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NEW DISCUSSION ON GLOBAL EXISTENCE AND ATTRACTIVITY OF MILD SOLUTIONS FOR NONAUTONOMOUS INTEGRODIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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ABSTRACT. This paper investigates a functional integral differential equation with state-dependent delay in Banach spaces. This equation's linear part depends on time and generates a linear evolution system. Using the resolvent operator and fixed-point methods theory, we formulate a new set of sufficient conditions for mild solutions of functional integral-differential equations with state-dependent delays. The next part of this study examines the attractiveness of mild solutions for the system under consideration. Finally, we give an example to illustrate the theoretical results.

1. INTRODUCTION

The focus of this investigation is the existence and attractivity of the mild solution to the following semi-linear evolution equation:

$$\begin{cases} z^{'}(t) = A(t)z(t) + \displaystyle\int_0^t B(t,s)z(s)ds + h(t, z_{\sigma(t,z_t)}), \quad t \in J = [0, \infty) \\ z(t) = \varphi(t) \in \mathcal{B}, \quad t \in (-\infty, 0], \end{cases} \tag{1}$$

where $z(\cdot)$ takes values in a Banach space $\mathbb{X}$, $A(t) : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ is a linear closed operator with domain $D(A)$ which is independent of t, $B(t,s) : D(B) \subset \mathbb{X} \longrightarrow \mathbb{X}$ is a closed linear operator with domain $D(B) \supset D(A)$, $h : J \times \mathcal{B} \longrightarrow \mathbb{X}$, $\sigma : J \times \mathcal{B} \longrightarrow \mathbb{R}$ are appropriate functions, $\mathcal{B}$ is an abstract phase space to be described later, and the history function $z_t : (-\infty, 0] \longrightarrow \mathbb{X}$, defined by $z_t(\theta) = z(t+\theta)$ for $\theta \in (-\infty, 0]$ is an element of $\mathcal{B}$.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years.

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Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [18], Kolmanovskii and Myshkis [22], Smith [25], and Wu [27], and the references therein. However, complicated situations in which the delay depends on unknown functions have been considered in recent years. These equations are frequently called equations with state-dependent delay, see, for instance, [13, 15]. Moreover, state-dependent delays are more prevalent and adequate in applications, some nice examples of state-dependent delay models are presented in [1,7] and the references therein. One of the important techniques to discuss these topics is the semigroup approach. In paper, [20] Hernández and McKibben have investigated the existence of mild solutions for the following abstract integrodifferential equations with state-dependent delay described by the form

$$\begin{cases} \dfrac{d}{dt}D(u_t) = AD(u_t) + F(t, u_{\sigma(t,u_t)}), \quad t \in J = [0, a] \\ u(t) = \varphi(t) \in \mathcal{B}, \quad t \in (-\infty, 0], \end{cases} \tag{2}$$

where $D\Psi = \Psi(0) - G(t, \Psi)$.

The nonlinear integrodifferential equation with resolvent operators was used as an abstract model of partial integrodifferential equations, which are found in many different kinds of physical events [16, 17]. The resolvent operator can be thought of as being analogous to the semigroup operator when it comes to abstract differential equations in Banach spaces. However, contrary to what is shown in [6, 23], the resolvent operator does not fulfill the requirements of the semigroup characteristics. Recently, there have been massive studies covering the existence and stability of solutions to partial integrodifferential equations with state-dependent delay. Bete et al. [5] proved the existence and attractivity of mild solutions for an integrodifferential system with state-dependent delay. Fall et al. [15] established the existence of mild solutions for a class of impulsive integrodifferential equations with state-dependent delay. Diop et al. [13] considered the existence and uniqueness of mild solutions, as well as controllability outcomes, for random integrodifferential equations with state-dependent.

Attractivity is a crucial concept in the theory of dynamical systems, initially articulated in his thesis by the Russian mathematician and engineer Lyapunov (see [24]). But, since its inception in 1892, this field has grown significantly. Many recent studies on the existence and attractivity of mild solutions for many sorts of differential problems in infinite dimensions spaces have been published. Various approaches were used to acquire results (for more details, see [2, 29]). However, it seems that there are few works concerned with the attractivity of evolution equations with state-dependent.

On the other hand, the linear component of (2) being dependent on the time t is something that happens quite often and is a fairly common occurrence. In point of fact, a significant number of partial functional differential equations can be rewritten as semilinear non-autonomous equations with the form $A = A(t)$. A significant amount of research has been conducted on the existence, asymptotic behavior, and controllability of deterministic nonautonomous partial functional differential equations with state-dependent delay. For an example, see the work that has been done on this topic [4] and the references therein.

However, very little is known about non-autonomous integrodifferential equations in abstract space, particularly for the case in which $A(t)$ is a family of unbounded operators on $\mathbb{X}$ with (common) dense domain such that it generates a linear evolution system. It is common knowledge that dealing with the equations describing non-autonomous evolution is significantly more challenging than dealing with the equations describing autonomous evolution. The research shown in paper [11] has given us ideas for how to do things here. That is, we assume that $\{A(t) : t \geq 0\}$ is a family of unbounded linear operators on $\mathbb{X}$ such that it generates a linear evolution system. So, we will use the theory of the linear evolution system, the theory of fixed points, and the resolvent operator in the sense developed by Grimmer to talk about the global existence and attractivity of mild solutions of equation (1), which is a topic that hasn't been looked into in depth yet.

