Electronic Journal of Mathematical Analysis and Applications
Vol. 10(2) July 2022, pp. 275-282.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

SOME INEQUALITIES FOR THE RATIONAL FUNCTIONS WITH PRESCRIBED POLES AND RESTRICTED ZEROS

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Abstract. Let $r(z)$ be a rational function with at most n poles $a_1, a_2, ..., a_n$ where $|a_j| > 1, \quad 1 \leq j \leq n$. This paper investigates the modulus of a derivative of a rational function $r(z)$ on the unit circle where $r(z) = (z - z_0)^\nu u(z)$. we establish an upper bound when $r(z)$ has ν zeros at z_0 where $|z_0| < 1$ and remaining zeros are outside the unit disc and a lower bound when $r(z)$ has ν zeros outside the disc $\{|z| \leq k, \quad k \leq 1\}$ and remaing zeros inside the disk $\{|z| \leq k, \quad k \leq 1\}$.

1. Introduction

Let $\mathcal{P}_n$ be the class of polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. Let D_{k-} denotes the region inside the circle $T_k = \{z; |z| = k > 0\}$ and D_{k+} the region outside T_k. For $a_j \in \mathbb{C}$ with $j = 1, 2, ..., n$, we write
$W(z) = \prod_{j=1}^{n} (z - a_j) \qquad ; \qquad B(z) = \prod_{j=1}^{n} \left(\frac{1 - \overline{a_j} z_j}{z - a_j} \right)$
and
$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, ..., a_n) = \left\{ \frac{P(z)}{W(z)} : P \in \mathcal{P}_n \right\},$
then $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, ..., a_n$ at most and with finite limit at infinity.We observe that $B(z) \in \mathcal{R}_n$. For f defined on T_k in the complex plane , we set

$$\max_{z \in T_k} |f(z)| = \sup_{z \in T_k} |f(z)|.$$

Throughout this paper, we also assume that all poles $a_1, a_2, ..., a_n$ are in D_{1+}.

The following famous result is due to Bernstein[7]

Theorem 1.1 If $P \in \mathcal{P}_n$ then $\max_{z \in T_1} |P'(z)| \leq n \max_{z \in T_1} |P(z)|$.

The following result was conjectured by Erdös and later proved by Lax [10]

Theorem 1.2 If $P \in \mathcal{P}_n$ and all the zeros of $P(z)$ lie in $T_1 \cup D_{1+}$ then for $z \in T_1$

2010 *Mathematics Subject Classification.* 26A84, 26D07.

Key words and phrases. Rational Functions, inequalities,polar derivative, zeros, Poles.
Submitted Feb. 6, 2022. Revised March 9, 2022.

we have

$$\max_{z\in T_1}|P'(z)| \leq \frac{n}{2}\max_{z\in T_1}|P(z)|. \tag{1}$$

Equality in (1) holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta|$.

Li, Mohapatra and Rodriguez [13] have proved Bernstein-type inequalities similar to Theorem 1.1 and Theorem 1.2 for rational functions with prescribed poles where they replaced z^n by Blaschkes product $B(z)$.Among other things they proved the following generalisation of Theorem 1.2:

Theorem 1.3 Suppose $r \in \mathcal{R}_n$ and all zeros of r lie in $T_1 \cup D_{1+}$, then for $z \in T_1$, we have

$$|r'(z)| \leq \frac{1}{2}|B'(z)|\max_{z\in T_1}|r(z)|. \tag{2}$$

Equality in (2) holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.

Theorem 1.4 Suppose $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, ..., a_n$ and all the zeros of r lie in $T_1 \cup D_{1-}$, then for $z \in T_1$,

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - (n-m)\right\}|r(z)| \tag{3}$$

where m is number of zeros of r.

Aziz and Shah [5] considered a class of rational functions $\mathcal{R}_n$ not vanishing in $T_k \cup D_{k+}$ where $k \leq 1$ and proved the following generalisation of Theorem 1.4.

