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ALMOST PERIODIC SOLUTION TO A NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENTS

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Abstract. In this article, we prove sufficient conditions for the existence of almost periodic solutions to a non-instantaneous impulsive differential equation with deviating argument. The results are established with the help of fixed point theorem. We also show that the solution is asymptotically stable. We conclude the article with an example to illustrate the main results.

1. Introduction

The differential equations with instantaneous impulses models phenomena with unanticipated changes in their state, such as natural calamities and changes experienced by the human body due to medical treatment. They have wide applications in dynamical systems, electrical engineering, biological and medical sciences, etc. In the case of instantaneous impulsive differential equations, the duration of sudden changes is negligible as compared to the occurrence of the whole event. So, they fail to provide enough data for the study of the dynamics of the event. In this scenario, we need to consider non-instantaneous impulses along with the governing differential equations where the abrupt changes begin impulsively and persist for a finite interval of time. Non-instantaneous impulsive differential equations are broadly studied and have a wide range of applications in life science, medical science, and many other fields. We refer to [16] for more details. These types of equations are suitable for describing impulsive action that starts at an arbitrary fixed point and remains active for a finite interval of time. Impulsive actions are described with the help of impulsive points and in between impulsive points, there are junction points that connect the impulsive points.

Non-instantaneous impulsive differential equations were introduced by Hernández and O'Regan [16] and after that, several authors [6, 10, 17, 19, 20] have studied this special class of equations over the last several years. Hernández et al. [14] applied fixed point techniques and the theory of semigroup in functional power

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space to study the existence of solutions for non-instantaneous impulsive equations. Pierri [19] developed the existence theory of asymptotically periodic solutions for non-instantaneous impulsive evolution equation with Hausdorff measure of non-compactness. Hernández et al.[15] proved several results related to the existence of solutions for a non local abstract Cauchy problem with non-instantaneous impulses. Liu et al. [17] extended the results to the existence and stability of solutions for a new class of generalized non-instantaneous impulsive evolution equation, which depends upon the state described by previous evolution equations. Abbas and Benchohra [2] studied partial fractional non-instantaneous impulsive differential equations. Recently, Tian et al. [20] established sufficient conditions for the existence of almost periodic solutions to a non-linear non-instantaneous impulsive differential equation. Later, Liu et al. [17] extended the study of non-instantaneous impulsive differential equations to a new (ω, c)-periodic class.

The study of different kinds of solutions to differential equations has been studied by many authors [5, 7, 8, 9, 11, 12, 14]. These kinds of differential equations are particular classes of differential equations where the unknown functions and their derivatives appear in their arguments. In this article, we prove the sufficient condition for existence of an almost periodic solution to the following non-instantaneous impulsive differential equation with deviating argument.

$$\begin{aligned} \frac{du}{dt} &= Au(t) + g(t, u(t), u(\phi(t, u(t)))), t \in (s_i, t_{i+1}] \qquad i \in \mathbb{Z} \\ u(t_i^+) &= (I + B_i)u(t_i^-) + J_i(t_i^+, u(t_i^-), u(K_i(t_i^+, u(t_i^-)))) \quad i \in \mathbb{Z} \setminus \{0\} \\ u(t) &= (I + B_i)u(t_i^-) + J_i(t_i, u(t_i^-), u(K_i(t_i, u(t_i^-)))), \quad i \in \mathbb{Z} \setminus \{0\} \\ u(s_i^+) &= u(s_i^-) \qquad\qquad i \in \mathbb{Z} \setminus \{0\} \end{aligned} \tag{1.1}$$

Here, $u(t) \in \mathbb{R}^n$ and A, B_i are real $n \times n$ matrix such that $AB_i = B_iA$, t_i acts as an impulsive point such that

$$t_0 = s_0 < t_1 < s_1 < t_2, ... < t_i < s_i < t_{i+1} .....,$$

t_i's are not bounded. $u(t_i^+)$ and $u(t_i^-)$ represent right and left hand limits of $u(t)$ at $t = t_i$ respectively. The functions g, ϕ are specified later. The existence results of almost periodic solution to non-instantaneous impulsive differential equation without any deviating arguments have been established by Tian et al. [20].

We have organized the article in the following way. In section 2, we define the almost periodicity concept and give a brief literature review on the existence of almost periodic solution to abstract Cauchy problem. In section 3, we define impulsive Cauchy matrix and some existence results are discussed with the help of impulsive Cauchy matrix. In section 4, we consider linear homogeneous, linear non-homogeneous and non-linear non-homogeneous non-instantaneous impulsive differential equations. The main results and their proofs are given in section 5. In section 6, we discuss an example to illustrate the main results.

2. Preliminaries

In this section, we will briefly discuss some basic definitions related to almost periodic function and some fundamental existence results of almost periodic solutions.

Definition 2.1. *A number P is said to be ϵ-Period for a function f if for all t*

$$|f(t) - f(t+P)| \leq \epsilon.$$

Definition 2.2. *A function is said to be an almost periodic function if for each $\epsilon > 0$ there exists a real number P such that each real interval with length P contains an ϵ-period.*

Bochner[3] introduced a new definition of almost periodic function, after which it was possible to connect almost periodic solutions with differential equations in the context of dynamics of topology. Bochner defined almost periodic function as similar to that of Bohr. According to Bochner "Let $f(x)$ be a continuous function in the interval $(-\infty, \infty)$. Then $f(x)$ is said to be Bochner almost periodic function if the family of functions $\{f(x+k) : -\infty < k < \infty\}$ is compact in the sense of uniform convergence on $(-\infty, \infty)$." That is if it is possible to select from each infinite sequence $f(x + h_k), k = 1, 2, ...,$ a subsequence which converges uniformly to $f(x)$ on $(-\infty, \infty)$. This definition acts as a starting point in the abstract generalization of almost periodicity concept.

