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GEOMETRIC PROPERTIES OF A NEW INTEGRAL OPERATOR ASSOCIATED WITH DINI FUNCTIONS

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Abstract. The main object of this work is to introduce a new integral operator which contains normalized Dini function. Sufficient criteria are investigated for this integral operator belonging to certain classes of starlike functions.

1. Introduction

Let $\mathcal{A}$ represent the class of functions consisting of the form

$$f(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k \tag{1}$$

that are analytic in the open unit disc $\mathbb{U} = \{\xi : \xi \in \mathbb{C} \; and \; |\xi|<1\}$. Further, we signify by S the subclass of $\mathcal{A}$ comprising of functions of the form (1) which are univalent in $\mathbb{U}$. A function $f(\xi)$ in $\mathcal{A}$ is supposed to be starlike of order $\vartheta(0 \le \vartheta<1)$ if the following analytic criteria is gratified

$$\Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \vartheta, \; (\xi \in \mathbb{U}). \tag{2}$$

Similarly, a function $f(\xi)$ in $\mathcal{A}$ is supposed to be convex of order $\vartheta(0 \le \vartheta<1)$ if the following analytic criteria is gratified

$$\Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) > \vartheta, \; (\xi \in \mathbb{U}), \tag{3}$$

we indicate by $S^*(\vartheta)$ the class of starlike functions of order ϑ and $C(\vartheta)$ the class of convex functions of order ϑ. Clearly $S^*(\vartheta) \subset S^*(0) \equiv S^*(0 \le \vartheta < 1)$, $S^* \subset S$ and $C(\vartheta) \subset C(0) \equiv C \subset S$

The Bessel function of the first kind of order ν is given by the series

$$J_\nu(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\xi}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)},$$

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where Γ denotes the Euler-Gamma function $\xi \in \mathbb{C}$ and $\nu \in \mathbb{R}$. The normalized Dini function

$$Q_\nu : \mathbb{U} \longrightarrow \mathbb{C}$$

defined by

$$Q_\nu(\xi) = 2^{\nu-1}\Gamma(\nu+1)\xi^{1-\frac{\nu}{2}}[(2-\nu)J_\nu(\xi^{\frac{1}{2}}) + \xi^{\frac{1}{2}}J'_\nu(\xi^{\frac{1}{2}})] \tag{4}$$

$$= \xi + \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)\Gamma(\nu+1)}{4^n n!\Gamma(n+\nu+1)}\xi^{n+1}, \quad (\xi \in \mathbb{U}). \tag{5}$$

The close-to-convexity of Dini functions are investigated by Baricz *et al.* [1]. Recently, Deniz and Gören [3] studied geometric properties of Dini functions. Motivated with the work of Deniz *et al.* [4], Din *et al.* [5], Frasin [6], Porwal and Breaz [8], Porwal and Kumar [9], Porwal *et al.* [10], we define a new integral operator associated with normalized Dini functions as follows

$$F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(\xi) = \int_0^\xi \prod_{i=1}^n \left(\frac{Q_{\nu_i}(t)}{t}\right)^{\alpha_i} dt, \tag{6}$$

and obtain some sufficient condition for the operator defined by (4) is in the class S^*.

2. Preliminary Results

To show our main results we shall necessitate the following lemmas:

Lemma 2.1. *([5]) Let* $\nu > \frac{-9+\sqrt{33}}{8}$ *and suppose the normalized Dini function* $Q_\nu : \mathbb{U} \longrightarrow \mathbb{C}$ *defined by* (4) *then the following inequality hold for all* $\xi \in \mathbb{U}$,

$$\left|\frac{\xi Q'_\nu(\xi)}{Q_\nu(\xi)} - 1\right| \leq \frac{4\nu+9}{2(4\nu^2+9\nu+3)}.$$

Lemma 2.2. *([11]) If* $f \in \mathcal{A}$ *satisfies*

$$\Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) < \frac{\vartheta+1}{2(\vartheta-1)}, \qquad (\xi \in \mathbb{U}),$$

for some $2 \leq \vartheta < 3$,
or

$$\Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) < \frac{5\vartheta-1}{2(\vartheta+1)}, \qquad (\xi \in \mathbb{U}),$$

for some $1 < \vartheta \leq 2$, *then* $f \in S^*$.

Lemma 2.3. *([11]) If* $f \in \mathcal{A}$ *satisfies*

$$\Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) > -\frac{\vartheta+1}{2\vartheta(\vartheta-1)}, \qquad (\xi \in \mathbb{U}),$$

for some $\vartheta \leq -1$,
or

$$\Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) > \frac{3\vartheta+1}{2\vartheta(\vartheta+1)}, \qquad (\xi \in \mathbb{U}),$$

for some $\vartheta > 1$, *then* $f \in S^*(\frac{\vartheta+1}{2\vartheta})$.

