_{15}u_{11}} = a_9, \qquad u_{32} = \frac{u_{12}}{-1 + u_{28}u_{24}u_{20}u_{16}u_{12}} = a_8,$$
$$u_{33} = \frac{u_{13}}{-1 + u_{29}u_{25}u_{21}u_{17}u_{13}} = a_7, \qquad u_{34} = \frac{u_{14}}{-1 + u_{30}u_{26}u_{22}u_{18}u_{14}} = a_6,$$
$$u_{35} = \frac{u_{15}}{-1 + u_{31}u_{27}u_{23}u_{19}u_{15}} = a_5, \qquad u_{36} = \frac{u_{16}}{-1 + u_{32}u_{28}u_{24}u_{20}u_{16}} = a_4,$$
$$u_{37} = \frac{u_{17}}{-1 + u_{33}u_{29}u_{25}u_{21}u_{17}} = a_3, \qquad u_{38} = \frac{u_{18}}{-1 + u_{34}u_{30}u_{26}u_{22}u_{18}} = a_2,$$
$$u_{39} = \frac{u_{19}}{-1 + u_{35}u_{31}u_{27}u_{23}u_{19}} = a_1, \qquad u_{40} = \frac{u_{20}}{-1 + u_{36}u_{32}u_{28}u_{24}u_{20}} = a_0.$$

The results are demonstrated by induction.

Theorem 10. *Equation* (7) *has two non-hyperbolic fixed points* $\overline{u} = 0$ *and* $\overline{u} = \sqrt[5]{2}$.

Proof. The proof is similar to the proof of Theorem 7, and will be omitted.

Theorem 11. *Equation* (7) *is periodic of period* 20 *if and only if* $\Psi_k = 2, \quad k = 0, 1, 2, 3$ *and the solutions have the form*

$$u_{20n-k} = a_k, \qquad k = 0, 1, ..., 19 \text{ and } n = 0, 1, 2, ... \ .$$

Proof. The proof can be easily done by using Theorem 9.

Theorem 12. *Let* $a_0, a_1, ..., a_{19} \in (0, 1)$. *Then, the solution* $\{u_n\}_{n=-19}^{\infty}$ *oscillates about the point* $\overline{u} = 0$, *with positive semicycles of length* 20, *and negative semicycles of length* 20.

Proof. Theorem 9 leads to $u_1, u_2, ..., u_{20} < 0$ and $u_{21}, u_{22}, ..., u_{40} > 0$, and the result is shown by induction.

5. The Difference Equation $u_{n+1} = \frac{u_{n-19}}{-1 - u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}$

This section introduces new exact solutions to the following equation:

$$u_{n+1} = \frac{u_{n-19}}{-1 - u_{n-3}u_{n-7}u_{n-11}u_{n-15}u_{n-19}}, \qquad n = 0, 1, 2, ..., \tag{9}$$

where the initial values $u_{-j}, \ j = 0, 1, 2, ..., 19$ are real numbers. Furthermore, we present some relevant theorems for this equation.

Theorem 13. *Every solution of Eq.* (9) *is periodic with period* 40.

Proof. The proof is omitted.

Theorem 14. *The periodic solution of Eq.* (9) *takes the form*

$$u_{40n-k} = \frac{a_{k-(q(k))(20)^{q(k)}}}{(-1 - \Psi_{k-20})^{\alpha(k)q(k)}}, \qquad k = 0, 1, ..., 39 \text{ and } n = 1, 2, ...$$

Here, $u_{-j} = a_j, \ \Psi_j = \prod_{i=0}^{4} a_{mod(j,4)+4i}$, where $\Psi_j \neq 1, \ j = 0, 1, 2, ..., 19, \ \Psi_{-l} = 0, l = 1, 2, ..., 20, \ \ q(k) = \frac{1}{2}((-1)^{[\frac{k}{20}]+1} + 1)$ and $\alpha(k) = (-1)^{[\frac{k}{4}]+1}$.

