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SOME INEQUALITIES FOR INTERVAL-R-L INTEGRALS VIA CONVEXITY

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Abstract. By using the convexity property some fractional integrals inequalities are obtained in a very simple way for interval-valued Riemann-Liouville fractional integrals. Furthermore, Hermite-Hadamard-type inequalities are obtained as consequence.

1. Introduction

It is known that inequalities play an important role in almost all branches of mathematics as well as in other areas of science.The study of inequalities has increased enormously since the classical treatise was published by Hardy, Littlewood and Polya.More recently, some of these inequalities have been extended to set-valued functions [7],[9] especially, to interval-valued functions by Chalco-Cano et al [2],[12],[13],[16],[19]. In this direction, recently several classical integral inequalities have been extended to the interval-valued context. In 2018, using the Kulisch-Miranker order on the space of real and compact intervals, Roman-Flores et al. [6] established interval Minkowski's inequality, interval Radon's inequality and interval Beckenbach's inequality. Others authors Meanwhile, Costa et al. (see [12],[13]), obtained new Jensen's inequality, Minkowski's and Gauss's integral inequalities for fuzzy-interval-valued functions and then they [14] established some Opial-type integral inequalities by using the concept of $gH-$ differentiability the authors in [14],[15],[19] shows an Ostrowski's inequality for interval-valued functions. By using the h-convex concept, Zhao et al. [16] presented new Jensen and Hermite-Hadamard type inequalities for interval-valued functions. Next, in 2019, Zhao et al. proved some integral inequalities on time scales, which generalize some previous inequalities presented by Costa [12] and Roman-Flores et al.[6].
To develope a theory of the fractional calculus for interval-valued-functions,several notions of derivative of interval-valued-function (gH-derivative,for example see [15]) were introduced.In ([17],section 3,4 and 5), the authors study main properties of Riemann-Liouville fractional integral,Riemann-fractional derivative,caputo derivative.

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The purpose of this work is to present some contributions on fractional integral inequalities in the interval-valued context.

2. Preliminaries

We start by recalling some basic notations, definitions and results on interval analysis. For more details on concepts (interval operations,interval-valued functions) and results on interval analysis. (see [[11], pp.131] ,[[15],[5],chap 4],[[9],[10],sect 2,th 2.3]).

2.1. **Interval operations.**

(1) A real interval A is the subset of $\mathbb{R}$ defined by

$$A = [\underline{a}, \bar{a}] = \{x \in \mathbb{R} : \underline{a} \leq x \leq \bar{a}\},$$

When $\underline{a} = \bar{a}$, the interval A is said to be degenerate or point interval. The mid-point representation of an interval $A = (\hat{a}, \tilde{a})$ where

$$m(A) := \hat{a} = \frac{\underline{a} + \bar{a}}{2}, \tilde{a} = \frac{\bar{a} - \underline{a}}{2}, \omega(A) = \bar{a} - \underline{a}.$$

Assuming the condition we define the comparison ratio:

$$\gamma_{A,B} = \frac{\tilde{a} - \tilde{b}}{\hat{a} - \hat{b}} = \gamma_{B,A}$$

whch is very useful in the characterization of different order relations for intervals;

$$\begin{aligned} \bar{b} - \bar{a} &= (\hat{b} - \hat{a})(\gamma_{A,B} + 1) \\ \underline{b} - \underline{a} &= (\hat{a} - \hat{b})(\gamma_{A,B} - 1) \end{aligned} \tag{1}$$

(2) $\mathbb{IR}$, $\mathbb{IR}^+$ and $\mathbb{IR}^-$, denote the set of all intervals,positive intervals and negative intervals of $\mathbb{R}$ respectively.The interval $A = [\underline{a}, \bar{a}]$ is positive (negative) if $\underline{a} > 0$ ($\bar{a} < 0$).

(3) Let $\odot$ be one of the four arithmetic operators $+, -, *, /$ we set

$$A \odot B = \{u \odot v : u \in A, v \in B\},$$

where $\odot$ can the quotient / is defined only if $0 \notin B$.
If $A, B, C,$ and D are intervals such that $A \subset B$ and $C \subset D$, then the following relation is valid $A \odot C \subset B \odot D$.