Theorem 1.5 Suppose $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, ..., a_n$ and all zeros of r lie in $T_k \cup D_{k-}$ where $k \leq 1$, then for $z \in T_1$, we have

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{2m - n(1+k)}{(k+1)}\right\}|r(z)| \tag{4}$$

where m is number of zeros of $r(z)$. The result is best possible and equality holds for $r(z) = \frac{(z+k)^m}{(z-a)^n}$ where $a > 1, k \leq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

Let $D_\alpha P(z)$ be an operator that carries n^{th} degree polynomial $P(z)$ to the polynomial

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z), \quad \alpha \in \mathbb{C}$$

of degree at most $(n-1)$. $D_\alpha P(z)$ generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z)$$

Aziz was among the first to extend these results to polar derivatives. Aziz [2] proved inequality (1) due to Lax [10] in terms of polar derivatives by showing that for $P \in \mathcal{P}_n$ having no zeros in D_{1-} and $|\alpha| \geq 1$,

$$|D_\alpha P(z)| \leq \frac{n}{2}(|\alpha z^{n-1}| + 1)\max_{z\in T_1}|P(z)| \quad \text{for} \quad z \in T \cup D_{1+}. \tag{5}$$

Xin Li [15] pointed out that inequalities involving polynomials and their polar derivatives are a special case of the inequalities for the rational functions by considering $a_i = \alpha$ for each $i = 1, 2, ..., n$, that is for $|a_i| = |\alpha| > 1$,

$$r'(z) = \left(\frac{P(z)}{(z-\alpha)^n}\right)' = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}} \tag{6}$$

2. Preliminaries

For the proof of the main Theorems we need the following Lemmas. The first Lemma which we need is due to Li, Mohapatra and Rodriguez [15].

Lemma 2.1 If $r \in \mathcal{R}_n$ and $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$ then for $z \in T_1$, we have

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)|||r||. \tag{7}$$

Equality in (7) holds in $r(z) = uB(z)$ with $u \in T_1$.

Lemma 2.2 If $z \in T_1$, then

$$Re\left(\frac{zW'(z)}{W(z)}\right) = \frac{n - |B'(z)|}{2}.$$

Lemma 2.2 is due to Aziz and Zargar [16].

Next Lemma is due to N. Arunrat and K. M. Nakprasit [1].

Lemma 2.3 Let $r \in \mathcal{R}_n$, where r has exactly n poles at $a_1, a_2, ..., a_n$ and all its zeros lie in $T_k \cup D_{k-}$ where $k \leq 1$, then for $z \in T_1$,

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{2t - n(1+k)}{1+k}\right](|r(z)| + m) \tag{8}$$

where t is the number of zeros of r with counting multiplicity and $m = \min_{z \in T_k} |r(z)|$.

3. Main Results

In this paper, we propose to relax the condition that all the zeros of the rational function $r(z)$ lie in $|z| \leq k, \quad k \leq 1$. In this direction we prove the following result which gives generalisation and refinement of (4).

Theorem 3.1 If $r \in \mathcal{R}_n$ has a zero of order ν at z_0 with $|z_0| > k, \quad k \leq 1$ and the remaining $t - \nu$ zeros in $T_k \cup D_{k-}$, then

$$\begin{aligned}\max_{z \in T_1} |r'(z)| \geq \frac{1}{2}\left\{\left(\frac{|1 - |z_0||}{1 + |z_0|}\right)^{\nu}\left[|B'(z)| + \frac{2(t-\nu) - n(1+k)}{1+k}\right] - \frac{2\nu}{(1 + |z_0|)}\right\}\max_{z \in T_1}|r(z)| \\ + \frac{1}{2}\left(\frac{|1 - |z_0||}{k + |z_0|}\right)^{\nu}\left[|B'(z)| + \frac{2(t-\nu) - n(1+k)}{1+k}\right]\min_{z \in T_k}|r(z)|.\end{aligned} \tag{9}$$