Proof. The definition of $q(k)$ gives

$$q(0) = q(1) = ... = q(19) = 0 \text{ and } q(20) = q(21) = ... = q(39) = 1.$$

Moreover,

$$\alpha(i+8r)=-1,\quad \alpha(i+4+8r)=1,\quad i=0,1,2,3 \text{ and } r=0,1,2,3.$$

Therefore,

$$\begin{aligned}
u_1 &= \frac{u_{-19}}{-1-u_{-3}u_{-7}u_{-11}u_{-15}u_{-19}} = -\frac{a_{19}}{1+a_3a_7a_{11}a_{15}a_{19}},\\
u_2 &= \frac{u_{-18}}{-1-u_{-2}u_{-6}u_{-10}u_{-14}u_{-18}} = -\frac{a_{18}}{1+a_2a_6a_{10}a_{14}a_{18}},\\
u_3 &= \frac{u_{-17}}{-1-u_{-1}u_{-5}u_{-9}u_{-13}u_{-17}} = -\frac{a_{17}}{1+a_1a_5a_9a_{13}a_{17}},\\
u_4 &= \frac{u_{-16}}{-1-u_0u_{-4}u_{-8}u_{-12}u_{-16}} = -\frac{a_{16}}{1+a_0a_4a_8a_{12}a_{16}},\\
u_5 &= \frac{u_{-15}}{-1-u_1u_{-3}u_{-7}u_{-11}u_{-15}} = -a_{15}(1+a_3a_7a_{11}a_{15}a_{19}),\\
u_6 &= \frac{u_{-14}}{-1-u_2u_{-2}u_{-6}u_{-10}u_{-14}} = -a_{14}(1+a_2a_6a_{10}a_{14}a_{18}),\\
u_7 &= \frac{u_{-13}}{-1-u_3u_{-1}u_{-5}u_{-9}u_{-13}} = -a_{13}(1+a_1a_5a_9a_{13}a_{17}),\\
u_8 &= \frac{u_{-12}}{-1-u_4u_0u_{-4}u_{-8}u_{-12}} = -a_{12}(1+a_0a_4a_8a_{12}a_{16}),\\
u_9 &= \frac{u_{-11}}{-1-u_5u_1u_{-3}u_{-7}u_{-11}} = -\frac{a_{11}}{1+a_3a_7a_{11}a_{15}a_{19}},\\
u_{10} &= \frac{u_{-10}}{-1-u_6u_2u_{-2}u_{-6}u_{-10}} = -\frac{a_{10}}{1+a_2a_6a_{10}a_{14}a_{18}},\\
u_{11} &= \frac{u_{-9}}{-1-u_7u_3u_{-1}u_{-5}u_{-9}} = -\frac{a_9}{1+a_1a_5a_9a_{13}a_{17}},\\
u_{12} &= \frac{u_{-8}}{-1-u_8u_4u_0u_{-4}u_{-8}} = -\frac{a_8}{1+a_0a_4a_8a_{12}a_{16}},\\
u_{13} &= \frac{u_{-7}}{-1-u_9u_5u_1u_{-3}u_{-7}} = -a_7(1+a_3a_7a_{11}a_{15}a_{19}),\\
u_{14} &= \frac{u_{-6}}{-1-u_{10}u_6u_2u_{-2}u_{-6}} = -a_6(1+a_2a_6a_{10}a_{14}a_{18}),\\
u_{15} &= \frac{u_{-5}}{-1-u_{11}u_7u_3u_{-1}u_{-5}} = -a_5(1+a_1a_5a_9a_{13}a_{17}),\\
u_{16} &= \frac{u_{-4}}{-1-u_{12}u_8u_4u_0u_{-4}} = -a_4(1+a_0a_4a_8a_{12}a_{16}),\\
u_{17} &= \frac{u_{-3}}{-1-u_{13}u_9u_5u_1u_{-3}} = -\frac{a_3}{1+a_3a_7a_{11}a_{15}a_{19}},\\
u_{18} &= \frac{u_{-2}}{-1-u_{14}u_{10}u_6u_2u_{-2}} = -\frac{a_2}{1+a_2a_6a_{10}a_{14}a_{18}},\\
u_{19} &= \frac{u_{-1}}{-1-u_{15}u_{11}u_7u_3u_{-1}} = -\frac{a_1}{1+a_1a_5a_9a_{13}a_{17}},\\
u_{20} &= \frac{u_0}{-1-u_{16}u_{12}u_8u_4u_0} = -\frac{a_0}{1+a_0a_4a_8a_{12}a_{16}},\\
u_{21} &= \frac{u_1}{-1-u_{17}u_{13}u_9u_5u_1} = a_{19}, \qquad u_{22} = \frac{u_2}{-1-u_{18}u_{14}u_{10}u_6u_2} = a_{18},
\end{aligned}$$