We consider the differential equation,

$$\frac{du}{dt} = Au + f(t), \tag{2.1}$$

where A is a $n \times n$ matrix and $f : \mathbb{R} \to \mathbb{R}^n$ is an almost periodic function.

Bohr and Neugebauer [4] studied almost periodic solutions of the above differential equation. They proved that there exists an almost periodic solution of (2.1) on $\mathbb{R}$ if and only if it is bounded on $\mathbb{R}$. Existence of solutions with almost periodicity of differential equations has been studied for many years. Among all those results we would like to mention some of the results for the following differential equation.

$$\frac{du}{dt} = Au + f(t), t \in \mathbb{R}, u \in \mathbb{C}^n \tag{2.2}$$

Here, A is $n \times n$ matrix and $f(t)$ is T-periodic, then the following results hold.

Theorem 2.3. *(Theorem 1.2.1* [1]*) The equation* (*2.2*) *has a T-periodic solution if and only if it has a bounded solution.*

Theorem 2.4. *(Theorem 1.2.2* [1]*) The equation* (*2.2*) *has a periodic solution with period T which is unique for every T-periodic if and only if* $1 \notin \sigma(e^{TA})$.

Further, Zaidman [21] discussed the existence of almost periodic mild solution to the following equation.

$$\frac{du}{dt} = Au + g(t),\ t \in \mathbb{R},\ u \in AP(X), \tag{2.3}$$

where $g : \mathbb{R} \to X$ is almost periodic function and A is the infinitesimal generator of a C_0 semigroup.

Naito [18] extended Theorem 2.3 and Theorem 2.4 for the following system.

$$\frac{du(t)}{dt} = Au(t) + M(t)u_t + g(t), \qquad u \in x, \quad t \in \mathbb{R}, \tag{2.4}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, $M(t)$ is a bounded linear operator.

Consider the quasi linear differential equation,

$$\frac{du(t)}{dt} = Au(t) + B(t, u_t), \tag{2.5}$$

where A is infinitesimal generator of a strongly continuous semigroup. Herniquez and Vasquez [13] proved the existence of almost periodic solution to (2.5). These types of equations are often referred to as the abstract retarded functional differential equation.

3. Almost periodic solution to non-instantaneous impulsive differential equation

In this section, we will discuss the system of non-instantaneous impulsive differential equation with the help of impulsive cauchy matrix. Tian [20] proved the sufficient conditions for the existence of almost periodic solution for such types of equations.

Let us consider a homogeneous linear differential equation with non-instantaneous impulsive conditions as follows,

$$\begin{aligned} \frac{du}{dt} &= Au(t), t \in (s_i, t_{i+1}] && i = 0, 1, 2.. \\ u(t_i^+) &= u(t_i^-) + B_i u(t_i^-) && i = 0, 1, 2.. \\ u(t) &= u(t_i^-) + B_i u(t_i^-), \quad t \in (t_i, s_i] && i = 1, 2.. \\ u(s_i^+) &= u(s_i^-) && i = 0, 1, 2.. \end{aligned} \tag{3.1}$$

Here, $u(t) \in \mathbb{R}^n$ and A, B_i are $n \times n$ matrix such that $AB_i = B_i A$, t_i acts as an impulsive point such that

$$t_0 = s_0 < t_1 < s_1 < t_2, ... < t_i < s_i < t_{i+1}....., t_i \to \infty$$

$u(t_i^+)$ and $u(t_i^-)$ represent the right and left hand limits of $u(t)$ at $t = t_i$ respectively.

Definition 3.1. *The non-instantaneous impulsive Cauchy matrix $W(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ of the equation (3.1) is defined as*

$$W(t, s) = \exp\left(A\left[(t - s_{i(t_i,t_0)})^+ - (s - s_{i(s,t_0)})^+ + \sum_{k=i(s,t_0)}^{i(t,t_0)-1} (t_{k+1} - s_k)\right]\right) \prod_{k=i(s,t_0)}^{i(t,t_0)-1} (I + B_k), \tag{3.2}$$

where I is the identity matrix. If $i(s, t_0) = i(t, t_0)$, then

$$\sum_{k=i(s,t_0)}^{i(t,t_0)-1} (t_{k+1} - s_k) = 0.$$

Definition 3.2. *Equation (3.1) is said to be exponentially stable if there exist constants $M \geq 1$ and $\alpha > 0$ such that*

$$\|W(t, s)\| \leq Me^{-\alpha(t-s)}, \qquad t_0 \leq s < t.$$

Remark 3.3. *Without the condition $u(s_i^+) = u(s_i^-)$, $i = 1, 2, \ldots,$ equation (3.1) can be written as*

$$\begin{aligned} \frac{du}{dt} &= Au(t), \qquad t \in (s_i, t_{i+1}], \qquad i = 0, 1, 2, \ldots, \\ u(t_i^+) &= u(t_i^-) + B_i u(t_i^-), \qquad\qquad i = 1, 2, \ldots, \\ u(t) &= u(t_i^-) + B_i u(t_i^-), \quad t \in (t_i, s_i], \qquad i = 1, 2, \ldots \end{aligned} \tag{3.3}$$

Equation (3.3) reduces to instantaneous impulsive equations as $s_i \to t_i$ for $i = 1, 2, \ldots$

$$\begin{aligned} \frac{du}{dt} &= Au(t), \;\; t \neq t_i \\ u(t_i^+) &= u(t_i^-) + B_i u(t_i^-). \end{aligned}$$

At the same time, non-instantaneous impulsive Cauchy matrix $W(\cdot, \cdot)$ can be reduced to instantaneous impulsive Cauchy matrix.