The rest of this paper is organized as follows: In Section 2, we give the theoretical concepts related to the resolvent operators and state-dependent delay. In Section 3, we prove the existence of mild solutions of system (1) by means of resolvent operator theory and Schauder's fixed point theorem. Section 4 is devoted to the attractivity of the solution. In the end, Section 5 offers a theoretical application to help our discussion be more productive.

2. Preliminaries

In this section, we will briefly review some of the notations, definitions, and technical facts used throughout this article. Let $\mathbb{X}$ be a real Banach space with the norm $\|\cdot\|$ and $BC(J, \mathbb{X})$ the Banach space of all bounded and continuous functions z mapping J into $\mathbb{X}$ with usual supremum norm

$$\|z\| = \sup\{\|z(t)\| : t \in J\}.$$

Let $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ denote the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$, where $\mathbb{Y}$ is a real Banach space. When $\mathbb{X} = \mathbb{Y}$ we write $\mathcal{L}(\mathbb{X})$ instead of $\mathcal{L}(\mathbb{X}, \mathbb{X})$. A measurable function $z : J \longrightarrow \mathbb{X}$ is Brochner integrable if and only if $|z|$ is Lebesgue integrable. (For the Bochner integrable properties, see the classical monograph of Yosida [28]). Let $L^1(J, \mathbb{X})$ denote the Banach space of measurable functions $z : J \longrightarrow \mathbb{X}$ which are Bochner integrable equipped with the norm

$$\|z\|_{L^1(J,\mathbb{X})} = \int_0^{\infty} |z(t)| dt.$$

Let $\mathcal{H}$ be the space defined by

$$\mathcal{H} = \{z : \mathbb{R} \longrightarrow \mathbb{X} \text{ such that } z|J \in BC(J, \mathbb{X}) \text{ and } z_0 \in \mathcal{B}\},$$

where $z|J$ is the restriction of z to J.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [19] and follow the terminology used in [21]. Thus,$(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$, and satisfying the following axioms:

(P_1): If $z : (-\infty, a) \longrightarrow \mathbb{X}$, $a > 0$, is continuous on $[0, a]$ and $z_0 \in \mathcal{B}$, then for every $t \in [0, a)$ the following conditions hold:
- **(1):** $z_t \in \mathcal{B}$;
- **(2):** There exists a positive constant L such that $|z(t)| \leq L\|z_t\|_{\mathcal{B}}$

(3): There exist two functions $H_1(\cdot), H_2(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ independent of z with H_1 continuous and H_2 locally bounded such that

$$\|z_t\|_{\mathcal{B}} \leq H_1(t) \sup\{|z(s)| : 0 \leq s \leq t\} + H_2(t)\|z_0\|_{\mathcal{B}}.$$

(P_2): For the function z in (P_1), z_t is a $\mathcal{B}$-valued continuous on $[0, a]$.
(P_3): The space $\mathcal{B}$ is complete.

Remark 2.1. *In the sequel we assume that H_1 and H_2 are bounded on J and*

$$\beta := \max\big\{ \sup_{t\in\mathbb{R}_+}\{H_1(t)\}, \sup_{t\in\mathbb{R}_+}\{H_2(t)\}\big\}.$$

For other details we refer, for example to the book by Hino et al. [21].

On the other hand, the theory of resolvent operator plays an important role in studying the existence of solutions of Eq. (1). Next, we collect definitions and some basic results about this theory.

In what follows, $\mathbb{X}_1$ is the Banach space $D(A)$ provided with the following norm :

$$\|y\|_{\mathbb{X}_1} := \|A(0)y\| + \|y\|$$

where $\|\cdot\|$ is the norm on $\mathbb{X}$. As the operators $A(t)$ and $B(t,s)$ are supposed to be closed, it follows that $A(t)$ and $B(t,s)$ belong to $\mathcal{L}(\mathbb{X}_1, \mathbb{X})$, for $0 \leq t \leq a$ and $0 \leq s \leq t \leq a$, respectively. Consider the following nonhomogeneous Cauchy equation:

$$\begin{cases} y'(t) = A(t)y(t) + \displaystyle\int_0^t B(t,s)y(s)ds, \quad \text{for } t \geq 0 \\ y(0) = y_0 \in \mathbb{X}. \end{cases} \tag{3}$$

Definition 2.1. [17] *The family $\{R(t,s) : 0 \leq s \leq t\}$ of bounded linear operator on $\mathbb{X}$ is called a resolvent operator for Eq.* (3)*, if the following properties are satisfied :*

(a) *$R(t,s)$ is strongly continuous in s and t, $R(s,s) = I$ (identity of $\mathbb{X}$), $0 \leq s \leq t$ and $\|R(t,s)\| \leq Me^{\alpha(t-s)}$ for some constants M and α.*
(b) *$R(t,s)\mathbb{X}_1 \subset \mathbb{X}_1$, $R(t,s)$ is strongly continuous in s and t on $\mathbb{X}_1$.*
(c) *For each $y \in \mathbb{X}_1, R(t,s)y$ is strongly continuously differentiable in t and s, nonempty and*

$$\begin{aligned} \frac{\partial R}{\partial t}(t,s)y &= A(t)R(t,s)y + \int_s^t B(t,r)R(r,s)ydr, \\ \frac{\partial R}{\partial s}(t,s)y &= -R(t,s)A(s)y - \int_s^t R(t,r)B(r,s)ydr \end{aligned}$$

with $\dfrac{\partial R}{\partial t}(t,s)y$ and $\dfrac{\partial R}{\partial s}(t,s)y$ are also strongly continuous on $0 \leq s \leq t \leq a$.