For simplicity we use notation

$$F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(\xi) = F_{\nu_i,\alpha_i}(\xi)$$

throughout the article.

3. Main Results

Theorem 3.1. *Let n be a natural number such that $\nu_i > (\frac{-9+\sqrt{33}}{8}); i = 1, 2, 3, \cdots n$. Consider the function $Q_{\nu_i} : \mathbb{U}\to\mathbb{C}$ defined by (4). Let $\nu = min\{\nu_1, \nu_2, ..., \nu_n\}$ and suppose that the inequality*

$$\frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{i=1}^{n}\alpha_i < \frac{3-\vartheta}{2(\vartheta-1)} \tag{7}$$

is satisfied. Then the function $F_{\nu_i,\alpha_i}(\xi)$ defined by (6) is in the class S^* for some $2 \leq \vartheta < 3$.

Proof. Since for all $i\in\{1, 2, \ldots, n\}$,

$$Q_{\nu_i}(0) = Q'_{\nu_i}(0) - 1 = 0,$$

this shows that $Q_{\nu_i}\in\mathcal{A}$. From (6) we have

$$F'_{\nu_i,\alpha_i}(\xi) = \prod_{i=1}^{n}\left(\frac{Q_{\nu_i}(\xi)}{\xi}\right)^{\alpha_i}.$$

Taking logarithmic differentiation we have

$$\frac{F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)} = \sum_{i=1}^{n}\alpha_i\left(\frac{Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)} - \frac{1}{\xi}\right)$$

$$1 + \frac{\xi F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)} = 1 + \sum_{i=1}^{n}\alpha_i\left(\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)} - 1\right)$$

$$\begin{aligned}\Re\left(1 + \frac{\xi F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)}\right) &= 1 + \sum_{i=1}^{n}\alpha_i\Re\left(\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)} - 1\right)\\ &\leq 1 + \sum_{i=1}^{n}\alpha_i\left|\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)} - 1\right|\\ &\leq 1 + \sum_{i=1}^{n}\alpha_i\left(\frac{4\nu_i+9}{2(4\nu_i{}^2+9\nu_i+3)}\right).\end{aligned} \tag{8}$$

For all $\xi\in\mathbb{U}$ and $\nu_1, \nu_2, ..., \nu_n > \frac{-9+\sqrt{33}}{8}$. Since the function $\phi : \left(\frac{-9+\sqrt{33}}{8}, \infty\right) \to \mathbb{R}$, defined by,

$$\phi(x) = \frac{4x+9}{8x^2+18x+6}$$

is decreasing and consequently for all $i\in\{1, 2, ..., n\}$. We have

$$\frac{4\nu_i+9}{8\nu_i{}^2+18\nu_i+6} \leq \frac{4\nu+9}{8\nu^2+18\nu+6}.$$

Using this result, inequality (8) can be written as

$$\Re\left\{1 + \frac{\xi F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)}\right\} \leq 1 + \frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{i=1}^{n}\alpha_i.$$

Since,

$$\begin{aligned}1+\frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{i=1}^{n}\alpha_i &< \frac{\vartheta+1}{2(\vartheta-1)}\\ \frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{i=1}^{n}\alpha_i &< \frac{\vartheta+1}{2(\vartheta-1)}-1\\ &= \frac{3-\vartheta}{2(\vartheta-1)}.\end{aligned}$$

Hence from Lemma 2.2 $F_{\nu_i,\alpha_i}(\xi)\in S^*$ for some $2\leq\vartheta<3$.

Thus, the proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let n be a natural number and $\nu_i>(\frac{-9+\sqrt{33}}{8}); i=1,2,3,\cdots n$. Consider the function $Q_{\nu_i}:\mathbb{U}\to\mathbb{C}$ defined by (4). Let $\nu=max\{\nu_1,\nu_2,.....,\nu_n\}$ be a positive real numbers and suppose that the inequality*

$$\frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{i=1}^{n}\alpha_i>\frac{2\vartheta^2-\vartheta+1}{2\vartheta(\vartheta-1)}, \tag{9}$$

is satisfied then $F_{\nu_i,\alpha_i}(\xi)$ defined by (6) is in the class $S^*(\frac{\vartheta+1}{2\vartheta})$ for some $\vartheta\leq-1$.