$$u(t, s) = exp\left(A \left[(t - t_{i(t_i,t_0)})^+ - (s - t_{i(s,t_0)})^+ + \sum_{k=i(s,t_0)}^{i(t,t_0)-1} (t_{k+1} - t_k) \right] \right) \prod_{k=i(s,t_0)}^{i(t,t_0)-1} (I + B_k).$$

Lemma 3.4. *(Lemma 2.3 [20]) If $u(t, s, u_0)$ is a solution of the equation (3.1) with the initial condition $u(s) = u_0 \in \mathbb{R}^n$, then*

$$u(t, s, u_0) = W(t, s)u_0$$

$$t_0 \leq s \leq t,$$

where, $W(\cdot, \cdot)$ is same as defined in (2).
Now,

$$u(t, t_0, u_0) = exp\left(A \left[(t - s_{i(t_i,t_0)})^+ + \sum_{k=0}^{i(t,t_0)-1} (t_{k+1} - s_k) \right] \right) \prod_{k=0}^{i(t,t_0)-1} (I + B_k)u_0. \tag{3.4}$$

Definition 3.5. *The solution $u(t, t_0, u_0)$ of equation (3.1) is said to be locally asymptotically stable if for any $v_0 \in \mathbb{R}^n$ there exists $\delta > 0$ such that $\|u_o - v_o\| < \delta$, then*

$$\lim_{t\to\infty} \|u(t, t_0, u_0) - u(t, t_0, v_0)\| = 0.$$

Remark 3.6. *Locally asymptotically stability and global stability coincide for equation (3.1).*

Lemma 3.7. *(Lemma 2.8 [20]) Let us consider $|\cdot|$ be a norm on $\mathbb{R}$ and B be a nth order matrix. For any $\epsilon > 0$ there exists $a_k \geq 1$ such that $\|I + B_k\| \leq a_k(1 + \rho(B_k) + \epsilon)$, k is non-negative integer and $\rho(B_k)$ is the spectral radius of B_k, $k = 1, 2, \ldots$*

Lemma 3.8. *(Lemma 2.9 [20]) For any $\epsilon > 0$ there exists $K_\epsilon \geq 1$ such that*

$$\|W(t, t_0)\| \geq K_\epsilon exp\left((\alpha(A) + \epsilon) \left[(t - s_{i(t,t_0)})^+ + \sum_{k=0}^{i(t,t_0)-1} (t_{k+1} - s_k) \right] \right) \prod_{k=0}^{i(t,t_0)-1} (1 + \rho(B_k) + \epsilon).$$

Theorem 3.9. *(Theorem 2.10 [20]) Assume that the distance between the impulsive point t_i and junction point s_i satisfies*

$$0 < \lambda_1 < t_{k+1} - s_k \leq \lambda_2, \qquad k = 0, 1, 2, \ldots$$

Define,

$$\begin{aligned} &\overline{\lambda_1} = \lambda_1, \quad \alpha(A) < 0 \qquad && \lambda_1 = 0 \quad t \in (t_i, s_i), \\ &\overline{\lambda_1} = \lambda_2, \quad \alpha(A) \geq 0 \qquad && \lambda_1 = \overline{\lambda_1} \quad t \in (s_i, t_{i+1}), \end{aligned}$$

where,

$$\alpha(A) = \max\{\mathbb{R}\theta | \theta \in \sigma(A)\}.$$

We denote $\rho_m = \max_{k=1,2,\ldots} \{\rho(B_k)\}$. Now if the following identity

$$\Xi = \alpha(a) + \frac{1}{\lambda_1} \ln(1 + \rho_m) < 0$$

holds then (3.1) is asymptotically stable.

Theorem 3.10. *(Theorem 2.11 [20]) Let us assume that*

$$\limsup_{t \to +\infty} \frac{i(t, t_0)}{(t - s_{i(t_i, t_0)})^+ + \sum_{k=0}^{i(t,t_0)-1} (t_{k+1} - s_k)} := \tilde{P} < +\infty.$$

Then equation (3.1) will be asymptotically stable provided

$$\gamma = \alpha(A) + \tilde{P} \ln(1 + \rho_m) < 0.$$

4. Almost periodic solution to differential equation with non-instantaneous impulses

In this section, we will discuss the sufficient condition for existence of almost periodic solution to linear homogeneous, linear non-homogeneous and non-linear differential equations with non instantaneous impulses. Next, we will state some basic definitions and lemma which will be required later.

Definition 4.1. *A set of sequence $\{x_i\}_{i \in \mathbb{Z}}$ is said to be almost periodic if for each $\epsilon > 0$ there exists a relatively dense set of ϵ-almost periods.*

Definition 4.2. *The set of sequences $\{x_i^j\}$, $\{x_i\} \in \mathbb{R}$ with $x_i < x_{i+1}$, and $i, j \in \mathbb{Z}$ is said to be uniformly almost periodic if for each $\epsilon > 0$ there exists a relatively dense set of ϵ-almost periods for any given sequence, where*

$$x_i^j = x_{i+j} - x_i.$$

Lemma 4.3. *(Lemma 2.9 [17]) Assume that $\{x_i^j\}$ for $i, j \in \mathbb{N}$ is uniformly almost periodic then, the following results hold.*

(i) *There exists a constant $K > 0$ such that*

$$\lim_{T \to \infty} \frac{i(t, t+T)}{T} = K.$$

(ii) *For $h > 0$, there exists a positive integer N such that every interval with length h contains no more than N elements of the sequence $\{x_i\}$, i.e*

$$i(s, t) \leq N(t - s) + N.$$

Next, we discuss the results for the linear homogeneous case.

