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ON THE TRANSCENDENTAL SOLUTION OF THE FERMAT TYPE Q-SHIFT EQUATION

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Abstract. In Nevanlinna's value distribution theory we considering some basic terms like $T(r,f), N(r,f)$, $m(r,f)$ etc., and let $f^m(z)+q(z)[f^n\Delta_\eta^q f]^{(k)}=p(z)$ be a non-linear q-th order difference equation and $f(z)$ be a transcendental meromorphic function with finite order m,n and k be a positive integers such that $m\geq(q+1)(nk+k+2)+3$, $p(z)$ be a meromorphic function satisfying $\overline{N}\left(r,\frac{1}{p(z)}\right)=S(r,f)$. The $q(z)$ be a non-zero meromorphic function satisfying that $T(r,q(z))=S(r,f)$, then $f(z)$ is not a solution of the non-linear q-th order difference equation. In this paper, we mainly investigate the uniqueness result of transcendental Fermat type q-shift equation by considering q-th order difference equation. Our result improves the results due to Abhijit Banerjee and Tania Biswas. In addition to that the example is exhibited to validate certain claims and justification of our main result.

1. **Introduction and Definitions**

We assume that the readers are familiar with the basic notations of Nevanlinna's value distribution theory of meromorphic function f, terms like $T(r,f), N(r,f), m(r,f)$ etc., we refer to [5, 11, 8, 2]. The notation $S(r,f)$, is defined to be any quantity logarithmic measure. The order of f is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

Let c be a nonzero complex constant, and let $f(z)$ be a meromorphic function. The shift operator of $f(z)$ is denoted by $f(z+c)$. Also, we use the notations $\Delta_c f$ and $\Delta_c^k f$ to denote the difference and k-th order difference operators of f, which are respectively defined as

$$\Delta_c f=f(z+c)-f(z),\Delta_c^k f=\Delta_c(\Delta_c^{k-1}f(z)),k\in\mathbb{N},k\geq 2.$$

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In this article, by linear q-th order difference operator $\Delta_\eta^q f(z)$ is defined by $\Delta_\eta^q f(z) = \Delta_\eta^{q-1}(\Delta_\eta f(z))$, where $q(\geq 2) \in \mathbb{N}$ and $\eta \in \mathbb{C}\{0\}$, while the difference polynomial of difference operator is given by $L(\Delta_\eta f) = \sum_{i=1}^q a_i \Delta_\eta^i f$, where $a_i (i = 1, 2, ..., q)$ are nonzero constants.

We can also deduce that,

$$\Delta_\eta^q f = \sum_{i=1}^{q} \binom{q}{i} f(z + (q-i)\eta). \tag{1.1}$$

For the existence of solutions of non-linear q-shift equation, in 2011, Qi [10] obtained the following theorems:
Theorem A. [10] Let $q(z), p(z)$ be polynomials and let n, m be distinct positive integers. Then the equation

$$f^m(z) + q(z) f(z+c)^n = p(z) \tag{1.2}$$

has no transcendental entire solutions of finite order.

In 2015, Qi-Liu-Yang [9] obtained the meromorphic variant of Theorem A and improved this as follows:
Theorem B. [9] Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+4, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of equation

$$f^m(z) + q(z) f(z+c)^n = p(z) \tag{1.3}$$

Theorem C. [9] Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+2, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of equation 1.3.

In 2021, A. Banerjee and T. Biswas [1] investigated the following result.
Theorem D. [1] Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq (\tau+1)(n+2)+2, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of the non-linear c-shift equation

$$f^m(z) + q(z)(L_c(z, f))^n = p(z). \tag{1.4}$$

In this article we extend Theorem-D at the expense of replacing $(L_c(z, f))^n$ by $[f^n \Delta_\eta^q f]^{(k)}$.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function with finite order, m, n and k be a positive integers such that $m \geq (q+1)(nk+k+2)+3, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of the non-linear q-th order difference equation*

$$f^m(z) + q(z)[f^n \Delta_\eta^q f]^{(k)} = p(z). \tag{1.5}$$

Corollary 1.1. *Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+2, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of equation (1.5).*

The next example show that if the condition $m \geq n+2$ is omitted then the equation (1.5) can admit a transcendental entire solution.
Considering $n=1, k=1$ and $m=1$ we have the following example.