Proof. Let $r(z) = (z - z_0)^{\nu} u(z) \in \mathcal{R}_n$ where $u(z) \in \mathcal{R}_n$ having all its $t - \nu$ zeros in $T_k \cup D_{k+}$ where $k \leq 1$. Then

$$r'(z) = (z - z_0)^{\nu} u'(z) + \nu(z - z_0)^{\nu - 1} u(z)$$

Or

$$\begin{aligned}|r'(z)| &= |(z - z_0)^{\nu} u'(z) + \nu(z - z_0)^{\nu - 1} u(z)| \\ &\geq |(z - z_0)^{\nu} u'(z)| - \nu|(z - z_0)^{\nu - 1} u(z)|.\end{aligned}$$

Which implies

$$\max_{z \in T_1} |r'(z)| \geq \max_{z \in T_1} |(z - z_0)^{\nu} u'(z)| - \nu \max_{z \in T_1} |(z - z_0)^{\nu - 1} u(z)|. \tag{10}$$

Using the fact that for $z \in T_1$,

$$|1 - |z_0|| \leq |z - z_0| \leq 1 + |z_0|$$

we obtain from (10)

$$\max_{z\in T_1}|r'(z)| \geq |1-|z_0||^{\nu}\max_{z\in T_1}|u'(z)| - \nu(1+|z_0|)^{\nu-1}\max_{z\in T_1}|u(z)|. \tag{11}$$

By Lemma 2.3, we have for $z \in T_1$

$$\max_{z\in T_1}|u'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{2(t-\nu)-n(1+k)}{1+k}\right](\max_{z\in T_1}|u(z)| + m') \tag{12}$$

where $m' = \min_{z\in T_k}|u(z)|$.

Using (12) in (11) we obtain for $z \in T_1$

$$\begin{aligned}\max_{z\in T_1}|r'(z)| &\geq \frac{|1-|z_0||^{\nu}}{2}\left[|B'(z)| + \frac{2(t-\nu)-n(1+k)}{1+k}\right](\max_{z\in T_1}|u(z)| + m') \\ &\quad - \nu(1+|z_0|)^{\nu-1}\max_{z\in T_1}|u(z)| \\ &= \left\{\frac{|1-|z_0||^{\nu}}{2}\left[|B'(z)| + \frac{2(t-\nu)-n(1+k)}{1+k}\right] - \nu(1+|z_0|)^{\nu-1}\right\}\max_{z\in T_1}|u(z)| \\ &\quad + \frac{|1-|z_0||^{\nu}}{2}\left[|B'(z)| + \frac{2(t-\nu)-n(1+k)}{1+k}\right]m'.\end{aligned} \tag{13}$$

The relation between $u(z)$ and $r(z)$ implies that

$$\begin{aligned}\max_{z\in T_1}|u(z)| &= \max_{z\in T_1}\left[\frac{1}{|z-z_0|^{\nu}}|r(z)|\right] \\ &\geq \frac{1}{(1+|z_0|)^{\nu}}\max_{z\in T_1}|r(z)|\end{aligned} \tag{14}$$

and

$$\begin{aligned}\min_{z\in T_k}|u(z)| &= \min_{z\in T_k}\left[\frac{1}{|z-z_0|^{\nu}}|r(z)|\right] \\ &\geq \frac{1}{(k+|z_0|)^{\nu}}\max_{z\in T_k}|r(z)|\end{aligned} \tag{15}$$

Using (14) and (15) in (13) we get inequality (9).