4.1. **Linear Homogeneous Equation.** Consider the following linear homogeneous differential equation with non-instantaneous impulses,

$$\begin{aligned}
&\frac{du}{dt} = Au(t), \qquad t \in (s_i, t_{i+1}] \qquad i = 0, \pm1, \pm2, \dots, \\
&u(t_i^+) = (I + B_i)u(t_i^-) \qquad\qquad i = \pm1, \pm2, \dots, \\
&u(t) = (I + B_i)u(t_i^-), \quad t \in (t_i, s_i] \quad i = \pm1, \pm2.. \\
&u(s_i^+) = u(s_i^-) \qquad\qquad\qquad i = \pm1, \pm2, \dots,
\end{aligned} \tag{4.1}$$

where, B_i is almost periodic for each $i = 0, \pm1, \pm2, \dots$

Now using Lemma 3.4, we can define the solution of (4.1) with initial condition $u(t_0) = u_0$ as

$$u(t, t_0, u_0) = W(t, t_0)u_0,$$

where, $W(\cdot, \cdot)$ is non-instantaneous Cauchy matrix.

We consider the following assumptions:

(A_1) Assume that, $W(t, s)$ is exponentially stable, i.e.

$$\|W(t, s)\| \leq Me^{-\alpha(t-s)}, s < t.$$

(A_2) The sequences $\{t_i^j\}$ and $\{s_i^j\}$ are uniformly almost periodic, where $i, j = 0, \pm1, \pm2, \dots$

(A_3) $\{B_i\}$ is almost periodic for $i = 0, \pm1, \pm2, \dots$

(A_4) Suppose $+\infty > v \geq s_i - t_i > 0$, $i = 0, \pm1, \pm2, \dots$

Lemma 4.4. *(Lemma 3.6 [20]) Let the assumptions $(A_1)-(A_4)$ hold. Then $W(t, s)$ is almost periodic. Thus for $h \in \Omega$, we have*

$$\|W(t + h, s + h) - W(t, s)\| \leq M(\epsilon)e^{-a(t-s)}\epsilon, \quad M(\epsilon) > 0.$$

Theorem 4.5. *(Theorem 3.7 [20]) Suppose that Lemma 4.4 holds. Then the homogeneous problem (4.1) with $u(t_0) = u_0$ has a asymptotically unique almost periodic solution.*

4.2. **Non-Homogeneous Linear Equation.** Consider the non-homogeneous linear differential equation with non-instantaneous impulse.

$$\begin{aligned}
&\frac{du}{dt} = Au(t) + g(t), t \in (s_i, t_{i+1}] \qquad i \in \mathbb{Z}, \\
&u(t_i^+) = (I + B_i)u(t_i^-) + c_i \qquad\qquad i \in \mathbb{Z} \setminus 0, \\
&u(t) = (I + B_i)u(t_i^-) + c_i, \quad t \in (t_i, s_i] \qquad i \in \mathbb{Z} \setminus 0, \\
&u(s_i^+) = u(s_i^-) \qquad\qquad\qquad\qquad i \in \mathbb{Z} \setminus 0,
\end{aligned} \tag{4.2}$$

where $g : \mathbb{R} \to \mathbb{R}^n$ is a given function. We will impose the following assumptions on g :

(A_5) The function $g : \mathbb{R} \to \mathbb{R}^n$ is an almost periodic function.

(A_6) The sequence $\{c_i\}$ is almost periodic for $i = 0, \pm1, \pm2, \dots$

Tian [20] proved that if Lemma 4.4 and the assumptions $(A_5) - (A_6)$ hold then the solution of (4.2) is uniquely determined and exponentially stable.

4.3. **Non-Homogeneous Non Linear Equation.** Consider the following non linear non-instantaneous impulsive differential equation.

$$\begin{aligned} \frac{du}{dt} &= Au(t) + g(t, u(t)), t \in (s_i, t_{i+1}], \qquad i = 0, \pm 1, \pm 2, \dots \\ u(t_i^+) &= (I + B_i)u(t_i^-) + I_i(t_i^+, u(t_i^-)), \qquad i \in \mathbb{Z} \setminus 0, \\ u(t) &= (I + B_i)u(t_i^-) + I_i(t, u(t_i^-)), \quad t \in (t_i, s_i], \qquad i \in \mathbb{Z} \setminus 0, \\ u(s_i^+) &= u(s_i^-), \qquad i \in \mathbb{Z} \setminus 0. \end{aligned} \tag{4.3}$$

We assume the following conditions:

(A_7) The function $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $I_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be almost periodic function on $\|u\| < r$ and $t \in \mathbb{R}$.

(A_8) Let $g \in C\big((s_i, t_{i+1}] \times \mathbb{R}^n, \mathbb{R}^n\big)$ and $I_i \in C\big((t_i, s_i] \times \mathbb{R}^n, \mathbb{R}^n\big)$. Suppose there exists a constant $L > 0$ such that

$$\|g(t, y_1) - g(t, y_2)\| + \|I_i(t, y_1) - I_i(t, y_2)\| \le L\|y_1 - y_2\|$$

for each $t \in \mathbb{R}$ and $\forall y_1, y_2 \in \mathbb{R}^n$.