Definition 2.2. [17] *Let $\{A(t)\}, 0 \leq t \leq a$, be a family of infinitesimal generators of C_0-semigroups. $\{A(t)\}$ is said stable if there are real constants $M \geq 1$ and α such that*

$$\|\prod_{k=1}^{n}(A(t_k) - \lambda I)^{-1}\| \leq M(\lambda - \alpha)^{-n}$$

for all $\lambda > \alpha, 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n, n = 1, 2, \ldots.$

Let $B_s(\cdot)$ be defined by

$$(B_s(t)y)(s) = B(t+s,t)y \quad \text{for } y \in \mathbb{X}_1 \text{ and } t,s \geq 0.$$

Let $BUC(\mathbb{R}^+, \mathbb{X})$ be the space of bounded uniformly continuous functions on $\mathbb{R}^+$ into $\mathbb{X}$ endowed with the sup norm. To get the existence of the resolvent operator of (3), we assume the following hypotheses due to [17]:

- **(C_1):** $(A(t))_{t\geq 0}$ is a stable family of infinitesimal generators of C_0-semigroups such that $A(t)y$ is strongly continuously differentiable on $[0,\infty)$ for $y \in \mathbb{X}_1$.
- **(C_2):** $B_s(t)$ is continuous on $[0;+\infty)$ into $\mathcal{L}(\mathbb{X}_1, \mathcal{F})$, where $\mathcal{F} \subset BUC(\mathbb{R}^+, \mathbb{X})$ is a Banach space with a norm stronger than the sup norm on $BUC(\mathbb{R}^+, \mathbb{X})$.
- **(C_3):** $B_s(t) : \mathbb{X}_1 \to D(D_s)$ for all $t \geq 0$, where D_s is infinitesimal generator of the C_0 semigroup $(T(t))_{t\geq 0}$ on $\mathcal{F}$ defined by $[T(t)f](s) = f(t+s)$ for $t,s \geq 0$.
- **(C_4):** $D_s B_s(t)$ is continuous on $\mathbb{R}^+$ into $\mathcal{L}(\mathbb{X}_1, \mathcal{F})$.

If all the above four conditions are satisfied, we say that condition **RO** is verified.

Theorem 2.1. *[17] Suppose that condition **RO** is satisfied. Then the linear part of* (3) *has a unique resolvent operator.*

Lemma 2.2. *[17] Assume that condition **RO** holds. If $y_0 \in D(A)$ and $q \in C^1([0,a], \mathbb{X})$, then* (3) *has a classical solution given by*

$$y(t) = R(t,0)y_0 + \int_0^t R(t,s)q(s)ds$$

Theorem 2.3. *[17] If $A(t) \equiv A$ and $B(t,s) \equiv B(t-s)$ and there exists a resolvent operator $R(t,s)$, then $R(t,s) = R(t-s)$.*

When $A(t) \equiv A$, we have the following important result.

Theorem 2.4. [9] *Assume that condition **RO** hold. The resolvent operator $(R(t))_{t\geq 0}$ is compact for $t > 0$ if only if the semigroup $(T(t))_{t\geq 0}$ is compact for $t > 0$.*

Now, consider the following Cauchy problem:

$$\begin{cases} y'(t) = A(t)y(t), & 0 \leq s \leq t \leq a, \\ y(s) = x \in \mathbb{X}. \end{cases} \tag{4}$$

Definition 2.3. *A family $\{S(t,s) : \ 0 \leq s \leq t \leq a\}$ of linear, bounded operators on a Banach space $\mathbb{X}$ is called an evolution family for* (4) *if:*

(1) *$S(t,t) = I$ and $S(t,s) = S(t,r)S(r,s)$ for every $0 \leq s \leq t \leq a$;*
(2) *the mapping $\mathcal{I} := \{(t,s) \in J \times J : \ 0 \leq s \leq t < \infty\} \ni (t,s) \mapsto S(t,s)$ is strongly continuous;*
(3) *$\|S(t,s)\| \leq N \exp(\omega(t-s))$ for some $N \geq 1$, $\omega \in \mathbb{R}$ and all $t \geq s \in \mathbb{R}^+$;*
(4) *for each $y \in \mathbb{X}_1$, $S(t,s)$ is strongly continuously differentiable in t and s with*

$$\frac{\partial}{\partial t}S(t,s)y = A(t)S(t,s)y,$$
$$\frac{\partial}{\partial t}S(t,s)y = -S(t,s)A(s)y.$$

In the next lemma, we consider the following perturbation of (4):

$$\begin{cases} y'(t) = A(t)y(t) + \int_s^t B(t,u)y(u)du, & 0 \leq s \leq t \leq a, \\ y(s) = x \in \mathbb{X}. \end{cases} \tag{5}$$

The variation of constants formula related to (4) combined with the resolvent operator of (5) gives a concise relation between the resolvent operator and the evolution family.

Lemma 2.5. [11] *Let $(A(t))_{t\in[0,a]}$ be the generator of an evolution family $\{S(t,s) : 0 \leq s \leq t \leq a\}$, and assume condition **RO** is satisfied. If $\{R(t,s) : 0 \leq s \leq t \leq a\}$ is the resolvent operator for the linear part of* (3)*, then we have the following representation*

$$R(t,s)y = S(t,s)y + \int_s^t S(t,s)Q(r)y\,dr \tag{6}$$

with

$$Q(r)y = B(r,r)\int_s^r R(\tau,s)y\,d\tau - \int_s^r \frac{\partial B(r,u)}{\partial u}\int_s^u R(\tau,s)yd\tau du$$

such that $\{Q(\cdot)y : \ r \in [0,a]\}$ are uniformly bounded and for each $\zeta \in \mathbb{X}$, $Q(\cdot)y \in C([0,a],\mathbb{X})$.