If we take $t = n$ in (9), then we have the following result:

Corollary 3.1 If $r \in \mathcal{R}_n$ has a zero of order ν at z_0 with $|z_0| > k$, $\quad k \leq 1$ and the remaining $n-\nu$ zeros in $T_k \cup D_{k-}$, then for $z \in T_1$

$$\begin{aligned}\max_{z\in T_1}|r'(z)| \geq \frac{1}{2}&\left\{\left(\frac{|1-|z_0||}{1+|z_0|}\right)^{\nu}\left[|B'(z)| + \frac{n(1-k)-2\nu}{1+k}\right] - \frac{2\nu}{(1+|z_0|)}\right\}\max_{z\in T_1}|r(z)| \\ &+ \frac{1}{2}\left(\frac{|1-|z_0||}{k+|z_0|}\right)^{\nu}\left[|B'(z)| + \frac{n(1-k)-2\nu}{1+k}\right]\min_{z\in T_k}|r(z)|.\end{aligned} \tag{16}$$

In particular if we consider $r(z) = \frac{P(z)}{(z-\alpha)^n}$ and noting that

$$r'(z) = \left(\frac{P(z)}{(z-\alpha)^n}\right)' = \frac{-D_{\alpha}P(z)}{(z-\alpha)^{n+1}}$$

and

$$B'(z) = n\frac{(\alpha^2-1)}{(z-\alpha)^2}\left(\frac{1-\alpha z}{z-\alpha}\right)^{n-1}.$$

Hence for $z \in T_1$

$$|B'(z)| = n\frac{(|\alpha|^2-1)}{|z-\alpha|^2}$$

we obtain the following result in terms of polar derivative.

Corollary 3.2 If $P \in \mathcal{P}_n$ has a zero of order ν at z_0 with $|z_0| > 1$ and the remaining $n-\nu$ zeros in $T_1 \cup D_{1-}$, then for any complex number α with $|\alpha| > 1$ and for $z \in T_1$

$$\begin{aligned}\max_{z\in T_1}|D_\alpha P(z)| \geq \frac{1}{2}\left\{\left(\frac{|1-|z_0||}{1+|z_0|}\right)^\nu (n-\nu)(|\alpha|-1) - \frac{2\nu}{(1+|z_0|)}(|\alpha|+1)\right\}\max_{z\in T_1}|P(z)| \\ + \frac{1}{2}\left(\frac{|1-|z_0||}{1+|z_0|}\right)^\nu (n-\nu)(|\alpha|-1)\left(\frac{|\alpha|-1}{|\alpha|+1}\right)^n \min_{z\in T_1}|P(z)|.\end{aligned} \tag{17}$$

Dividing both of (17) by α and letting $|\alpha| \to \infty$, we get the following result:

Corollary 3.3 If $P \in \mathcal{P}_n$ has a zero of order ν at z_0 with $|z_0| > 1$ and the remaining $n-\nu$ zeros in $T_1 \cup D_{1-}$, then for $z \in T_1$

$$\begin{aligned}\max_{z\in T_1}|P'(z)| \geq \frac{(n-\nu)}{2}\left\{\left(\frac{|1-|z_0||}{1+|z_0|}\right)^\nu - \frac{2\nu}{(1+|z_0|)}\right\}\max_{z\in T_1}|P(z)| \\ + \frac{(n-\nu)}{2}\left(\frac{|1-|z_0||}{1+|z_0|}\right)^\nu \min_{z\in T_1}|P(z)|.\end{aligned} \tag{18}$$

Theorem 3.2 If $r \in \mathcal{R}_n$ has a zero of order ν at z_0 with $|z_0| < 1$ and the remaining $n-\nu$ zeros in $T_1 \cup D_{1+}$, then for $z \in T_1$

$$\max_{z\in T_1}|r'(z)| \leq \frac{1}{2}\left(\frac{1+|z_0|}{1-|z_0|}\right)^\nu\left(|B'(z)| + \frac{\nu(1-|z_0|)}{1+|z_0|}\right)\max_{z\in T_1}|r(z)|. \tag{19}$$

Proof. Let $r(z) = (z-z_0)^\nu u(z) \in \mathcal{R}_n$ where $u(z) \in \mathcal{R}_n$ having all its $n-\nu$ zeros in $T_1 \cup D_{1+}$. Then

$$r'(z) = (z-z_0)^\nu u'(z) + \nu(z-z_0)^{\nu-1}u(z).$$

Or

$$\begin{aligned}|r'(z)| &= |(z-z_0)^\nu u'(z) + \nu(z-z_0)^{\nu-1}u(z)| \\ &\leq |(z-z_0)^\nu u'(z)| + \nu|(z-z_0)^{\nu-1}u(z)|.\end{aligned}$$