(A_9) Suppose

$$H = \sup_{t \in \mathbb{R},\ \|x\| \le \delta} \|g(t, x)\| + \sup_{t \in \mathbb{R},\ \|x\| \le \delta,\ i \in \mathbb{Z}} \|I_i(t, x)\| < \infty.$$

Lemma 4.6. *(Theorem 3.6 [20]) Suppose that Theorem 4.5 holds. Then equation (4.3) is asymptotically stable and has a unique periodic solution provided the following three conditions hold.*

$$MH\Big(\frac{1}{\alpha} + 2N + \frac{2N}{e^\alpha - 1}\Big) < r.$$

$$ML\Big(\frac{1}{\alpha} + 2N + \frac{2N}{e^\alpha - 1}\Big) < 1.$$

$$\alpha - ML - \tilde{P}\ln(1 + ML) > 0.$$

5. Main Result

We will prove the sufficient condition for the existence of an almost periodic solution to a non-instantaneous impulsive differential equation with a deviating argument. To establish the result, we will use some of the already discussed results. Consider differential equations with deviating arguments.

$$\begin{aligned} \frac{du}{dt} &= Au(t) + g(t, u(t), u(\phi(t, u(t)))), t \in (s_i, t_{i+1}] \qquad i \in \mathbb{Z} \\ u(t_i^+) &= (I + B_i)u(t_i^-) + J_i(t_i^+, u(t_i^-), u(K_i(t_i^+, u(t_i^-)))) \quad i \in \mathbb{Z}\backslash 0 \\ u(t) &= (I + B_i)u(t_i^-) + J_i(t_i, u(t_i^-), u(K_i(t_i, u(t_i^-)))), \quad i \in \mathbb{Z}\backslash 0 \\ u(s_i^+) &= u(s_i^-) \qquad i \in \mathbb{Z}\backslash 0 \end{aligned} \tag{5.1}$$

where $\{t_i\}, \{s_i\}, B_i$ are same as defined earlier.

Let us consider the following assumptions

(A_{10}) $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$,$J_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $K_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are almost periodic $\quad \forall t \in \mathbb{R}$

(A_{11}) g, K_i, J_i are continuous functions in $(s_i, t_{i+1}] \times \mathbb{R}^n \to \mathbb{R}^n$ then there exist $R_1, R_2 > 0$ such that

$$\|g(t, y_1, z_1) - g(t, y_2, z_2)\| \leq R_1(\|y_1 - y_2\| + \|z_1 - z_2\|), \tag{5.2}$$

$$\|J_i(t, y_1, K_i(t, y_1)) - J_i(t, y_2, K_i(t, y_2))\| \leq R_2\|y_1 - y_2\|$$
$$\forall y_1, y_2, z_1, z_2 \in \mathbb{R}^n$$

and each $t \in \mathbb{R}$

(A_{12}) The function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is almost periodic in t and satisfies,

$$\|\phi(t, w_1) - \phi(t, w_2)\| \leq R_3(\|w_1 - w_2\|) \tag{5.3}$$

(A_{13}) Assume that,

$$S = \sup\|g(t, u, u(\phi(t, u(t))))\| + \sup\|J_i(t, u, u(K_i(t, u(t))))\| < +\infty.$$

Here, we will prove the sufficient condition for existence of almost periodic solution.

Theorem 5.1. *Let the assumptions $(A_{10}) - (A_{12})$ hold along with Lemma 4.4. Then the system (8) has a unique almost periodic solution which is asymptotically stable provided*

$$MR_3(\frac{1}{\alpha} + 2N + \frac{2N}{e^{\alpha} - 1}) < r \tag{5.4}$$

$$\alpha - MR - \tilde{P}\ln(1 + MR) > 0 \tag{5.5}$$

Where

$$R := min\{R_1(1 + R_3), R_2\}$$

Proof. Suppose that,
X is the space of almost periodic functions.
Define, $X = \{y(t) \mid y(t)$ is almost periodic and is not continuous at t_i and $s_i\}$
$y(.) \in$X$\subseteq PC(\mathbb{R}, \mathbb{R}^n)$ and

$$||y||_{PC} = \sup_{t\in\mathbb{R}} ||y(t)|| < r$$

We define a operator ω :X$\to PC(\mathbb{R}, \mathbb{R}^n)$ such that

$$\omega(y(t)) = \sum_{s_k<t_{k+1}<s_i<t} \int_{s_k}^{t_k+1} W(t,s)g(s, y(s), y(\phi(s, y(s))))ds + \int_{s_i}^{t} W(t,s)g(s, y(s), y(\phi(s, y(s))))$$
$$+ \sum_{s_k<t} W(t, s_k^+)J_k(s_k, y(t_k^-), y(K_k(s_k, y(t_k^-))))$$

Claim: $||\omega(y(t))|| < r$.

$$||\omega(y(t))|| < \int_{-\infty}^{t} ||W(t,s)||||g(t, y(t), y(\phi(t, y(t))))||ds + \sum_{s_k<t} ||W(t, s_k^+)||||J_k(s_k, y(t_k^-), y(K_k(s_k, y(t_k^-))))||$$

$$\leq MR_3 \int_{-\infty}^{t} e^{-\alpha(t-s)}ds + MR_3 \left(\sum_{0<t-s_k<1} e^{-\alpha(t-s_k)} + \sum_{j=1}^{\infty} \sum_{0<t-s_k<j+1} e^{-\alpha(t-s_k)} \right)$$

$$\leq MR_3\left(\frac{1}{\alpha}+2N+\frac{2N}{e^{\alpha}-1}\right)<r$$

which is given by (17).