Definition 2.4. *The resolvent operator $\{R(t,s)\}_{0\leq s\leq t\leq a}$ is said to be norm-continuous if the function $(t,s) \mapsto R(t,s)$ is continuous by operator norm for $0 \leq s < t \leq a$.*

Theorem 2.6. [11] *Assume that $A(t)$ is a family of linear operators generating an evolution family $\{S(t,s) : 0 \leq s \leq t \leq a\}$ and condition **RO** hold. Then the resolvent operator $\{R(t,s) : 0 \leq s \leq t \leq a\}$ for* (5) *is norm-continuous for $0 \leq s \leq t \leq a$ if the evolution family $\{S(t,s) : 0 \leq s \leq t \leq a\}$ is norm-continuous for $0 \leq s \leq t \leq a$.*

Lemma 2.7. [12] *Assume that condition **RO** hold. Then, there exists a constant ρ such that*

$$\|R(t,t-\tau)R(t-\tau,s) - R(t,s)\| \leq \rho\tau, \forall\, 0 \leq s \leq t,\ 0 < \tau \leq t.$$

Remark 2.2. *According to the representation of the resolvent operator given by Lemma 2.5, it is easy to establish a relation between the compactness of the evolution family $S(t,s)$ and the resolvent operator $\{R(t,s),\ 0 \leq s \leq t\}$. Then the resolvent operator $R(t,s)$ is compact for $0 \leq s < t$ if the evolution family $\{S(t,s),\ 0 \leq s \leq t\}$ is compact for $0 \leq s < t$.*

Lemma 2.8 (Corduneanu). [8] *Let $\mathbb{F} \subset BC(J,\mathbb{X})$ be a set satisfying the following conditions:*

- **(i):** *$\mathbb{F}$ is bounded in $BC(J,\mathbb{X})$;*
- **(ii):** *the functions belonging to $\mathbb{F}$ are equicontinuous on any compact interval of J;*
- **(iii):** *the set $\mathbb{F}(t) := \{z(t) : z \in \mathbb{F}\}$ is relatively compact on any compact interval of J;*
- **(iv):** *the functions from $\mathbb{F}$ are equiconvergent, i.e., given $\epsilon > 0$, there corresponds $N(\epsilon) > 0$ such that $|z(t) - z(+\infty)| < \epsilon$ for any $t \geq N(\epsilon)$ and $z \in \mathbb{F}$.*

Then $\mathbb{F}$ is is relatively compact in $BC(J,\mathbb{X})$.

Theorem 2.9 (Schauder's fixed point theorem). *[14] Let $\mathbb{F}$ be a nonempty closed convex bounded subset of a Banach space $\mathbb{X}$. Then every continuous compact mapping $S : \mathbb{F} \longrightarrow \mathbb{F}$ has a fixed point.*

3. Existence results

In this section, we prove the existence of mild solutions for Eq. (1). Before stating and proving the main result, we give the definition of mild solutions for Eq. (1).

Definition 3.1. *A mild solution of Eq.* (1) *is a function* $z \in \mathcal{H}$ *satisfying the relation*

$$z(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ R(t,0)\varphi(0) + \int_0^t R(t,s)h(s, z_{\sigma(s,z_s)})ds, & t \in J. \end{cases} \quad (7)$$

To prove our existence results, we need the following hypotheses:

(A_1): The resolvent operator $R(t,s)$ is compact for every $t > s$ and there exist constants $M \geq 1$ and $\alpha > 0$ such that

$$\|R(t,s)\|_{\mathcal{L}(\mathbb{X})} \leq Me^{-\alpha(t-s)} \text{ for every } t \geq s \geq 0.$$

(A_2): The function $h : J \times \mathcal{B} \longrightarrow \mathbb{X}$ is Carathéodory and there exist a function $\mu \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\Theta : J \longrightarrow (0, \infty)$ and such that

$$|h(t,\xi)| \leq \mu(t)\Theta(\|\xi\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } \xi \in \mathcal{B}.$$

(A_3): For each $(t,s) \in \mathcal{I}$, we have: $\lim\limits_{t\to+\infty} \int_0^t e^{-\alpha(t-s)}\mu(s)ds = 0$.

(A_4): There exists a constant $q > 0$ such that

$$M\|\mu\|_{L^1}\Theta(\lambda q + [\lambda(M+1)L^{\varphi}]\|\varphi\|_{\mathcal{B}}) \leq q.$$

(A_5): Let $\mathcal{R}(\sigma^-) = \{\sigma(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \quad \sigma(s,\varphi) \leq 0\}$, and $\varphi_t : \mathcal{R}(\sigma^-) \longrightarrow \mathcal{B}$ continuous function and there exists a continuous and bounded function $J^{\varphi} : \mathcal{R}(\sigma^-) \longrightarrow (0, \infty)$ such that

$$\|\varphi_t\|_{\mathcal{B}} \leq J^{\varphi}(t)\|\varphi\|_{\mathcal{B}} \text{ for all } t \in \mathcal{R}(\sigma^-).$$

Remark 3.1. *[17] Note that the following conditions ensure the exponential decay of* $R(t,s)$ *in the assumption* (A_1)*:*

(i): $A(t)$ *generates an evolution family* $(S(t,s))_{t\geq s\geq 0}$.