Example 1.1. The function $f(z) = ze^{\frac{\pi iz}{\eta}}$ satisfies the equation $f(z) + \frac{1}{z+1}[f^n \Delta_\eta^q f] = \frac{z(z+2)}{z+1} e^{\frac{\pi iz}{\eta}}$, where the coefficients of $\Delta_\eta^q f$ is chosen such that they satisfy simultaneously the equations

$$\begin{cases} a_0 - a_1 + a_2 - ... + (-1)^q a_q = 1, \\ -a_1 + 2a_2 - 3a_3 - ... + q(-1)^q a_q = 0. \end{cases}$$

2. Lemmas

To proceed further we require the following lemmas:

Lemma 2.1. [3] *Let $f(z)$ be a finite order meromorphic function and $\varepsilon > 0$, then $T(r, f(z+c)) = T(r, f(z)) + o(r^{\sigma-1+\varepsilon}) + O(\log r)$ and $\sigma(f(z+c)) = \sigma(f(z))$. Thus, if $f(z)$ is a transcendental meromorphic function with finite order, then we know $T(r, f(z+c)) = T(r, f) + S(r, f)$.*

Lemma 2.2. [4] *Let $f(z)$ be a meromorphic function with finite order, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then $m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right) = S(r, f)$.*

Lemma 2.3. [6] *Let f be a non-constant meromorphic function with finite order and $c \in \mathbb{C}$. Then*

$$N(r, \infty; f(z+c)) \leq N(r, \infty; f(z)) + S(r, f),$$
$$N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f).$$

3. Main Result

Proof of Theorem 1.1. Suppose by contradiction that $f(z)$ is a transcendental meromorphic function with finite order satisfying equation (1.5).
If $T(r, p(z)) = S(r, f)$, then applying Lemma 2.1 to equation (1.5), we have

$$\begin{aligned} m.T(r, f) &= T(r, f^m) \\ &= T(r, p(z) - q(z)[f^n \Delta_\eta^q f]^{(k)}) \\ &= T(r, [f^n \Delta_\eta^q f]^{(k)}) + S(r, f) \\ &= kT(r, f^n) + kT(r, \Delta_\eta^q f) + S(r, f) \\ &\leq (nk + (q+1)k)T(r, f) + S(r, f), \end{aligned}$$

which contradicts the assumption that $m \geq (q+1)(nk+k+2)+3$.
If $T(r, p(z)) = S(r, f)$, differentiating equation (1.5), we get

$$(f^m)' + (q(z)[f^n \Delta_\eta^q f]^{(k)})' = p'(z). \tag{3.1}$$

Next dividing (3.1) by (1.5) we have

$$p'(z)[f^m(z) + q(z)[f^n \Delta_\eta^q f]^{(k)}] = p(z)[(f^m)' + (q(z)[f^n \Delta_\eta^q f]^{(k)})']$$

$$f^m(z) = \frac{\frac{p'(z)}{p(z)}q(z)[f^n\Delta_\eta^q f]^{(k)} - (q(z)[f^n\Delta_\eta^q f]^{(k)})'}{\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}}. \tag{3.2}$$

First observe that $\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}$ cannot vanish identically. Indeed, if $\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)} \equiv 0$, then we get $p(z) = \beta f^m(z)$, where β is a non-zero constant. Substituting the above equality to equation (1.5), we have $q(z)[f^n\Delta_\eta^q f]^{(k)} = (\beta - 2)f^m(z)$. From Lemma 2.1 and above equation, we immediately see as above that $mT(r,f) \leq (nk + (q+1)k)T(r,f) + S(r,f)$, which is a contradiction to $m \geq (q+1)(nk+k+2)+3$. From equation (3.2), we know

$$\begin{aligned} mT(r,f) &= T(r,f^m) \\ &\leq m(r, q(z)[f^n\Delta_\eta^q f]^{(k)}) + m\left(r, \frac{p'(z)}{p(z)} - \frac{([f^n\Delta_\eta^q f]^{(k)})'}{[f^n\Delta_\eta^q f]^{(k)}}\right) \\ &+ N\left(r, \frac{p'(z)}{p(z)}q(z)[f^n\Delta_\eta^q f]^{(k)} - (q(z)([f^n\Delta_\eta^q f]^{(k)}))'\right) \end{aligned}$$

$$+m\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + S(r,f). \tag{3.3}$$

As Lemma 2.1 together with equation (3.2) implies that

$$(m - nk - (q+1)k)T(r,f) + S(r,f) \leq T(r,p(z)) \leq (m + nk + (q+1)k)T(r,f) + S(r,f),$$

we conclude that

$$S(r,p(z)) = S(r,f). \tag{3.4}$$

Applying Lemmas 2.1, 2.2 and (3.4) to equation (3.3), we obtain that
$mT(r,f) \leq k(n+1)m(r,f)+$

$$N\left(r, \frac{p'(z)}{p(z)}q(z)[f^n\Delta_\eta^q f]^{(k)} - (q(z)([f^n\Delta_\eta^q f]^{(k)}))'\right) + N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + S(r,f). \tag{3.5}$$