Now we will prove $\omega(y(t))$ to be almost periodic. For any $k\in\Omega$ and $t\in\mathbb{R}$ we have

$$\begin{aligned}||\omega(y(t+k))-\omega(y(t))|| \leq &\sum_{s_k<t_{k+1}<s_i<t}\int_{s_k}^{t_{k+1}}||W_1(t,s)g(s+k,y(s+k),y(\phi(s+k,y(s+k))))\\ &-W(t,s)g(s,y(s),y(\phi(s,y(s))))||ds\\ &+\int_{s_i}^{t}||W_1(t,s)g(s+k,y(s+k),y(\phi(s+k,y(s+k))))||ds\\ &+\sum_{s_k<t}||W_1(t,s_k)J_{s+q}(s_{k+q},y(t_{k+q}^-)y(K_{k+q}(s_{k+q},y_{k+q}^-)))||\\ &\leq Q_1+Q_2+Q_3.\end{aligned}$$

Q_1,Q_2,Q_3 are all bounded and it is obvious from asymptotically stability and finiteness of supremum norm from assumption (A_{12}).
So,

$$||\omega(y_1)-\omega(y_2)||_{PC}<||y_1-y_2||_{PC}$$

i.e ω is contraction operator in X and there exists a almost periodic solution y of system (5.6), which is unique.
As a last step of the proof, we have to show y is asymptotically stable. Let y_1 and y_2 are two solutions of (5.6) with initial condition $y_1(t_0)$, $y_2(t_0)$.

$$\begin{aligned}||y_1(t)-y_2(t)|| \leq &||W(t,t_0)||||y_1(t_0)-y_2(t_0)||+\sum_{s_k<t_{k+1}<s_i<t}\int_{s_k}^{t_k+1}||W(t,s)||||g(s,y_1(s),y_1(\phi(s,y_1(s))))\\ &-g(s,y_2(s),y_2(\phi(s,y_2(s))))||ds\\ &+\int_{s_i}^{t}||W(t,s)||||g(s,y_1(s),y_1(\phi(s,y_1(s))))-g(s,y_2(s),y_2(\phi(s,y_2(s))))||ds\\ &+\sum_{s_k<t}||W(t,s_k)||||J_k(s_k,y_1(t_k^-),y_1(K_k(s_k,y_1(t_k^-))))-J_k(s_k,y_2(t_k^-),y_2(K_k(s_k,y_2(t_k^-))))||\end{aligned}$$

$$\begin{aligned}\|y_1(t)-y_2(t)\| \leq &Me^{-\alpha(t-t_0)}||y_1(t_0)-y_2(t_0)||+MR_1\sum_{s_k<t_{k+1}<s_i<t}\int_{s_K}^{t_{k+1}}e^{-\alpha(t-s)}(||y_1(s)-y_2(s)||\\ &+||y_1(\phi(s,y_1(s)))-y_2(\phi(s,y_2(s)))||)ds\\ &+MR_1\int_{s_i}^{t}e^{-\alpha(t-s)}(||y_1(s)-y_2(s)||+||y_1(\phi(s,y_1(s)))-y_2(\phi(s,y_2(s)))||)ds+\\ &MR_2\sum_{s_k<t}e^{-\alpha(t-s_k)}||y_1(t_k^-)-y_2(t_k^-)||\end{aligned}$$

Multiplying $e^{\alpha t}$ on both sides we get,

$\|y_1(t)-y_2(t)\|e^{\alpha t}$

$$\leq Me^{\alpha t_0}||y_1(t_0) - y_2(t_0)|| + MR_1(1+R_3) \sum_{s_k<t_{k+1}<s_i<t} \int_{s_K}^{t_{k+1}} e^{\alpha s}||y_1(s) - y_2(s)||ds$$

$$+ MR_1(1+R_3)\int_{s_i}^{t} e^{\alpha s}||y_1(s) - y_2(s)||ds + MR_2 \sum_{s_k<t} e^{\alpha s_k}||y_1(t_k^-) - y_2(t_k^-)||$$

$$\leq Me^{\alpha t_0}||y_1(t_0) - y_2(t_0)|| + MR_1(1+R_3)\int\limits_{t_0}^{t} ||y_1(s) - y_2(s)||e^{\alpha s}ds + \sum_{t_k<t} MR_2 e^{\alpha s_k}||y_1(t_k^-) - y_2(t_k^-)||$$

$$\leq Me^{\alpha t_0}||y_1(t_0) - y_2(t_0)|| \prod_{t_0<t_k<t}(1+MR)e^{MR(t-t_0)}.$$

which implies

$$\begin{aligned}\|y_1(t) - y_2(t)\| &\leq M||y_1(t_0) - y_2(t_0)||(1+MR)^{i(t,t_0)}e^{(-\alpha+MR)(t-t_0)}\\ &\leq M||y_1(t_0) - y_2(t_0)||e^{-[\alpha - MR - \tilde{P}ln(1+MR)](t-t_0)} \quad \to 0 \quad \text{as } t \to +\infty\end{aligned}$$

which is due to the condition (18). □

Hence the proof is complete.

5.1. **Almost periodic solution of integro-differential equation.** Now, we will extend our study to non-instantaneous impulsive integro-differential equation with deviating arguments.

Consider the non-instantaneous impulsive integro-differential equation

$$\begin{aligned}\frac{du}{dt} &= Au(t) + g(t, u(t), u(\phi(t, u(t)))) + \int\limits_0^t a(t,\tau)\Psi(\tau)\,d\tau, t \in (s_i, t_{i+1}] \qquad i \in \mathbb{Z}\\ u(t_i^+) &= (I + B_i)u(t_i^-) + J_i(t_i^+, u(t_i^-), u(K_i(t_i^+, u(t_i^-)))) + T_i(t_i^+) \qquad i \in \mathbb{Z}\backslash 0\\ u(t) &= (I + B_i)u(t_i^-) + J_i(t_i, u(t^{-i}), u(K_i(t_i, u(t^{-i})))) + T_i(t_i), \quad i \in \mathbb{Z} \setminus \{0\}\\ u(s_i^+) &= u(s_i^-), \qquad\qquad i \in \mathbb{Z} \setminus \{0\}\end{aligned} \tag{5.6}$$

where, $\int_0^t a(t,\tau)\Psi(\tau)\,d\tau = T(t)$

Let us consider the following assumptions:

(A_{14}) Assume that, $\Psi : [a, b] \to \mathbb{R}^n$ is continuous and $a(t, \tau)$ is bounded so that T(t) is well defined.