$$u_{23} = \frac{u_3}{-1 - u_{19}u_{15}u_{11}u_7u_3} = a_{17}, \qquad u_{24} = \frac{u_4}{-1 - u_{20}u_{16}u_{12}u_8u_4} = a_{16},$$
$$u_{25} = \frac{u_5}{-1 - u_{21}u_{17}u_{13}u_9u_5} = a_{15}, \qquad u_{26} = \frac{u_6}{-1 - u_{22}u_{18}u_{14}u_{10}u_6} = a_{14},$$
$$u_{27} = \frac{u_7}{-1 - u_{23}u_{19}u_{15}u_{11}u_7} = a_{13}, \qquad u_{28} = \frac{u_8}{-1 - u_{24}u_{20}u_{16}u_{12}u_8} = a_{12},$$
$$u_{29} = \frac{u_9}{-1 - u_{25}u_{21}u_{17}u_{13}u_9} = a_{11}, \qquad u_{30} = \frac{u_{10}}{-1 - u_{26}u_{22}u_{18}u_{14}u_{10}} = a_{10},$$
$$u_{31} = \frac{u_{11}}{-1 - u_{27}u_{23}u_{19}u_{15}u_{11}} = a_9, \qquad u_{32} = \frac{u_{12}}{-1 - u_{28}u_{24}u_{20}u_{16}u_{12}} = a_8,$$
$$u_{33} = \frac{u_{13}}{-1 - u_{29}u_{25}u_{21}u_{17}u_{13}} = a_7, \qquad u_{34} = \frac{u_{14}}{-1 - u_{30}u_{26}u_{22}u_{18}u_{14}} = a_6,$$
$$u_{35} = \frac{u_{15}}{-1 - u_{31}u_{27}u_{23}u_{19}u_{15}} = a_5, \qquad u_{36} = \frac{u_{16}}{-1 - u_{32}u_{28}u_{24}u_{20}u_{16}} = a_4,$$
$$u_{37} = \frac{u_{17}}{-1 - u_{33}u_{29}u_{25}u_{21}u_{17}} = a_3, \qquad u_{38} = \frac{u_{18}}{-1 - u_{34}u_{30}u_{26}u_{22}u_{18}} = a_2,$$
$$u_{39} = \frac{u_{19}}{-1 - u_{35}u_{31}u_{27}u_{23}u_{19}} = a_1, \qquad u_{40} = \frac{u_{20}}{-1 - u_{36}u_{32}u_{28}u_{24}u_{20}} = a_0.$$

The result follows by induction.

Theorem 15. *Equation* (9) *has two non-hyperbolic fixed points* $\overline{u} = 0$ *and* $\overline{u} = -\sqrt[5]{2}$.

Proof. The proof is similar to the proof of Theorem 7, and will be omitted.