(4) For $s \in \mathbb{R}$ and $A \in \mathbb{IR}$, we set

$$s\,A = \begin{cases} [s\,\bar{a}, s\,\underline{a}] & \text{if } s < 0. \\ 0 & \text{if} s = 0, \\ [s\,\underline{a}, s\,\bar{a}], & \text{if } s > 0, \end{cases}$$

(5) The Kulisch-Miranker order relation $\leq_{ku}$ is defined on $\mathcal{K}_c$ as follows [8]:

$$[\underline{a}, \overline{a}] \leq_{ku} [\underline{b}, \overline{b}] \iff \quad a \leq b \quad \text{and} \quad a \leq b.$$

2.2. **Interval valued functions.** ([[11],[1] ,[15]]). A function $F : \mathbb{I} = [a,b] \to \mathbb{IR}$ is said to be an interval-valued function(in short i-v-f) of t on $\mathbb{I} = [a,b]$ if it assigns a nonempty interval $F(t)$ to each $t \in [a,b]$,

$$F(t) = [\underline{F}(t), \overline{F}(t)],$$

where $\underline{F}(t)$ and $\overline{F}(t)$ are single-real-valued functions with $\underline{F}(t) \le \overline{F}(t)$ for all $t \in [a,b]$. The set of all such functions is denoted by $\mathcal{F}_{[a,b]}$.

1) F is said $\omega-$ increasing ($\omega-$ decreasing) in $\mathbb{I}$ if the single-valued function $\omega_F(t) := \omega(F(t))$ is increasing (decreasing) in $\mathbb{I}$, respectively.Or equivalently if $\underline{F}$ and $\overline{F}$ are increasing (decreasing) on $[a,b]$.
2) **Integral of interval-valued functions.**(see [[1],th 2.3],[[11], p. 131])
   - The $\mathbb{IR}-$integral: The i-v-function F is (Riemann) integrable in $[a,b], a \le b$ if the functions $\underline{F}$ and $\overline{F}$ are Riemann-integrable. And the collection of all functions that are $\mathbb{IR}-$ integrable on $[a,b], a \le b$ is denoted by $\mathbb{IR}_{([a,b])}$. In this case we define

$$(\mathbb{IR})\int_a^b F(t)dt = \left[(\mathcal{R})\int_a^b \underline{F}(t)dt, (\mathcal{R})\int_a^b \overline{F}(t)dt\right],$$

   where $\mathcal{R}([a,b])$ denotes the Riemann-integrable functions .
   - The $\mathbb{IL}-$integral: The i-v-function F is (Lebesgue) integrable in $[a,b], a \le b$ if the functions $\underline{F}$ and $\overline{F}$ are Lebesgue integrable(see [[15]]). In this case we define

$$(\mathcal{IL})\int_a^b f(t)dt = \left[(\mathcal{L})\int_a^b \underline{F}(t)dt, (\mathcal{L})\int_a^b \overline{F}(t)dt\right].$$

3) **Derivative for interval-valued functions**.
   Differentiation of interval functions is considered in ([15],sec 5.,th 3.,th 4.,p 328).In general the differentiability($gH-$ differentiability) of $F = [\underline{F}, \overline{F}]$ does not imply generally differentiability of the single-valued functions $\underline{F}$ and $\overline{F}$. But differentiability($gH-$ differentiability) of $F = [\underline{F}, \overline{F}]$ is equivalent to differentiability of both $\underline{F}$ and $\overline{F}$ if F is $\omega-$ increasing function and

$$F'(x) = [\underline{F}'(x), \overline{F}'(x)].$$

4) **C**onvexity
   Definition 1 (see ([2],[3],[7],[16],[15],[19])
   Let $\mathbb{I}$ be an interval in $\mathbb{IR}$. Let $F : \mathbb{I} \to \mathbb{IR}^+$ be an interval-valued-functionis,we say that F is convex if for all $x, y \in \mathbb{I}$ and all $r \in [0,1]$, then

$$rF(x) + (1-r)F(y) \subseteq F(rx + (1-r)y) \tag{2}$$

   holds and F is said to be concave interval-valued functions if set inclusion (2) is reversed.Here $F(x) = [\underline{F}(x), \overline{F}(x)]$ and $\underline{F}(x), \bar{F}(x)$ are single-valued functions.
   Remark 1.(see [[16],theorem 3.7]) F is convex if and only if $\underline{F}$ is convex and $\overline{F}$ is concave.

3. Main Results

In this section, we mention some concepts related to fractional integration and we prove some inequalities by considering the Riemann-Liouville fractional integrals for real-valued functions [[5],[17]]. In the following the notations $\mathbb{IR}, \mathbb{R}, \mathbb{IL}, \mathbb{L}$ in front of the integral symbol have been omited for simplicity.