Proof. We observe that for all $i\in\{1,2,3,\ldots,n\}$, we have $Q_{\nu_i}\in\mathcal{A}$ i.e.

$$Q_{\nu_i}(0)=Q'_{\nu_i}(0)-1=0.$$

From (6) we have

$$F'_{\nu_i,\alpha_i}(\xi)=\prod_{i=1}^{n}\left(\frac{Q_{\nu_i}(\xi)}{\xi}\right)^{\alpha_i}.$$

Taking logarithmic differentiation, we have

$$\Re\left(1+\frac{\xi F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)}\right)=\sum_{i=1}^{n}\alpha_i\Re\left(\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)}-1\right)+1.$$

From Lemma 2.1, we have

$$\left|\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)}-1\right|\leq\frac{4\nu_i+9}{8{\nu_i}^2+18\nu_i+6}.$$

Since $\Re(\xi) \leq |\xi|$,

$$\begin{aligned}
\Re\left(1-\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)}\right) &\leq \frac{4\nu_i+9}{8{\nu_i}^2+18\nu_i+6}\\
\Re\left(\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)}\right) &\geq 1-\frac{4\nu_i+9}{8{\nu_i}^2+18\nu_i+6}\\
\Re\left(1+\frac{\xi F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)}\right) &\geq \sum_{i=1}^{n}\alpha_i\left(1-\frac{4\nu_i+9}{8{\nu_i}^2+18\nu_i+6}\right)+1-\sum_{i=1}^{n}\alpha_i\\
&= -\left(\frac{4\nu+9}{8\nu^2+18\nu+6}\right)\sum_{i=1}^{n}\alpha_i+1\\
&> -\frac{\vartheta+1}{2\vartheta(\vartheta-1)},\ \text{from (9)}.
\end{aligned}$$

Hence from Lemma 2.3 , $F_{\nu_i,\alpha_i}(\xi) \in S^*(\frac{\vartheta+1}{2\vartheta})$ for some $\vartheta \leq -1$.
Thus, the proof of Theorem 3.2 is complete.

□

Theorem 3.3. *Let n be a natural number such that $\nu_i > (\frac{-9+\sqrt{33}}{8}); i = 1, 2, 3, \cdots n$. Consider the function $Q_{\nu_i} : \mathbb{U} \to \mathbb{C}$ defined by (4). Let $\nu = min\{\nu_1, \nu_2, ..., \nu_n\}$ and suppose that the inequality*

$$\frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{i=1}^{n}\alpha_i < \frac{3(\vartheta-1)}{2(\vartheta+1)} \tag{10}$$

is satisfied. Then the function $F_{\nu_i,\alpha_i}(\xi)$ defined by (6) is in the class S^* for some $1 < \vartheta \leq 2$.

Proof. The proof is much akin to proof of Theorem 3.1, hence we omit the details involved.

□

Theorem 3.4. *Let n be a natural number and $\nu_i > (\frac{-9+\sqrt{33}}{8}); i = 1, 2, 3, \cdots n$. Consider the function $Q_{\nu_i} : \mathbb{U} \to \mathbb{C}$ defined by (4). Let $\nu = max\{\nu_1, \nu_2, ..., \nu_n\}$ be positive real numbers and suppose that the inequality*

$$\frac{4\nu+9}{8\nu^2+18\nu+6}\sum_{i=1}^{n}\alpha_i > \frac{\vartheta+1-2\vartheta^2}{2\vartheta(\vartheta+1)} \tag{11}$$

is satisfied then $F_{\nu_i,\alpha_i}(\xi)$ defined by (6) is in the class $S^*(\frac{\vartheta+1}{2\vartheta})$ for some $\vartheta > 1$.

Proof. From (6) we have

$$F'_{\nu_i,\alpha_i}(\xi) = \prod_{i=1}^{n}\left(\frac{Q_{\nu_i}(\xi)}{\xi}\right)^{\alpha_i}.$$

Taking logarithmic differentiation, we have

$$\Re\left(1+\frac{\xi F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)}\right) = \sum_{i=1}^{n}\alpha_i\Re\left(\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)}-1\right)+1.$$

From Lemma 2.1, we have

$$\left|\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)} - 1\right| \leq \frac{4\nu_i + 9}{8{\nu_i}^2 + 18\nu_i + 6}.$$