Which implies

$$\max_{z\in T_1}|r'(z)| \leq \max_{z\in T_1}|(z-z_0)^\nu u'(z)| + \nu\max_{z\in T_1}|(z-z_0)^{\nu-1}u(z)|. \tag{20}$$

Using the fact that for $z \in T_1$ and $|z_0| < 1$,

$$1-|z_0| \leq |z-z_0| \leq 1+|z_0|$$

we obtain from (20)

$$\max_{z\in T_1}|r'(z)| \leq (1+|z_0|)^\nu \max_{z\in T_1}|u'(z)| + \nu(1+|z_0|)^{\nu-1}\max_{z\in T_1}|u(z)|. \tag{21}$$

Let $u(z)=\frac{h(z)}{W(z)}\in\mathcal{R}_n$ where $h(z)=\sum\limits_{j=0}^{n-\nu}a_jz^j$. If $b_1,b_2,...,b_{n-\nu}$ are the zeros of $h(z)$, then $|b_j|\geq 1, j=1,2,...,n-\nu$ and we have

$$\begin{aligned}\frac{zu'(z)}{u(z)}&=\frac{zh'(z)}{h(z)}-\frac{zW'(z)}{W(z)}\\&=\sum_{j=1}^{n-\nu}\frac{z}{z-b_j}-\frac{zW'(z)}{W(z)}\end{aligned}$$

For $z\in T_1$, this gives with the help of Lemma 2.2, that

$$\begin{aligned}Re\frac{zu'(z)}{u(z)}&=Re\sum_{j=1}^{n-\nu}\frac{z}{z-b_j}-Re\frac{zW'(z)}{W(z)}\\&=Re\sum_{j=1}^{n-\nu}\frac{z}{z-b_j}-\left(\frac{n-|B'(z)|}{2}\right)\end{aligned}\tag{22}$$

Now $Re\left(\frac{z}{z-b_j}\right)\leq\frac{1}{2}$ for $|b_j|\geq 1,\quad j=1,2,...,n-\nu$. Using this in (22), we get for $z\in T_1$,

$$\begin{aligned}Re\frac{zu'(z)}{u(z)}&\leq\frac{n-\nu}{2}-\left(\frac{n-|B'(z)|}{2}\right)\\&=\frac{|B'(z)|-\nu}{2}.\end{aligned}$$

Hence for $z\in T_1$ we have [[15], p.529],

$$\begin{aligned}\left|\frac{z(u^*(z))'}{u(z)}\right|^2&=\left||B'(z)|-\frac{zu'(z)}{u(z)}\right|^2\\&=|B'(z)|^2+\left|\frac{zu'(z)}{u(z)}\right|^2-2|B'(z)|Re\frac{zu'(z)}{u(z)}\\&\geq|B'(z)|^2+\left|\frac{zu'(z)}{u(z)}\right|^2-2|B'(z)|\left(\frac{|B'(z)|-\nu}{2}\right)\\&=\left|\frac{zu'(z)}{u(z)}\right|^2+\nu|B'(z)|.\end{aligned}$$

This implies for $z\in T_1$,

$$\left\{|u'(z)|^2+\nu|u(z)|^2|B'(z)|\right\}^{\frac{1}{2}}\leq|(u^*(z))'|\tag{23}$$

Combining (23) with Lemma 2.1, we get

$$|u'(z)|+\left\{|u'(z)|^2+\nu|u(z)|^2|B'(z)|\right\}^{\frac{1}{2}}\leq|B'(z)|\max_{z\in T_1}|u(z)|.$$

or equivalently

$$\begin{aligned}|u'(z)|^2+\nu|u(z)|^2|B'(z)|&\leq\left\{|B'(z)|\max_{z\in T_1}|u(z)|-|u'(z)|\right\}^2\\&=|B'(z)|^2\left(\max_{z\in T_1}|u(z)|\right)^2-2|B'(z)||u'(z)|\max_{z\in T_1}|u(z)|+|u'(z)|^2\end{aligned}$$

which after a simplification yields for $z \in T_1$ that

$$|u'(z)| \leq \frac{|B'(z)|}{2} \max_{z\in T_1} |u(z)| - \frac{\nu}{2} \frac{|u(z)|^2}{\max_{z\in T_1} |u(z)|}$$