3.1. **Interval-valued R-L fractional integral. Definition** 2 [15]. For $1 \leq p \leq \infty$ we denote by $L_p := L_p([a,b], \mathbb{IR}^+)$ the set of all Lebesgue measurable interval-valued functions $F : [a,b] \to \mathbb{IR}^+$ such that

$$\|F\|_p = \begin{cases} \left(\int_a^b |F(x)|^p dx\right)^{\frac{1}{p}} < \infty, & \text{if } 1 \leq p < \infty \\ ess\sup_{x\in[a,b]} |F(x)|, & \text{if } p = \infty. \end{cases} \tag{3}$$

Definition 3 [18]. Let $z > 0, r, s > 0$. The gamma and the beta functions are defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt \;\; z > 0,$$

$$B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}\, dt.$$

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}.$$

First, we recall that the Riemann-Liouville fractional integral of a single-valued function f is defined as follows

Definition 4 ([18],[4]) Let $-\infty < a < b < +\infty$. The left and right-sided Riemann-Liouville fractional integral operators $\mathbf{J}^\alpha$ of order $\alpha \geq 0$ of function $f(x) \in L_1[a,b]$ are defined by

$$\mathbf{J}_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \;\; x > a; \tag{4}$$

and

$$\mathbf{J}_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \;\; x < b. \tag{5}$$

We set $\mathbf{J}_{a+}^0 f = \mathbf{J}_{b-}^0 f = f$.

Let $\alpha > 0, F \in L_1([a,b], \mathbb{IR})$. The interval-valued function $t \to (x-t)^{\alpha-1}F(t)$ ($t \to (t-x)^{\alpha-1}F(t)$) is Lebesgue integrable on $[a,x]$ ($[x,b]$) for all $t \in [a,b]$. Hence

$$\mathcal{J}_{a+}^\alpha F(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} F(t)dt, \;\; x > a; \tag{6}$$

and

$$\mathcal{J}_{b-}^\alpha F(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} F(t)dt, \;\; x < b \tag{7}$$

exist for all $t \in [a,b]$. The formulas (6),(7) (see [5]) are called the left(respectively right) interval-valued Riemann-Liouville fractional integral of order α of the interval-valued-function F.

As consequence and by the theorem (see [15]) giving a relation between $\mathbb{IR}-$ integrable and $\mathcal{R}-$ Riemann-integrable, we have for all $x \in [a, b]$

$$\mathcal{J}_{a+}^{\alpha} F(x) = [\mathbf{J}_{a+}^{\alpha} \underline{F}(x), \mathbf{J}_{a+}^{\alpha} \overline{F}(x)]$$

and

$$\mathcal{J}_{b-}^{\alpha} F(x) = [\mathbf{J}_{b-}^{\alpha} \underline{F}(x), \mathbf{J}_{b-}^{\alpha} \overline{F}(x)].$$

Now let $\alpha_1, \alpha_2 \geq 1$ and $F : [a; b] \to \mathbb{IR}$ be a non-negative convex-i-v-function. Then for all $t \in [a, x]$, we have

$$F(t) \supseteq \frac{x-t}{x-a} F(a) + \frac{t-a}{x-a} F(x). \tag{8}$$

Multiplying (Scalar multiplication of interval) both side of (8) by $(x-t)^{\alpha_1 - 1}$ and integrating the resulting inequality with respect to t over $[a, x]$, we obtain

$$\begin{aligned}\int_a^x (x-t)^{\alpha_1-1} F(t) dt \supseteq \frac{1}{x-a} \times \\ \left[F(a) \int_a^x (x-t)^{\alpha_1} dt + F(x) \int_a^x (t-a)(x-t)^{\alpha_1-1} dt \right] \\ = (x-a)^{\alpha_1} \left[\frac{1}{\alpha_1+1} F(a) + B(\alpha_1, 2) F(x) \right],\end{aligned}$$

Therefore, in view of the definition of the interval-valued left sided RiemannLiouville fractional integrals, we get

$$\Gamma(\alpha_1) \mathcal{J}_{a+}^{\alpha_1} F(x) \supseteq (x-a)^{\alpha_1} \left[\frac{1}{\alpha_1+1} F(a) + B(\alpha_1, 2) F(x) \right]. \tag{9}$$

Now we consider the i-v function F on the interval $[x, b], x \in (a, b)$. Similarly,we have

$$\Gamma(\alpha_2) \mathcal{J}_{b-}^{\alpha_2} F(x) \supseteq (b-x)^{\alpha_2} \left[\frac{1}{\alpha_2+1} F(b) + B(\alpha_2, 2) F(x) \right], \tag{10}$$

by adding (9) and (10),we obtain

$$\begin{aligned}\Gamma(\alpha_1) \mathcal{J}_{a+}^{\alpha_1} F(x) + \Gamma(\alpha_2) \mathcal{J}_{b-}^{\alpha_2} F(x) \supseteq \\ [B(\alpha_1, 2)(x-a)^{\alpha_1} + B(\alpha_2, 2)(b-x)^{\alpha_2}] F(x) \\ + \frac{(x-a)^{\alpha_1}}{1+\alpha_1} F(a) + \frac{(b-x)^{\alpha_2}}{1+\alpha_2} F(b).\end{aligned}$$

This allows the following result.