(ii): $S(t,s)$ *decays exponentially for* $t \geq s \geq 0$.

Remark 3.2. *For more details about* (A_5) *see ([21], Lemma 3.4 and Proposition 7.1.1).*

Moreover, we have the following lemma, which plays an important role to prove our results.

Lemma 3.1. *([20], Lemma 3.1) Let* $z : (-\infty, a] \to \mathbb{X}$ *be continuous on* $[0,a]$ *and* $z_0 = \varphi$. *If* (A_5) *hold, then*

$$\|z_s\|_{\mathcal{B}} \leq (H_2^* + H^{\varphi})\|\varphi\|_{\mathcal{B}} + H_1^* \sup\{|z(\theta)|; \theta \in [0, \max\{0,s\}]\}, \quad s \in \mathcal{R}(\sigma^-) \cup J,$$

where $H^{\varphi} = \sup_{t\in\mathcal{R}(\sigma^-)} H^{\varphi}(t)$, $H_1^* = \sup_{t\in\mathbb{R}_+} H_1(t)$, *and* $H_2^* = \sup_{t\in\mathbb{R}_+} H_2(t)$.

Theorem 3.2. *Assume that* ***RO*** *and* $(A_1) - (A_5)$ *are fulfilled. Then Eq.* (1) *has at least one mild solution.*

Proof. We transform the problem (1) into a fixed point problem. Consider the operator $\Phi : \mathcal{H} \longrightarrow \mathcal{H}$ defined by

$$\Phi z(t) = \begin{cases} \varphi(t), \quad t \in (-\infty, 0], \\ R(t,0)\varphi(0) + \int_0^t R(t,s)h(s, z_{\sigma(s,z_s)})ds, \quad t \in J. \end{cases} \tag{8}$$

Note that the fixed points of the operator Φ are mild solutions of Eq. (1). For $\varphi \in \mathcal{B}$, we introduce the following function $\xi : \mathbb{R} \longrightarrow \mathbb{X}$ by

$$\xi(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, 0], \\ R(t,0)\varphi(0), & \text{if } t \in J. \end{cases}$$

Then $\xi_0 = \varphi$. For each function $\zeta \in \mathcal{H}$, set

$$z(t) = \xi(t) + \zeta(t).$$

Clearly, z fulfills (7) if and only if ζ satisfies $\zeta_0 = 0$ and

$$\zeta(t) = \int_0^t R(t,s)h(s, \zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds, \text{ for } t \in J.$$

Let

$$\mathcal{H}_0 = \{\zeta \in \mathcal{H} : \zeta_0 = 0\}.$$

Then $\mathcal{H}_0$ is a Banach space with norm

$$\|\zeta\|_{\mathcal{H}_0} = \sup_{t\in J} |\zeta(t)| + \|\zeta_0\|_{\mathcal{B}} = \sup_{t\in J} |\zeta(t)|.$$

Now, define the operator $\overline{\Phi} : \mathcal{H}_0 \longrightarrow \mathcal{H}_0$ by

$$\overline{\Phi}\zeta(t) = \int_0^t R(t,s)h(s, \zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds, \text{ for } t \in J.$$

It is evident that the operator Φ has a fixed point if and only if $\overline{\Phi}$ does as well. In order to do this, we begin with the following estimate.

For each $\zeta \in \mathcal{H}_0$ and $t \in \mathcal{R}(\sigma^-) \cup J$, from Lemma 3.1, it follows that

$$\begin{aligned} \|\zeta_t + \xi_t\|_{\mathcal{B}} &\le \|\zeta_t\|_{\mathcal{B}} + \|\xi_t\|_{\mathcal{B}} \\ &\le H_1(t)|\zeta(t)| + H_1(t)\|R(t,0)\|_{\mathbb{B}(\mathbb{X})}\|\varphi\|_{\mathcal{B}} + (H_2(t) + H^{\varphi}(t))\|\varphi\|_{\mathcal{B}} \\ &\le \beta\|\zeta\|_{\mathcal{H}_0} + \beta M e^{-\alpha t}\|\varphi\|_{\mathcal{B}} + (\beta + H^{\varphi})\|\varphi\|_{\mathcal{B}} \\ &\le \beta\|\zeta\|_{\mathcal{H}_0} + [\beta(M+1)H^{\varphi}]\|\varphi\|_{\mathcal{B}}. \end{aligned} \tag{9}$$

Then for $t \in J$, we have

$$\begin{aligned} |(\overline{\Phi}\zeta)(t)| &\le \int_0^t \|R(t,s)\|_{\mathcal{L}(\mathbb{X})}|h(s, \zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})|ds \\ &\le M\int_0^t e^{-\alpha(t-s)}\mu(s)\Theta(\|\zeta_s + \xi_s\|_{\mathcal{B}})ds \\ &\le M\|\mu\|_{L^1}\Theta(\beta\|\zeta\|_{\mathcal{H}_0} + [\beta(M+1)H^{\varphi}]\|\varphi\|_{\mathcal{B}}) \\ &\le M\|\mu\|_{L^1}\Theta(\beta\|\zeta\|_{\mathcal{H}_0} + [\beta(M+1)H^{\varphi}]\|\varphi\|_{\mathcal{B}}). \end{aligned}$$

By (A_4), there exists $q > 0$ such that

$$\mathbb{Y}_q = \{\zeta \in \mathcal{H}_0 : \|\zeta\|_{\mathcal{H}_0} \le q\}$$

a bounded, convex and closed set of $\mathcal{H}_0$.