Theorem 16. *Equation* (9) *is periodic of period* 20 *if and only if* $\Psi_k = -2$, $k = 0, 1, 2, 3$ *and the solutions have the form*

$$u_{20n-k} = a_k, \quad k = 0, 1, ..., 19 \text{ and } n = 0, 1, 2, ... \, .$$

Proof. It can be easily shown from Theorem 14.

Theorem 17. *Let* $a_0, a_1, ..., a_{19} \in [0, \infty)$. *Then, the solution* $\{u_n\}_{n=-19}^{\infty}$ *oscillates about the point* $\overline{u} = 0$, *with positive semicycles of length* 20, *and negative semicycles of length* 20.

Proof. Theorem 14 gives $u_1, u_2, ..., u_{20} < 0$ and $u_{21}, u_{22}, ..., u_{40} > 0$. Hence, the result can be done by induction.

6. Numerical Examples

This section is added to verify the obtained theoretical results. We present some 2D figures plotted by using MATLAB for the stability, periodicity, and bounded solutions of the proposed equations.

Example 1. The behavior of the solutions of Eq. (1) when $u_{-19} = 0.5$, $u_{-18} = 0.7$, $u_{-17} = 0.52$, $u_{-16} = -0.1$, $u_{-15} = 0.3$, $u_{-14} = 0.1$, $u_{-13} = 1$, $u_{-12} = 2.2$, $u_{-11} = 0.1$, $u_{-10} = 0.1$, $u_{-9} = -0.2$, $u_{-8} = 0.52$, $u_{-7} = -0.2$, $u_{-6} = 0.8$, $u_{-5} = 0.9$, $u_{-4} = 0.3$, $u_{-3} = 0.2$, $u_{-2} = -0.5$, $u_{-1} = 0.1$ and $u_0 = 0.2$ is shown in Figure 1 (right).

Example 2. The behavior of the solutions of Eq. (4) is depicted in Figure 1 (left) under the initial data $u_{-19} = 0.1$, $u_{-18} = 0.2$, $u_{-17} = 0.2$, $u_{-16} = -0.1$, $u_{-15} = 0.3$, $u_{-14} = 0.3$, $u_{-13} = 0.1$, $u_{-12} = -1$, $u_{-11} = 2.2$, $u_{-10} = 0.1$, $u_{-9} = -0.2$, $u_{-8} = 0.52$, $u_{-7} = 0.2$, $u_{-6} = 0.8$, $u_{-5} = 0.9$, $u_{-4} = 0.3$, $u_{-3} = 0.2$, $u_{-2} = -0.01$, $u_{-1} = 0.3$ and $u_0 = 0.1$.

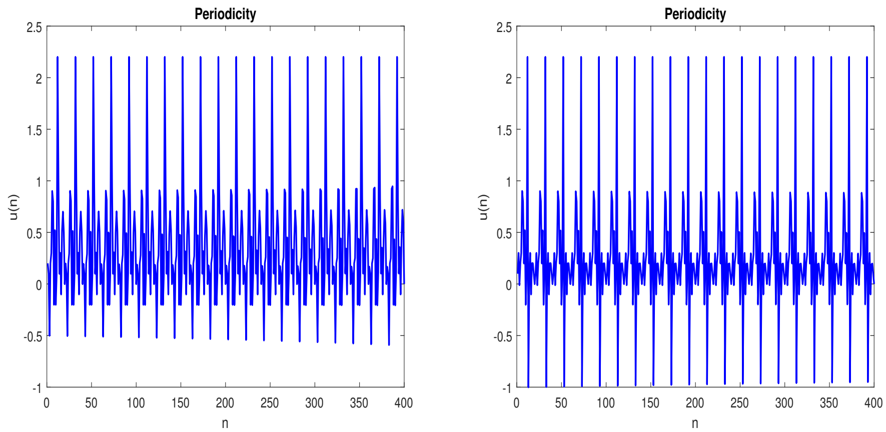

Figure 1. The left graph demonstrates the periodicity of Eq. (1) while the right graph illustrates the periodicity of Eq. (4).