Theorem 1 Let $\alpha_1, \alpha_2 \geq 1$. Let $F : [a; b] \to \mathbb{IR}$ be a non-negative convex interval-valued function for all $x \in (a, b)$, $F \in L_1([a, b], \mathcal{IR})$. Then

$$\begin{aligned}\Gamma(\alpha_1) \mathcal{J}_{a+}^{\alpha_1} F(x) + \Gamma(\alpha_2) \mathcal{J}_{b-}^{\alpha_2} F(x) \supseteq \\ [B(\alpha_1, 2)(x-a)^{\alpha_1} + B(\alpha_2, 2)(b-x)^{\alpha_2}] F(x) \\ + \frac{(x-a)^{\alpha_1}}{1+\alpha_1} F(a) + \frac{(b-x)^{\alpha_2}}{1+\alpha_2} F(b)\end{aligned} \tag{11}$$

holds .In particular if $\alpha_1 = \alpha_2 = \alpha$,then

$$\begin{aligned}\Gamma(\alpha) \left[\mathcal{J}_{a+}^{\alpha} F(x) + \mathcal{J}_{b-}^{\alpha} F(x) \right] \supseteq B(\alpha, 2) [(x-a)^{\alpha} + (b-x)^{\alpha}] F(x) \\ + \frac{(x-a)^{\alpha} F(a) + (b-x)^{\alpha} F(b)}{\alpha+1}.\end{aligned}$$

Corollary 1. By setting $\alpha_1 = \alpha_2 = 1$ and $x = b$ or $x = a$, we have

$$\frac{1}{b-a}\int_a^b F(t)dt \supseteq \frac{F(a)+F(b)}{2}.$$

Corollary 2 By taking $\alpha_1 = \alpha_2 = 1$ and $x = \dfrac{a+b}{2}$, we have

$$\frac{1}{b-a}\int_a^b F(t)dt \supseteq F\left(\frac{a+b}{2}\right) + \frac{F(a)+F(b)}{2}.$$

Corollary 3 If $F$is degenerate i.e $\underline{F}(t) = \overline{F}(t) = F(t)$, then Theorem 1 reduces to the result

$$\Gamma(\alpha_1)\mathbf{J}_{a+}^{\alpha_1} F(x) + \Gamma(\alpha_2)\mathbf{J}_{b-}^{\alpha_2} F(x) \leq \frac{(x-a)^{\alpha_1} + (b-x)^{\alpha_2}}{2} F(x) + \frac{(x-a)^{\alpha_1}F(a) + (b-x)^{\alpha_2}F(b)}{2}$$

holds . If $\alpha_1 = \alpha_2 = \alpha$ then

$$\Gamma(\alpha)(\mathbf{J}_{a+}^{\alpha} F(x) + \mathbf{J}_{b-}^{\alpha} F(x)) \leq \frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{2} F(x) + \frac{(x-a)^{\alpha}F(a) + (b-x)^{\alpha}F(b)}{2}$$

Remark 2. The result expressed in Corollary 3. is the same to that of theorem 1 and corollary 1 see [3].

Remark 3 If F is a concave interval valued function, for $x \in (a,b)$, then the set inclusion (11) is reversed .

Lemma 1 Let $0 < \lambda_1 < \lambda_2, A = [\underline{a}, \bar{a}] := (\hat{a}, \tilde{a}),\quad B = [\underline{b}, \bar{b}] = (\hat{b}, \tilde{b})$ such that $A \subseteq B$.

1. If $\lambda_1\hat{a} > \lambda_2\hat{b}$, *then* $\lambda_1 A \subset \lambda_2 B$.
2. If $\lambda_1\hat{a} \leq \lambda_2\hat{b}$, two cases
   - (a) If $-1 < \gamma_{\lambda_1 A\lambda_2 B} < 1$ *then* $\lambda_1 A \not\subset \lambda_2 B$.
   - (b) If $\gamma_{\lambda_1 A\lambda_2 B} \geq 1$, *then* $\lambda_1 A \subset \lambda_2 B$.

Proof. Since $A = [\underline{a}, \bar{a}] \subseteq B = [\underline{b}, \bar{b}]$, then $\overline{a} \leq \bar{b}$, $\underline{b} \leq \underline{a}$ it follows that $\lambda_1\overline{a} \leq \lambda_2\overline{b}$ but $\lambda_2\underline{b} \leq \lambda_1\underline{a}$ does not hold.

1. Suppose that $\lambda_1\hat{a} > \lambda_2\hat{b}$, that is

$$\lambda_2\underline{b} - \lambda_1\underline{a} \leq \lambda_1\overline{a} - \lambda_2\overline{b} \leq 0$$

   which leads to $\lambda_2\underline{b} \leq \lambda_1\underline{a}$ consequently $\lambda_1 A \subset \lambda_2 B$.
2. If $\lambda_1\hat{a} \leq \lambda_2\hat{b}$, two cases
   - (a) If $-1 < \gamma_{\lambda_1 A\lambda_2 B} < 1$ then $\gamma_{\lambda_1 A\lambda_2 B} + 1 > 0, \gamma_{\lambda_1 A\lambda_2 B} - 1 < 0$ from (1) we deduce that $\lambda_1 A \not\subset \lambda_2 B$.
   - (b) If $\gamma_{\lambda_1 A\lambda_2 B} \geq 1$, *then* $\gamma_{\lambda_1 A\lambda_2 B} \pm 1 \leq 0$. From (1) it follows that $\lambda_1 A \subset \lambda_2 B$.