Then for $\zeta \in \mathbb{Y}_q$, by (9) and (A_4), we obtain

$$\|\overline{\Phi}\zeta\|_{\mathcal{H}_0} \leq M\Theta(\omega_q)\|\mu\|_{L^1} \leq q,$$

with

$$\omega_q := \beta\|\zeta\|_{\mathcal{H}_0} + [\beta(M+1)H^{\varphi}]\|\varphi\|_{\mathcal{B}}.$$

Hence, $\overline{\Phi}$ maps $\mathbb{Y}_q$ into itself.

In the following discussion, we are going to show, with the assistance and backing of Schauder's fixed point theorem, that $\overline{\Phi}$ possesses a fixed point. In order to demonstrate this, the evidence will be segmented into the following stages:
Step 1: $\overline{\Phi}$ is continuous.
Let $(\zeta^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}_0$ such that $\zeta^n \to \zeta$ in $\mathcal{X}_0$. From (P_1), we have that $\zeta^n_s \to \zeta_s$ uniformly for $(-\infty, a]$, as $n \to \infty$. By the continuity of h, we get

$$h(s, \zeta^n_{\sigma(s,\zeta^n_s+\xi_s)} + \xi_{\sigma(s,\zeta^n_s+\xi_s)}) \to h(s, \zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)}).$$

Let $s \in J$ be such that $\sigma(s,\zeta_s) > 0$. Then, we have

$$\begin{aligned}\|\zeta^n_{\sigma(s,\zeta^n_s)} - \zeta_{\sigma(s,\zeta_s)}\|_{\mathcal{B}} &\leq \|\zeta^n_{\sigma(s,\zeta^n_s)} - \zeta_{\sigma(s,\zeta^n_s)}\|_{\mathcal{B}} + \|\zeta_{\sigma(s,\zeta^n_s)} - \zeta_{\sigma(s,\zeta_s)}\|_{\mathcal{B}} \\ &\leq H^*_1\|\zeta^n - \zeta\|_{\mathcal{H}_0} + \|\zeta_{\sigma(s,\zeta^n_s)} - \zeta_{\sigma(s,\zeta_s)}\|_{\mathcal{B}}.\end{aligned}$$

By the continuity of σ, we deduce that the right hand of the above inequality converges to zero as $n \to \infty$, and we conclude that $\zeta^n_{\sigma(s,\zeta^n_s)} \to \zeta_{\sigma(s,\zeta_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ satisfying $\sigma(s,\zeta_s) > 0$. Similarly, for $\sigma(s,\zeta_s) < 0$, we find

$$\|\zeta^n_{\sigma(s,\zeta^n_s)} - \zeta_{\sigma(s,\zeta_s)}\|_{\mathcal{B}} = \|\varphi^n_{\sigma(s,\zeta^n_s)} - \varphi_{\sigma(s,\zeta_s)}\|_{\mathcal{B}} = 0,$$

which proves that $\zeta^n_{\sigma(s,\zeta^n_s)} \to \zeta_{\sigma(s,\zeta_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ satisfying $\sigma(s,\zeta_s) < 0$. Combining the previous arguments, we can prove that $\zeta^n_{\sigma(s,\zeta^n_s)} \to \varphi$ for every $s \in J$ such that $\sigma(s,\zeta_s) = 0$. For every $t \in J$, we have

$$\begin{aligned}&|\overline{\Phi}(\zeta^n)(t) - \overline{\Phi}(\zeta)(t)| \\ &\leq \int_0^t \|R(t,s)\|_{\mathcal{L}(\mathbb{X})}|h(s, \zeta^n_{\sigma(s,\zeta^n_s+\xi_s)} + \xi_{\sigma(s,\zeta^n_s+\xi_s)}) - h(s, \zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})|ds \\ &\leq M\int_0^t e^{-\alpha(t-s)}|h(s, \zeta^n_{\sigma(s,\zeta^n_s+\xi_s)} + \xi_{\sigma(s,\zeta^n_s+\xi_s)}) - h(s, \zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})|ds.\end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem and the continuity of h, we have

$$\|\overline{\Phi}\zeta^n - \overline{\Phi}\zeta\|_{\mathcal{H}_0} \to 0 \text{ as } n \to +\infty.$$

As a result, $\overline{\Phi}$ is continuous.
Step 2: $\overline{\Phi}(\mathbb{Y}_q)$ is an equicontinuous set on each closed bounded interval $[0, a]$ in J.

Let $t_1, t_2 \in [0,a]$ with $t_2 > t_1$ and $\zeta \in \mathbb{Y}_q$. Then using (A_4) and (9), we find

$$\begin{aligned}
&|\overline{\Phi}(\zeta)(t_2) - \overline{\Phi}(\zeta)(t_1)| \\
&= \Big| \int_0^{t_2} R(t_2,s)h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds - \int_0^{t_1} R(t_1,s)h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds \Big| \\
&\leq \int_0^{t_1} \|R(t_2,s) - R(t_1,s)\|_{\mathcal{L}(\mathbb{X})} |h(s, s_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})|ds \\
&\quad + M\int_{t_1}^{t_2} e^{-\alpha(t_2-s)} |h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})|ds \\
&\leq \Theta(\omega_q) \int_0^{t_1} \|R(t_2,s) - R(t_1,s)\|_{\mathcal{L}(\mathbb{X})} \mu(s)ds + M\Theta(\omega_q)\int_{t_1}^{t_2} \mu(s)ds.
\end{aligned}$$

Since $R(t,s)$ is compact, it is continuous in the uniform operator topology in J. Therefore, the right-hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$, which implies the equicontinuity of $\overline{\Phi}(\mathbb{Y}_q)$.