(A_{15}) Assume that, there exists a constant R_4 such that $||T_i(t_i)|| < R_4$

Theorem 5.2. *Let the assumptions $(A_{10}) - (A_{15})$ hold along with Lemma 4.4. Then the system (8) has a unique almost periodic solution which is asymptotically stable provided*

$$MR_3(\frac{1}{\alpha} + 2N + \frac{2N}{e^\alpha - 1}) + MR_4 < r \tag{5.7}$$

$$\alpha - MR - \tilde{P}\ln(1 + MR) > 0 \tag{5.8}$$

Where

$$R := \min\{R_1(1 + R_3), R_2, R_4\}$$

Proof. We will prove this theorem in a similar way as that of previous one. Here, X is the space of almost periodic functions with discontinuities at point of sequence t_i and s_i. $y(.) \in$X$\subseteq PC(\mathbb{R}, \mathbb{R}^n)$ and

$$||y||_{PC} = \sup_{t\in\mathbb{R}} ||y(t)|| < r$$

We define a operator ω :X$\rightarrow PC(\mathbb{R}, \mathbb{R}^n)$ such that

$$\omega(y(t)) = \sum_{s_k<t_{k+1}<s_i<t} \int_{s_k}^{t_k+1} W(t,s)g(s,y(s),y(\phi(s,y(s))))ds + \int_{s_i}^{t} W(t,s)g(s,y(s),y(\phi(s,y(s))))$$
$$+ \sum_{s_k<t} W(t,s_k^+)J_k(s_k,y(t_k^-),y(K_k(s_k,y(t_k^-)))) + W(t,s)T(t_k)$$

Now we will verify $||\omega(y(t))|| < r$.

$$||\omega(y(t))|| < \int_{-\infty}^{t} ||W(t,s)||||g(t,y(t),y(\phi(t,y(t))))||ds + \sum_{s_k<t} ||W(t,s_k^+)||||J_k(s_k,y(t_k^-),y(K_k(s_k,y(t_k^-)))$$

$$+||W(t,s)||||T(t_k)||$$

$$\leq MR_3 \int_{-\infty}^{t} e^{-\alpha(t-s)}ds + MR_3\left(\sum_{0<t-s_k<1} e^{-\alpha(t-s_k)} + \sum_{j=1}^{\infty}\sum_{0<t-s_k<j+1} e^{-\alpha(t-s_k)}\right) + MR_4$$
$$\leq MR_3\left(\frac{1}{\alpha} + 2N + \frac{2N}{e^{\alpha}-1}\right) + MR_4 < r$$

which is given in condition (20).

Now we will prove $\omega(y(t))$ to be almost periodic. For $h \in \Omega \subseteq$X, $t \in \mathbb{R}$ we have,

$$||\omega(y(t+h)) - \omega(y(t))|| \leq \sum_{s_k<t_{k+1}<s_i<t} \int_{s_k}^{t_{k+1}} ||W_1(t,s)g(s+h,y(s+h),y(\phi(s+h,y(s+h))))$$
$$- W(t,s)g(s,y(s),y(\phi(s,y(s))))||ds$$
$$+ \int_{s_i}^{t} ||W_1(t,s)g(s+h,y(s+h),y(\phi(s+h,y(s+h))))||ds$$
$$+ \sum_{s_k<t} ||W_1(t,s_k)J_{s+q}(s_{k+q},y(t_{k+q}^-)y(K_{k+q}(s_{k+q},y_{k+q}^-)))|| + ||W_1(t,s)||||T(t_k)||$$
$$\leq Q_1 + Q_2 + Q_3 + Q_4.$$

Q_1, Q_2, Q_3, Q_4 are all bounded and it is obvious from asymptotically stability and finiteness of supremum norm from assumption (A_{12}).
So,

$$||\omega(y_1) - \omega(y_2)||_{PC} < ||y_1 - y_2||_{PC}$$

i.e ω is contraction operator in X and there exists a almost periodic solution y of system (5.6), which is unique.
As a last step of the proof, we have to show y is asymptotically stable. Let y_1 and y_2 are two solutions of (5.6) with initial condition $y_1(t_0)$, $y_2(t_0)$.

$$||y_1(t) - y_2(t)|| \leq ||W(t,t_0)||||y_1(t_0) - y_2(t_0)|| + \sum_{s_k<t_{k+1}<s_i<t} \int_{s_k}^{t_k+1} ||W(t,s)||||g(s,y_1(s),y_1(\phi(s,y_1(s))))$$

$$- g(s, y_2(s), y_2(\phi(s, y_2(s))))||ds$$

$$+ \int_{s_i}^{t} ||W(t,s)||||g(s, y_1(s), y_1(\phi(s, y_1(s)))) - g(s, y_2(s), y_2(\phi(s, y_2(s))))||ds$$

$$+ \sum_{s_k<t} ||W(t,s_k)||||J_k(s_k, y_1(t_k^-), y_1(K_k(s_k, y_1(t_k^-)))) - J_k(s_k, y_2(t_k^-), y_2(K_k(s_k, y_2(t_k^-))))|| + ||W_1(t,s)||||T_k(t_k^-)||$$

$$\|y_1(t) - y_2(t)\| \le Me^{-\alpha(t-t_0)}||y_1(t_0) - y_2(t_0)|| + MR_1 \sum_{s_k<t_{k+1}<s_i<t} \int_{s_K}^{t_{k+1}} e^{-\alpha(t-s)}(||y_1(s) - y_2(s)||$$