Theorem 2 Let $0 < \alpha_1, \alpha_2 \leq 1$. Let $F = [\underline{F}, \bar{F}] = (\hat{F}, \tilde{F}) : [a;b] \to \mathbb{IR}_{\mathbb{I}}$ be a non-negative convex interval-valued function for all $x \in (a,b)$, $F \in L_1([a,b], \mathbb{IR})$ satisfying for all $\lambda \in [0,1], t \in [a,x]$

$$(x-t)^{\alpha_1 - 1}\hat{F}(t) \leq (x-a)^{\alpha_1 - 1}(\lambda \hat{F}(a) + (1-\lambda)\hat{F}(x)) \tag{12}$$

and for all $t \in [x,b]$,

$$(t-x)^{\alpha_2 - 1}\hat{F}(t) \leq (b-x)^{\alpha_2 - 1}(\lambda \hat{F}(b) + (1-\lambda)\hat{F}(x)). \tag{13}$$

Then

$$\begin{aligned}\Gamma(\alpha_1)\mathcal{J}_{a+}^{\alpha_1} F(x) + \Gamma(\alpha_2)\mathcal{J}_{b-}^{\alpha_2} F(x) \supseteq & \frac{(x-a)^{\alpha_1} + (b-x)^{\alpha_2}}{2} F(x) \\ & + \frac{(x-a)^{\alpha_1}F(a) + (b-x)^{\alpha_2}F(b)}{2}\end{aligned} \tag{14}$$

holds .If $\alpha_1 = \alpha_2 = \alpha$, then

$$\begin{aligned}\Gamma(\alpha)\left[\mathcal{J}_{a+}^{\alpha} F(x) + \mathcal{J}_{b-}^{\alpha} F(x)\right] \supseteq & \frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{2} F(x) \\ & + \frac{(x-a)^{\alpha}F(a) + (b-x)^{\alpha}F(b)}{2}.\end{aligned}$$

Proof. For all $x \in [a,b]$, for all $t \in [a,x]$ and $0 < \alpha_1 \leq 1$ the following inequality holds

$$(x-t)^{\alpha_1 - 1} \geq (x-a)^{\alpha_1 - 1}. \tag{15}$$

Since F is convex therefore for all $t \in [a,x]$, we have

$$F(t) \supseteq \frac{x-t}{x-a}F(a) + \frac{t-a}{x-a}F(x). \tag{16}$$

By multiply (Scalar multiplication of interval) (15), (16) takin in to acount of (12), and integrating the result with respect to t over $[a,x]$, we obtain

$$\begin{aligned}\int_a^x (x-t)^{\alpha_1 - 1}F(t)dt \supseteq & \frac{(x-a)^{\alpha_1 - 1}}{x-a} \\ & \times \left[F(a)\int_a^x (x-t)dt + F(x)\int_a^x (t-a)dt\right] \\ & = (x-a)^{\alpha_1}\frac{F(x)+F(a)}{2},\end{aligned}$$

Consequently, we have

$$\Gamma(\alpha_1)\mathcal{J}_{a+}^{\alpha_1}F(x) \supseteq (x-a)^{\alpha_1}\frac{F(x)+F(a)}{2}. \tag{17}$$

And similarly by reasoning for all $t \in [x,b]$, we get

$$\Gamma(\alpha_2)\mathcal{J}_{b-}^{\alpha_2}F(x) \supseteq (b-x)^{\alpha_2}\frac{F(x)+F(b)}{2}. \tag{18}$$

By adding (17) and (18),we obtain the desired inequality.

Corollary 4. By taking $\alpha_1 = \alpha_2 = 1$ and $x = b$ or $x = a$, we have

$$\frac{1}{b-a}\int_a^b F(t)dt \supseteq \frac{F(a)+F(b)}{2}.$$

Corollary 5. If $\alpha_1 = \alpha_2 = 1$ and $x = \dfrac{a+b}{2}$, then

$$\frac{1}{b-a}\int_a^b F(t)dt \supseteq F\left(\frac{a+b}{2}\right) + \frac{F(a)+F(b)}{2}.$$