Step 3: $\overline{\Phi}(\mathbb{Y}_q)$ is relatively compact.

For this end, we will show that $\mathbb{Y}_q(t) = \{(\overline{\Phi}\zeta)(t) : \zeta \in \mathbb{Y}_q\}$ is relatively compact in $\mathbb{X}$.

Let $t \in J$ be fixed, and let τ be a real number satisfying $0 < \tau < t \leq a$. For $\zeta \in \mathbb{Y}_q$, we define

$$\begin{aligned}
\overline{\Phi}_\tau(t) &= R(t,t-\tau)\int_0^{t-\tau} R(t-\tau,s)h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds, \\
\overline{\Phi}^*_\tau(t) &= \int_0^{t-\tau} R(t,s)h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds.
\end{aligned}$$

From the compactnes of the operator $R(t,t-\tau)$, we derive that the set $\mathbb{Y}_q^\tau(t) = \{(\overline{\Phi}_\tau\zeta)(t) : \zeta \in \mathbb{Y}_q\}$ is precompact in $\mathbb{X}$ for every $\tau \in (0,t)$. Moreover, for $\zeta \in \mathbb{Y}_q$ by Lemma 2.7 and (A_2), we have

$$\begin{aligned}
&|\overline{\Phi}_\tau(\zeta)(t) - \overline{\Phi}^*_\tau(\zeta)(t)| \\
&\quad = \Big| R(t,t-\tau)\int_0^{t-\tau} R(t-\tau,s)h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds \\
&\qquad - \int_0^{t-\tau} R(t,s)h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})ds \Big| \\
&\quad \leq \int_0^{t-\tau} \|R(t,t-\tau)R(t-\tau,s) - R(t,s)\| |h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})|ds \\
&\quad \leq \rho\tau \int_0^{t-\tau} |h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})|ds \\
&\quad \leq \rho\tau \int_0^{t-\tau} \Theta(\|\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)}\|_{\mathcal{B}})ds \\
&\quad \leq \rho\tau\Theta(\omega_q)\int_0^{t-\tau} \mu(s)ds \to 0 \text{ as } \tau \to 0.
\end{aligned}$$

Consequently, the set $\mathbb{Y}^*_\tau(t) := \{(\overline{\Phi}^*_\tau\zeta)(t) : \zeta \in \mathbb{Y}\}$ is relatively compact in $\mathbb{X}$ thanks to the total boundedness.

Applying this method again, we obtain

$$\begin{aligned}|\overline{\Phi}(\zeta)(t)-\overline{\Phi}_\tau(\zeta)(t)| &\leq \int_{t-\tau}^{t}\|R(t,s)\||h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)}+\xi_{\sigma(s,\zeta_s+\xi_s)})|ds\\ &\leq M\int_{t-\tau}^{t}e^{-\alpha(t-s)}\Theta(\|\zeta_{\sigma(s,\zeta_s+\xi_s)}+\xi_{\sigma(s,\zeta_s+\xi_s)}\|_{\mathcal{B}})ds\\ &\leq M\Theta(\omega_q)\int_{t-\tau}^{t}\mu(s)ds\to 0 \text{ as } \tau\to 0.\end{aligned}$$

Therefore, $\overline{\Phi}\zeta$ converges uniformly to $\overline{\Phi}_\tau\zeta$, which implies that $\mathbb{Y}_q(t)$ is relatively compact in $\mathbb{X}$.

Step 4: $\overline{\Phi}$ is equiconvergent. Let $\zeta\in\mathbb{Y}_q$, then from (A_2),(A_3) and (9), we have

$$\begin{aligned}|(\overline{\Phi}\zeta)(t)| &\leq \int_0^t\|R(t,s)\|_{\mathcal{L}(\mathbb{X})}|h(s,\zeta_{\sigma(s,\zeta_s+\xi_s)}+\xi_{\sigma(s,\zeta_s+\xi_s)})|ds\\ &\leq M\int_0^t e^{-\alpha(t-s)}\mu(s)\Theta(\|\zeta_s+\xi_s\|_{\mathcal{B}})ds\\ &\leq M\Theta(\omega_q)\int_0^t e^{-\alpha(t-s)}\mu(s)ds,\end{aligned}$$

Based on (9), we see that $|(\overline{\Phi}\zeta)(t)|\to 0$ as $t\to+\infty$.

Therefore,

$$\lim_{t\to+\infty}|(\overline{\Phi}\zeta)(t)-(\overline{\Phi}\zeta)(+\infty)|=0,$$

that is, $\overline{\Phi}$ is equiconvergent.

Hence, we deduce from steps 1-4 and Lemma 2.8 that the operator $\overline{\Phi}$ is continuous and compact. As a consequence of Schauder's fixed point theorem, we conclude that the operator $\overline{\Phi}$ has at least one fixed point which is a mild solution of Eq. (1). □

4. Attractivity of solutions

Here we examine the attractivity of solutions of system (1).