$$+ ||y_1(\phi(s, y_1(s))) - y_2(\phi(s, y_2(s)))||)ds$$

$$+ MR_1 \int_{s_i}^{t} e^{-\alpha(t-s)}(||y_1(s) - y_2(s)|| + ||y_1(\phi(s, y_1(s))) - y_2(\phi(s, y_2(s)))||)ds +$$

$$MR_2 \sum_{s_k<t} e^{-\alpha(t-s_k)}||y_1(t_k^-) - y_2(t_k^-)|| + MR_4$$

Multiplying $e^{\alpha t}$ on both sides we get,

$$\|y_1(t) - y_2(t)\|e^{\alpha t}$$

$$\le Me^{\alpha t_0}||y_1(t_0) - y_2(t_0)|| + MR_1(1 + R_3) \sum_{s_k<t_{k+1}<s_i<t} \int_{s_K}^{t_{k+1}} e^{\alpha s}||y_1(s) - y_2(s)||ds$$

$$+ MR_1(1 + R_3) \int_{s_i}^{t} e^{\alpha s}||y_1(s) - y_2(s)||ds + MR_2 \sum_{s_k<t} e^{\alpha s_k}||y_1(t_k^-) - y_2(t_k^-)|| + MR_4$$

$$\le Me^{\alpha t_0}||y_1(t_0) - y_2(t_0)|| + MR_1(1 + R_3) \int_{t_0}^{t} ||y_1(s) - y_2(s)||e^{\alpha s}ds + \sum_{t_k<t} MR_2e^{\alpha s_k}||y_1(t_k^-) - y_2(t_k^-)|| + MR_4$$

$$\le Me^{\alpha t_0}||y_1(t_0) - y_2(t_0)|| \prod_{t_0<t_k<t} (1 + MR)e^{MR(t-t_0)}.$$

which implies

$$\|y_1(t) - y_2(t)\| \le M||y_1(t_0) - y_2(t_0)||(1 + MR)^{i(t,t_0)}e^{(-\alpha+MR)(t-t_0)}$$

$$\le M||y_1(t_0) - y_2(t_0)||e^{-[\alpha - MR - \tilde{P}ln(1+MR)](t-t_0)} \qquad \to 0 \qquad as\ t \to +\infty$$

which is due to the condition (21). □

6. Application

In this section, we will verify the theorem with the help of an example [20]. We consider the impulsive point and junction point as $\{t_i\}$ and $\{s_i\}$.

We set $t_0 = 0$, the sequence $\{t_i\}$,$\{s_i\}$ are given as :
$t_i = i + \frac{1}{5} \mid sin(i) - sin(i\sqrt{3}) \mid$,
$s_i = i + \frac{1}{3} \mid sin(i) - sin(i\sqrt{3}) \mid$

We choose the matrix A and B_i to be as follows:

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

$$B_i = \begin{pmatrix} sin(i\sqrt{3}) & 0 \\ 0 & sin(i\sqrt{3}) \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

Consider, $g(t, u(t), u(h(t, u(t)))) = \begin{pmatrix} \frac{u_1 sin(\sqrt{5}t)}{60} \\ \frac{u_2 cos(\sqrt{5}t)}{60} \end{pmatrix} \in \mathbb{R}^2,$

$$J_i(t, u(t), u(K_i(t, u(t)))) = \begin{pmatrix} \frac{u_1}{60}(1 + sin(\sqrt{3}t)) \\ \frac{u_2}{60}(1 + cos(\sqrt{3}t)) \end{pmatrix} \in \mathbb{R}^2$$

We can obtain, $\alpha(A) = -2, \rho_m = 1$.
(Since ρ_m is the maximum spectral radius)
$\{t_i\}$ and $\{s_i\}$ are uniformly almost periodic sequence and $\{B_i\}, i \in \mathbb{Z}$ is almost periodic.So,(A_2) and (A_3) hold.

It is obvious, $\mid sin(i) - sin(i\sqrt{3}) \mid \leq 2$.

so,$\frac{4}{15} = \nu \geq s_i - t_i > 0$,
Thus (A_4) holds.

$$\limsup_{t\to+\infty} \frac{i(t, t_0)}{(t - s_{i(t_i,t_0)})^+ + \sum_{k=0}^{i(t,t_0)-1}(t_{k+1} - s_k)} := \tilde{P} = 2 < +\infty.$$

Again,

$$\gamma = \alpha(A) + \tilde{P}\ln(1 + \rho_m) = -2 + 2ln2 < 0.$$

Thus,$W(t, s)$ is asymptotically stable.So (A_1) holds.
We can choose,
$M = 1$ and $\alpha = 1 - ln2 > 0$, $L = \frac{1}{30}$
Next we choose, $r = 1$ and $N = \tilde{P} = 2$.

Again, the function g and J_i are almost periodic.

$$S = \sup||g(t, u(t), u(h(t, u(t))))|| + \sup||J_i(t, u(t), u(K_i(t, u(t))))|| = \frac{1}{20}.$$

Therefore,

$$MS(\frac{1}{\alpha} + 2N + \frac{2N}{e^{\alpha} - 1}) = \frac{1}{20}(\frac{1}{1 - ln2} + 4 + \frac{4}{e^{1-ln2} - 1}) \approx= 0.9198 < 1.$$

Now,

$$\alpha - MR - \tilde{P}\ln(1 + MR) = 1 - ln2 - \frac{1}{30} - 2ln(1 + \frac{1}{30}) \approx 0.2079 > 0$$

So all the Assumption has been satisfied.
Therefore, by previous theorem system has a unique almost periodic solution which is exponentially stable.

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