Corollary 6. If $F$is degenerate i.e $\underline{F}(t) = \overline{F}(t) = F(t)$, then Theorem 2. reduces to the result

$$\begin{aligned}\Gamma(\alpha_1)\mathbf{J}^{\alpha_1}_{a+} F(x) + \Gamma(\alpha_2)\mathbf{J}^{\alpha_2}_{b-} F(x) \leq \\ \frac{(x-a)^{\alpha_1} + (b-x)^{\alpha_2}}{2} F(x) + \\ \frac{(x-a)^{\alpha_1}F(a) + (b-x)^{\alpha_2}F(b)}{2}\end{aligned}$$

holds . If $\alpha_1 = \alpha_2 = \alpha$ then

$$\begin{aligned}\Gamma(\alpha)(\mathbf{J}^{\alpha}_{a+} F(x) + \mathbf{J}^{\alpha}_{b-} F(x)) \leq \\ \frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{2} F(x) + \\ \frac{(x-a)^{\alpha}F(a) + (b-x)^{\alpha}F(b)}{2}\end{aligned}$$

Theorem 3 Let $\alpha_1, \alpha_2 \geq 0$. Let $F = [\underline{F}, \overline{F}] : [a;b] \to \mathbb{IR}$ be a non-negative interval-valued function such that the real valued functions $\underline{F}, \overline{F}$ are differentiable .If the $gH-$ derivative F' is non-negative convex interval-valued function, then

$$\begin{aligned}\Gamma(\alpha_1)\left(\mathcal{J}^{\alpha_1}_{a+}F(x)\right) + \Gamma(\alpha_2)\left(\mathcal{J}^{\alpha_2}_{b-}\right)F(x) - \\ ((x-a)^{\alpha_1}F(a) + (b-x)^{\alpha_2}F(b)) \supseteq \\ F'(x)\left(B(\alpha_2+1,2)(b-x)^{\alpha_2+1} + B(\alpha_1+1,2)(x-a)^{\alpha_1+1}\right) + \\ \frac{(b-x)^{\alpha_2+1}F'(b)}{\alpha_2+2} + \frac{(x-a)^{\alpha_1+1}F'(a)}{\alpha_1+2}.\end{aligned} \tag{19}$$

holds. If $\alpha_1 = \alpha_2 = \alpha$, then

$$\begin{aligned}\Gamma(\alpha)\left(\mathcal{J}^{\alpha}_{a+} F(x) + \mathcal{J}^{\alpha}_{b-} F(x)\right) - \\ ((x-a)^{\alpha}F(a) + (b-x)^{\alpha}F(b)) \supseteq \\ B(\alpha+1,2)\left((b-x)^{\alpha+1} + (x-a)^{\alpha+1}\right)F'(x) + \\ \left(\frac{(b-x)^{\alpha+1}F'(b) + (x-a)^{\alpha+1}F'(a)}{\alpha+2}\right).\end{aligned}$$

Proof. Let $\alpha_1 \geq 0$ and $x \in (a,b)$, then for all $t \in [a,x]$, F' existts and theremore $F' = [\underline{F}', \overline{F}']$ (see[15]). Since F' is convex, we have

$$F'(t) \supseteq \frac{x-t}{x-a}F'(a) + \frac{t-a}{x-a}F'(x) \tag{20}$$

Multiplying (20) by $(x-t)^{\alpha_1}$ and integrating the resulting inequality with respect to t over $[a,x]$, we get

$$\int_a^x (x-t)^{\alpha_1} F'(t)dt \supseteq \frac{1}{x-a} \times \left[F'(a) \int_a^x (x-t)^{\alpha_1+1} dt + F'(x) \int_a^x (t-a)(x-t)^{\alpha_1} dt \right] = (x-a)^{\alpha_1+1} \left[\frac{1}{\alpha_1+2} F'(a) + B(\alpha_1+1,2) F'(x) \right],$$

since

$$\int_a^x (x-t)^{\alpha_1} F'(t)dt = -F(a)(x-a)^{\alpha_1} + \Gamma(\alpha_1) \mathcal{J}_{a+}^{\alpha_1} F(x),$$

hence

$$\Gamma(\alpha_1) \mathcal{J}_{a+}^{\alpha_1} F(x) - F(a)(x-a)^{\alpha_1} \supseteq (x-a)^{\alpha_1+1} \times \left[\frac{1}{\alpha_1+2} F'(a) + B(\alpha_1+1,2) F'(x) \right]. \tag{21}$$

On the other hand, using the convexity of F', for all $t \in [x,b]$, we have

$$F'(t) \supseteq \frac{t-x}{b-x} F'(b) + \frac{b-t}{b-x} F'(x). \tag{22}$$

Also,multiply (22) by $(t-x)^{\alpha_2}$, we get

$$\Gamma(\alpha_2) \mathcal{J}_{b-}^{\alpha_2} F(x) - F(b)(b-x)^{\alpha_2} \supseteq (b-x)^{\alpha_2+1} \times \left[\frac{1}{\alpha_2+2} F'(b) + B(\alpha_2+1,2) F'(x) \right]. \tag{23}$$

from (21) and (23),we obtain the required estimation.