Since $\Re(\xi) \leq |\xi|$,

$$\begin{aligned}
\Re\left(1 - \frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)}\right) &\leq \frac{4\nu_i + 9}{8{\nu_i}^2 + 18\nu_i + 6} \\
\Re\left(\frac{\xi Q'_{\nu_i}(\xi)}{Q_{\nu_i}(\xi)}\right) &\geq 1 - \frac{4\nu_i + 9}{8{\nu_i}^2 + 18\nu_i + 6} \\
\Re\left(1 + \frac{\xi F''_{\nu_i,\alpha_i}(\xi)}{F'_{\nu_i,\alpha_i}(\xi)}\right) &\geq \sum_{i=1}^{n} \alpha_i \left(1 - \frac{4\nu_i + 9}{8{\nu_i}^2 + 18\nu_i + 6}\right) + 1 - \sum_{i=1}^{n} \alpha_i \\
&= -\left(\frac{4\nu + 9}{8\nu^2 + 18\nu + 6}\right) \sum_{i=1}^{n} \alpha_i + 1 \\
&> \frac{3\vartheta + 1}{2\vartheta(\vartheta + 1)}, \text{ from (11).}
\end{aligned}$$

Hence from Lemma 2.3 , $F_{\nu_i,\alpha_i}(\xi) \in S^*(\frac{\vartheta+1}{2\vartheta})$ for some $\vartheta > 1$.
Thus, the proof of Theorem 3.4 is complete.

□

Example 3.1. *If* $0 \leq \frac{11}{17}\alpha < \frac{3-\vartheta}{2(\vartheta-1)}$ *then*

$$\int_0^{\xi} \left[\frac{1}{2}\left(\frac{sin\sqrt{t}}{\sqrt{t}} + cos\sqrt{t}\right)\right]^{\alpha} dt \in S^*$$

for some $2 \leq \vartheta < 3$.

Example 3.2. *If* $0 \leq \frac{5}{17}\alpha < \frac{3-\vartheta}{2(\vartheta-1)}$ *then*

$$\int_0^{\xi} \left[\frac{3}{2t\sqrt{t}}\left((t-1)\, sin\sqrt{t} + \sqrt{t}cos(\sqrt{t})\right)\right]^{\alpha} dt \in S^*$$

for some $2 \leq \vartheta < 3$.

Example 3.3. *If* $0 \leq \frac{11}{17}\alpha < \frac{2\vartheta^2-\vartheta+1}{2\vartheta(\vartheta-1)}$ *then*

$$\int_0^{\xi} \left[\frac{1}{2}\left(\frac{sin\sqrt{t}}{\sqrt{t}} + cos\sqrt{t}\right)\right]^{\alpha} dt \in S^*\left(\frac{\vartheta+1}{2\vartheta}\right)$$

for some $\vartheta \leq -1$.

Example 3.4. *If* $0 \leq \frac{5}{17}\alpha < \frac{2\vartheta^2-\vartheta+1}{2\vartheta(\vartheta-1)}$ *then*

$$\int_0^{\xi} \left[\frac{3}{2t\sqrt{t}}\left((t-1)\, sin\sqrt{t} + \sqrt{t}cos(\sqrt{t})\right)\right]^{\alpha} dt \in S^*\left(\frac{\vartheta+1}{2\vartheta}\right)$$

for some $\vartheta \leq -1$.

Example 3.5. *If* $0 \leq \frac{11}{17}\alpha < \frac{3(\vartheta-1)}{2(\vartheta+1)}$ *then*

$$\int_0^\xi \left[\frac{1}{2}\left(\frac{sin\sqrt{t}}{\sqrt{t}} + cos\sqrt{t}\right)\right]^\alpha dt \in S^*$$

for some $1 < \vartheta \leq 2$.

Example 3.6. *If* $0 \leq \frac{5}{17}\alpha < \frac{3(\vartheta-1)}{2(\vartheta+1)}$ *then*

$$\int_0^\xi \left[\frac{3}{2t\sqrt{t}}\left((t-1)\, sin\sqrt{t} + \sqrt{t}cos(\sqrt{t})\right)\right]^\alpha dt \in S^*$$

for some $1 < \vartheta \leq 2$.

Example 3.7. *If* $0 \leq \frac{11}{17}\alpha < \frac{\vartheta+1-2\vartheta^2}{2\vartheta(\vartheta+1)}$ *then*

$$\int_0^\xi \left[\frac{1}{2}\left(\frac{sin\sqrt{t}}{\sqrt{t}} + cos\sqrt{t}\right)\right]^\alpha dt \in S^*\left(\frac{\vartheta+1}{2\vartheta}\right)$$

for some $\vartheta > 1$.

Example 3.8. *If* $0 \leq \frac{5}{17}\alpha < \frac{\vartheta+1-2\vartheta^2}{2\vartheta(\vartheta+1)}$ *then*

$$\int_0^\xi \left[\frac{3}{2t\sqrt{t}}\left((t-1)\, sin\sqrt{t} + \sqrt{t}cos(\sqrt{t})\right)\right]^\alpha dt \in S^*\left(\frac{\vartheta+1}{2\vartheta}\right)$$

for some $\vartheta > 1$.

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