Let

$$\Phi(z) = \frac{p'(z)}{p(z)}q(z)[f^n\Delta_\eta^q f]^{(k)} - (q(z)([f^n\Delta_\eta^q f]^{(k)}))' \tag{3.6}$$

and

$$\Psi(z) = \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}. \tag{3.7}$$

First of all, we deal with $N(r,\Phi(z))$. From (1.5) and (3.6), we know the poles of $\Phi(z)$ are at the zeros of $p(z)$ and at the poles of $f(z), f(z+(q-i)\eta), (i = 1,2,...,q)$ and $q(z)$. Poles of $p(z)$ will not contribute towards the poles of $\Phi(z)$ as from the equation (1.5) we know that the poles of $p(z)$ should be at the poles of $f(z), f(z+(q-i)\eta), (i = 1,2,...,q)$ and $q(z)$. We note that $T(r,q(z)) = S(r,f)$.

If z_0 is a zero of $p(z)$ then by (3.6), z_0 is at most a simple pole of $\Phi(z)$. If z_0 is a pole of $f(z)$ of multiplicity t but not a pole of $f(z+(q-i)\eta), (i = 1,2,...,q)$, then z_0 will be a pole of $\Phi(z)$ of multiplicity at most $tnk+1$. Next suppose z_1 be any pole of $f(z)$ of multiplicity t_0 and a pole of at least one $f(z+(q-i)\eta), (i = 1,2,...,q)$, of multiplicity $t_i \geq 0$. Then z_1 may or may not be a pole of $f^n\Delta_\eta^q f$. From the above arguments and our assumption, we conclude that

$$N(r,\Phi) \leq \overline{N}\left(r, \frac{1}{p(z)}\right) + kN(r,f^n) + kN(r,\Delta_\eta^q f) + \overline{N}(r,f) + \overline{N}(r,\Delta_\eta^q f) + S(r,f)$$

$$N(r,\Phi) \leq nkN(r,f) + kN(r,\Delta_\eta^q f) + ((q+1)+1)\overline{N}(r,f) + S(r,f). \tag{3.8}$$

Next, we turn our attention towards the poles of $\Psi(z)$ are at the zeros of $p(z)$ and $f(z)$ and at the poles of $f(z), f(z+(q-i)\eta), (i = 1, 2, ..., q)$. If z_0 is a zero of $p(z)$, zero of $f(z)$, or pole of $f(z), f(z+(q-i)\eta), (i = 1, 2, ..., q)$, then by (3.7) we know z_0 will be at most a simple pole of $\Psi(z)$. If z_0 is a pole of $f(z)$ but not a pole of $f(z), f(z+(q-i)\eta), (i = 1, 2, ..., q)$, then by the Laurent expansion of $\Psi(z)$ at z_0, we obtain that $\Psi(z)$ is analytic at z_0. Therefore, from our assumption and the discussions above, we know

$$N(r,\Psi) \leq \overline{N}\left(r, \frac{1}{p(z)}\right) + \overline{N}(r,f) + \overline{N}(r,\Delta_\eta^q f) + \overline{N}(r, \frac{1}{f}) + S(r,f)$$

$$N(r,\Psi) \leq \overline{N}(r,f) + \overline{N}(r,\Delta_\eta^q f) + \overline{N}(r, \frac{1}{f}) + S(r,f). \tag{3.9}$$

Using Lemma 2.3, from equations (3.5), (3.8) and (3.9) we have

$$\begin{aligned}
mT(r,f) &\leq (nk+k)m(r,f) + nkN(r,f) + kN(r,\Delta_\eta^q f) + ((q+1)+1)\overline{N}(r,f) \\
&+ \overline{N}(r,f) + \overline{N}(r,\Delta_\eta^q f) + \overline{N}(r, \frac{1}{f}) + S(r,f) \\
&\leq (nk+k)m(r,f) + nkN(r,f) + k(q+1)N(r,f) + ((q+1)+1)\overline{N}(r,f) + \overline{N}(r,f) \\
&+ (q+1)\overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) + S(r,f) \\
&\leq (nk+k)m(r,f) + ((q+1)k+nk)N(r,f) + ((q+1)+1)\overline{N}(r,f) + \overline{N}(r,f) + (q+1)\overline{N}(r,f) \\
&+ \overline{N}(r, \frac{1}{f}) + S(r,f) \\
&\leq \{(q+1)(nk+k+2)+2\}T(r,f) + S(r,f),
\end{aligned}$$

which contradicts the assumption that $m \geq (q+1)(nk+k+2)+3$. This completes the proof of the theorem.

4. Conclusion

When the $f(z)$ be a transcendental meromorphic function and $f^m(z)+q(z)[f^n\Delta_\eta^q f]^{(k)} = p(z)$ be a non-linear q-th order difference equation. Then there exists uniqueness result between the functions and it's conditions.

Declarations:

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this paper.

Human/animals participants The authors declare that there is no research involving human participants and/or animals in the contained of this paper.

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