Corollary 7. Taking $x = \frac{a+b}{2}$ and $\alpha_1 = \alpha_2 = 1$ in (19), this reduces to

$$\frac{2}{b-a} \int_a^b F(t)dt - (F(b) + F(a)) \supseteq \frac{b-a}{3} \left(F'\left(\frac{a+b}{2}\right) + \frac{F'(b) + F'(a)}{2} \right).$$

An interval valued-function $F = [\underline{F}, \overline{F}]$ is said to be symmetric about $\frac{a+b}{2}$, if we have $F(a+b-x) = F(x)$.

Remark 4. F symmetric about $\frac{a+b}{2} \Leftrightarrow \underline{F}, \overline{F}$ are symmetric about $\frac{a+b}{2}$.

Lemma 2. Let $F : \mathbb{I} = [a,b] \to \mathbb{IR}$, be a convex i-v-function.If F is symmetric about $\frac{a+b}{2}$, then

$$F\left(\frac{a+b}{2}\right) \supseteq F(x), \ \ x \in [a,b].$$

Proof. We have

$$\frac{a+b}{2} = \frac{1}{2}\left(b\frac{x-a}{b-a} + a\frac{b-x}{b-a} \right) + \frac{1}{2}\left(a\frac{x-a}{b-a} + b\frac{b-x}{b-a} \right)$$

by convexity of F ,we have

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &\supseteq \frac{1}{2}\left[F\left(b\frac{x-a}{b-a}+a\frac{b-x}{b-a}\right)\right] \\ &+\frac{1}{2}\left[F\left(a\frac{x-a}{b-a}+b\frac{b-x}{b-a}\right)\right] \\ &=\frac{1}{2}F(x)+\frac{1}{2}F(a+b-x) \\ &=F(x). \end{aligned}$$

Theorem 4 Let $\alpha_1,\alpha_2 \geq 1$. Let $F:[a,b]\to \mathbb{IR}$, be a convex interval-valued function and $F\in L_1([a,b],\mathbb{IR})$,. If F is symmetric about $\frac{a+b}{2}$, then

$$\begin{gathered} \left[\frac{(b-a)^{\alpha_1+1}}{(\alpha_1+1)}+\frac{(b-a)^{\alpha_2+1}}{(\alpha_2+1)}\right]F\left(\frac{a+b}{2}\right)\supseteq \\ \Gamma(\alpha_1+1)\mathcal{J}_{b-}^{\alpha_1+1}F(a)+\Gamma(\alpha_2+1)\mathcal{J}_{a+}^{\alpha_2+1}F(b)\supseteq \\ \left[\frac{(b-a)^{\alpha_1+1}}{\alpha_1+2}+B(\alpha_1+1,2)(b-a)^{\alpha_1+1}\right]F(a)+ \\ \left[\frac{(b-a)^{\alpha_2+1}}{\alpha_2+2}+B(\alpha_2+1,2)(b-a)^{\alpha_2+1}\right]F(b) \end{gathered} \tag{24}$$

holds.If $\alpha_1=\alpha_2=\alpha$, then

$$\begin{aligned} \frac{1}{(\alpha+1)}F\left(\frac{a+b}{2}\right)&\supseteq \\ \Gamma(\alpha+1)\frac{\mathcal{J}_{b-}^{\alpha+1}F(a)+\mathcal{J}_{a+}^{\alpha+1}F(b)}{2(b-a)^{\alpha+1}}&\supseteq \\ \left[\frac{1}{\alpha+2}+B(\alpha+1,2)\right]\frac{F(a)+F(b)}{2}&. \end{aligned}$$

Proof. Let $\alpha_1,\alpha_2\geq 1$. Since F is convex, we have for $t\in[a,b]$

$$F(t)\supseteq \frac{t-a}{b-a}F(a)+\frac{b-t}{b-a}F(b). \tag{25}$$

Multiplying (25) by $(t-a)^{\alpha_1}$ and integrating the result with respect to t over $[a,b]$, we obtain

$$\Gamma(\alpha_1+1)\mathcal{J}_{b-}^{\alpha_1+1}F(a)\supseteq (b-a)^{\alpha_1+1}\left[\frac{1}{\alpha_1+2}F(a)+B(\alpha_1+1,2)F(b)\right]. \tag{26}$$

Now multiplying (25) by $(b-t)^{\alpha_2}$ and integrating with respect to t over $[a,b]$, we obtain

$$\Gamma(\alpha_2+1)\mathcal{J}_{a+}^{\alpha_2+1}F(b)\supseteq (b-a)^{\alpha_2+1}\left[\frac{1}{\alpha_2+2}F(b)+B(\alpha_2+1,2)F(a)\right]. \tag{27}$$