Example 3. The solutions of Eq. (7) are shown in Figure 2 (left) when $u_{-19} = 4$, $u_{-18} = 0.5$, $u_{-17} = 0.2$, $u_{-16} = -5$, $u_{-15} = -2$, $u_{-14} = 2$, $u_{-13} = -3$, $u_{-12} = -10$, $u_{-11} = 1$, $u_{-10} = -7$, $u_{-9} = -3.5$, $u_{-8} = 5$, $u_{-7} = 5$, $u_{-6} = -2$, $u_{-5} = -2$, $u_{-4} = 0.8$, $u_{-3} = -0.2$, $u_{-2} = 0.1$, $u_{-1} = -8$ and $u_0 = -3$.

Example 4. Figure 2 (right) shows the behavior of the solutions of Eq. (9) under the conditions $u_{-19} = 4$, $u_{-18} = 0.5$, $u_{-17} = 0.2$, $u_{-16} = -5$, $u_{-15} = -2$, $u_{-14} = 2$, $u_{-13} = -3$, $u_{-12} = -10$, $u_{-11} = 1$, $u_{-10} = -7$, $u_{-9} = -3.5$, $u_{-8} = 5$, $u_{-7} = -2$, $u_{-6} = 1$, $u_{-5} = 8$, $u_{-4} = -4$, $u_{-3} = 1$, $u_{-2} = 1$, $u_{-1} = 8$ and $u_0 = -3$.

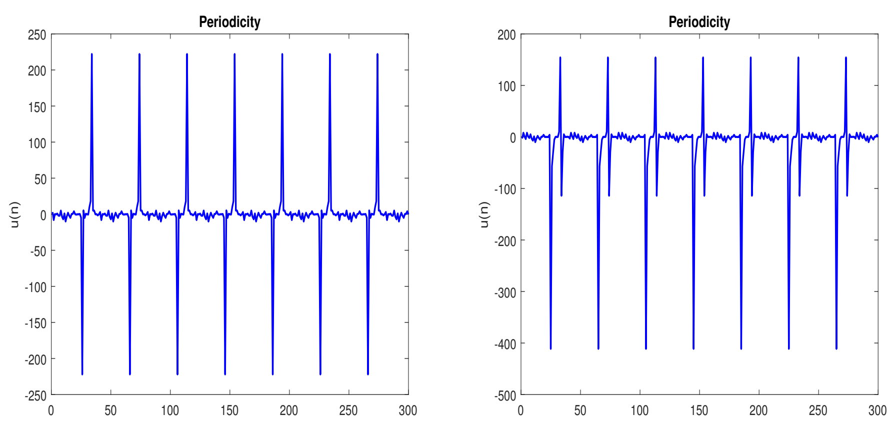

Figure 2. The left sketch presents the periodicity of Eq. (7) while the right figure illustrates the periodicity of Eq. (9).

7. Conclusion

This article has discussed some new solutions and theorems for novel difference equations. We have presented the obtained rational solutions using the modulus operator. The solutions of Eq. (1) are found bounded. Furthermore, the equilibrium point of Eq. (1) is locally and globally stable. In Theorem 7, we presented that Eq. (4) has a unique non-hyperbolic fixed point $\overline{u} = 0$, while Theorem 8 proves that every solution of Eq. (7) is periodic with period 40. Moreover, we proved that Eq. (7) is periodic of period 20 if and only if $\Psi_k = 2$. In Theorem 13, we showed that every solution of Eq. (9) is periodic with period 40. Equation (9) is periodic of period 20 if and only if $\Psi_k = -2$. Finally, we have confirmed the constructed theoretical results in the presented figures. For example, Figure 2 (right) shows the periodic solutions of Eq. (9). The used methods can be utilized to solve some high order nonlinear equations.

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