Or

$$\max_{z\in T_1} |u'(z)| \leq \left(\frac{|B'(z)| - \nu}{2}\right) \max_{z\in T_1} |u(z)|. \tag{24}$$

using (24) in (21) we obtain for $z \in T_1$

$$\begin{aligned}
\max_{z\in T_1} |r'(z)| &\leq (1 + |z_0|)^\nu \left(\frac{|B'(z)| - \nu}{2}\right) \max_{z\in T_1} |u(z)| + \nu(1 + |z_0|)^{\nu-1} \max_{z\in T_1} |u(z)| \\
&= \frac{(1 + |z_0|)^\nu}{2} \left(|B'(z)| - \nu + \frac{2\nu}{1 + |z_0|}\right) \max_{z\in T_1} |u(z)| \\
&= \frac{(1 + |z_0|)^\nu}{2} \left(|B'(z)| + \frac{\nu(1 - |z_0|)}{1 + |z_0|}\right) \max_{z\in T_1} |u(z)|.
\end{aligned} \tag{25}$$

Further,

$$\begin{aligned}
\max_{z\in T_1} |u(z)| &= \max_{z\in T_1} \left[\frac{1}{|z - z_0|^\nu} |r(z)|\right] \\
&\leq \frac{1}{(1 - |z_0|)^\nu} \max_{z\in T_1} |r(z)|.
\end{aligned} \tag{26}$$

(25) together with (26) gives the desired result.

If we take $z_0 = 0$, we get the following result:

Corollary 3.4 If $r \in \mathcal{R}_n$ has ν-fold zeros at origin and the remaining $n - \nu$ zeros in $T_1 \cup D_{1+}$, then for $z \in T_1$

$$\max_{z\in T_1} |r'(z)| \leq \frac{1}{2} (|B'(z)| + \nu) \max_{z\in T_1} |r(z)|. \tag{27}$$

The result is sharp and equality holds for $r(z) = \frac{z^\nu (z+1)^{n-\nu}}{(z+a)^n}$ where $a > 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

By considering $r(z) = \frac{P(z)}{(z-\alpha)^n}$, we get the following result:

Corollary 3.5 If $P \in \mathcal{P}_n$ has a zero of order ν at z_0 with $|z_0| < 1$ and the remaining $n-\nu$ zeros in $T_1 \cup D_{1+}$, then for any complex number α with $|\alpha| > 1$ and for $z \in T_1$

$$\max_{z\in T_1} |D_\alpha P(z)| \leq \frac{(|\alpha| - 1)}{2} \left(\frac{1 + |z_0|}{1 - |z_0|}\right)^\nu \left(n + \frac{\nu(1 - |z_0|)}{1 + |z_0|}\right) \max_{z\in T_1} |P(z)|. \tag{28}$$

Dividing both sides of (28) by α and letting $|\alpha| \to \infty$, we get the following result:

Corollary 3.6 If $P \in \mathcal{P}_n$ has a zero of order ν at z_0 with $|z_0| < 1$ and the remaining $n - \nu$ zeros in $T_1 \cup D_{1+}$, then for $z \in T_1$

$$\max_{z\in T_1} |P'(z)| \leq \frac{1}{2} \left(\frac{1 + |z_0|}{1 - |z_0|}\right)^\nu \left(n + \frac{\nu(1 - |z_0|)}{1 + |z_0|}\right) \max_{z\in T_1} |P(z)|. \tag{29}$$

4. Acknowledgement

The authors are extremely grateful to the referee for his valuable suggestion.

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