By adding (26) and (27), we get

$$\begin{aligned}&\Gamma(\alpha_1+1)\mathcal{J}_{b-}^{\alpha_1+1}F(a)+\Gamma(\alpha_2+1)\mathcal{J}_{a+}^{\alpha_2+1}F(b)\supseteq\\&\left[\frac{(b-a)^{\alpha_1+1}}{\alpha_1+1}+B(\alpha_1+1,2)(b-a)^{\alpha_1+1}\right]F(a)+\\&\left[\frac{(b-a)^{\alpha_2+1}}{\alpha_2+1}+B(\alpha_2+1,2)(b-a)^{\alpha_2+1}\right]F(b).\end{aligned}\tag{28}$$

Using lemma 2.,we have

$$F\left(\frac{a+b}{2}\right)(t-a)^{\alpha_1}\supseteq F(t)(t-a)^{\alpha_1},\ F\left(\frac{a+b}{2}\right)(b-t)^{\alpha_2}\supseteq F(t)(b-t)^{\alpha_2}.\tag{29}$$

Integrating (29) with respect to t over $[a,b]$, we obtain

$$\frac{(b-a)^{\alpha_1+1}}{\alpha_1+1}F\left(\frac{a+b}{2}\right)\supseteq\Gamma(\alpha_1+1)\,\mathcal{J}_{b-}^{\alpha_1+1}F(a).\tag{30}$$

and

$$\frac{(b-a)^{\alpha_2+1}}{\alpha_2+1}F\left(\frac{a+b}{2}\right)\supseteq\Gamma(\alpha_2+1)\mathcal{J}_{a+}^{\alpha_2+1}F(b).\tag{31}$$

Adding (30) and (31), we have

$$\begin{aligned}&\left[\frac{(b-a)^{\alpha_1+1}}{\alpha_1+1}+\frac{(b-a)^{\alpha_2+1}}{\alpha_2+1}\right]F\left(\frac{a+b}{2}\right)\supseteq\\&\Gamma(\alpha_1+1)\mathcal{J}_{b-}^{\alpha_1+1}F(a)+\Gamma(\alpha_2+1)\mathcal{J}_{a+}^{\alpha_2+1}F(b).\end{aligned}\tag{32}$$

Combining (28) and (32),we obtain the desired inequality.

Theorem 5 Let $0<\alpha_1,\alpha_2\le 1$. Let $F=[\underline{F},\bar{F}]=(\hat{F},\tilde{F}):[a;b]\to\mathbb{IR}$ be a non-negative convex interval-valued function , $F\in L_1([a,b],\mathbb{IR})$ satisfying for all $t\in[a,b],\lambda\in[0,1]$

$$(b-t)^{\alpha_1-1}\hat{F}(t)\le(b-a)^{\alpha_1-1}(\lambda\hat{F}(a)+(1-\lambda)\hat{F}(b)),$$

and

$$(t-a)^{\alpha_2-1}\hat{F}(t)\le(b-a)^{\alpha_2-1}(\lambda\hat{F}(a)+(1-\lambda)\hat{F}(b)).$$

Then

$$\begin{aligned}&\left[\frac{(b-a)^{\alpha_1+1}}{\alpha_1+1}+\frac{(b-a)^{\alpha_2+1}}{\alpha_2+1}\right]F\left(\frac{a+b}{2}\right)\supseteq\\&\Gamma(\alpha_1+1)\mathcal{J}_{b-}^{\alpha_J+1}F(a)+\Gamma(\alpha_2+1)\mathcal{J}_{a+}^{\alpha_2+1}F(b)\supseteq\\&\left((b-a)^{\alpha_1+1}+(b-a)^{\alpha_2+1}\right)\frac{F(a)+F(b)}{2}\end{aligned}\tag{33}$$

holds .In particular if $\alpha_1=\alpha_2=\alpha$, then

$$\begin{aligned}&\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}F\left(\frac{a+b}{2}\right)\supseteq\\&\frac{\mathcal{J}_{b-}^{\alpha+1}F(a)+\mathcal{J}_{a+}^{\alpha+1}F(b)}{2}\supseteq\\&\frac{(b-a)^{\alpha+1}(F(a)+F(b))}{2\Gamma(\alpha+1)}.\end{aligned}\tag{34}$$

Proof. The proof is similar to that of Theorem 2. Using lemma 2. taking into account the convexity of F and the fact that for all $t \in [a, b]$ and $0 < \alpha_1, \alpha_2 \leq 1$

$$(t-a)^{\alpha_1 - 1} \geq (b-a)^{\alpha_1 - 1}, (b-t)^{\alpha_2 - 1} \geq (b-a)^{\alpha_2 - 1}. \tag{35}$$

Corollary 8 ([5],remark 2.6) If $\alpha \to 0$, then from above inequality, we get Hermite-Hadamard's inequality for interval-valued function

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\int_a^b F(t)dt \supseteq \frac{F(a)+F(b)}{2}.$$

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