Definition 4.1. [3] *We say that solutions of Eq.* (1) *are locally attractive if there exists a closed ball* $B^{'}(\zeta^*,\tau)$ *close ball* $\mathcal{H}_0$ *for some* $\zeta^*\in\mathcal{H}$ *such that for arbitrary solutions* ζ *and* $\bar{\zeta}$ *of* (1) *belonging to* $B^{'}(\zeta^*,\tau)$ *we have that*

$$\lim_{t\to+\infty}(\zeta(t)-\bar{\zeta}(t))=0.$$

Under the hypotheses of Section 3, let ζ^* be a solution of Eq. (1) and $B^{'}(\zeta^*,\tau)$ the closed ball which τ satisfies the following inequality

$$2M\Theta(\omega_\tau)\|\mu\|_{L^1}\leq\tau,$$

with $\omega_\tau:=\beta\tau+[\beta(M+1)H^\varphi]\|\varphi\|_{\mathcal{B}}$.

Then, for $\zeta \in B'(\zeta^*, \tau)$ by $(A_1) - (A_2)$ and (9), we have

$$\begin{aligned}|(\overline{\Phi}\zeta)(t) - \zeta^*(t)| &= |(\overline{\Phi}\zeta)(t) - (\overline{\Phi}\zeta^*)(t)| \\ &\leq \int_0^t \|R(t,s)\|_{\mathcal{L}(\mathbb{X})} |h(s, \zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)}) \\ &\quad - h(s, \zeta^*_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})| ds \\ &\leq M \int_0^t e^{-\alpha(t-s)} \mu(s) (\Theta(\|\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)}\|_{\mathcal{B}}) ds \\ &\quad + \Theta(\|\zeta_{\sigma(s,\zeta_s+\xi_s)} + \xi_{\sigma(s,\zeta_s+\xi_s)})\|_{\mathcal{B}}) \\ &\leq 2M\Theta(\omega_\tau) \int_0^t \mu(s) ds.\end{aligned}$$

As a result, we obtain $\overline{\Phi}(B'(\zeta^*, \tau)) \subset B'(\zeta^*, \tau)$.

According to (A_3), for each $z \in B'(\zeta^*, \tau)$ solution of Eq. (1) and $t \in J$, we have

$$\begin{aligned}|\zeta(t) - \bar{\zeta}(t)| &\leq |(\overline{\Phi}\zeta)(t) - (\overline{\Phi}\bar{\zeta})(t)| \\ &\leq 2M\Theta(\omega_\tau) \int_0^t e^{-\alpha(t-s)} \mu(s) ds \to 0, \text{ as } t \to +\infty.\end{aligned}$$

Thus, the solutions of Eq. (1) are locally attractive.

5. Example

In this section, we provide an example to illustrate our obtained results. Consider the following partial integrodifferential with state-dependent delay:

$$\begin{cases} \dfrac{\partial}{\partial t} y(t,\nu) = \big[\dfrac{\partial^2}{\partial \nu^2} + \hat{\sigma}(t)\big] y(t,\nu) + \displaystyle\int_0^t \gamma e^{-\lambda(t-s)} \dfrac{\partial^2}{\partial \nu^2} y(s,\nu) ds \\ \qquad\qquad + \dfrac{e^{-2t}}{1+t^2} \displaystyle\int_{-\infty}^t e^{2(s-t)} y\Big(s - \sigma_1(s)\sigma_2\Big(\int_0^\pi a(\xi)|y(s,\xi)|^2 d\xi\Big), \nu\Big) ds \\ y(t,0) = y(t,\pi) = 0, \quad t \geq 0, \\ y(\theta,\nu) = y_0(\theta,\nu), \quad \theta \in (-\infty, 0], \nu \in [0,\pi], \end{cases} \tag{10}$$

where $\gamma > 0$, $\lambda \in [0,\pi]$, $\sigma_i : [0,\infty) \longrightarrow [0,\infty)$, $i = 1, 2$, are continuous, the function a is continuous and positive, $\hat{\sigma}$ is a positive function and is Hölder continuous in t with parameter $0 < \delta < 1$.

Let $\mathbb{X} = L^2([0,\pi], \mathbb{R})$ and define the operator $\hat{A} : D(\hat{A}) \longrightarrow \mathbb{X}$ given by $\hat{A}\omega = \omega''$ with domain

$$D(\hat{A}) = H^2([0,\pi]) \cap H^1_0([0,\pi]).$$

It is well known that in this case $\hat{A}$ has a discrete spectrum which is given by $-n^2, n \in \mathbb{N}$. Furthermore, $\mathbb{X}$ has a completely orthonormal base formed by eigenfunctions of $\hat{A}$ associated with the eigenvalues $-n^2$, which is given by $e_n(\nu) = \sqrt{\frac{2}{\pi}} \sin(n\nu)$, $n \in \mathbb{N}$. This implies the following statements:

(a): For each $\omega \in \mathbb{X}$, $\omega(\nu) = \sum_{n=1}^{\infty} \langle \omega, e_n \rangle e_n(\nu)$.

(b): For each $\omega \in D(\hat{A})$, we have $\hat{A}\omega(\nu) = -\sum_{n=1}^{\infty} n^2 \langle \omega, e_n \rangle e_n(\nu)$, where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{X}$. From [[26], p.79], we know that the operator $\hat{A}$ generates a compact semigroup $(T(t))_{t\geq 0}$, which is described by

$$T(t)\omega(\nu)=\sum_{n=1}^{\infty}e^{-n^2t}\langle\omega,e_n\rangle e_n(\nu),\ \text{for } t\geq 0, \nu\in[0,\pi].$$

We define

$$A(t)z=\hat{A}z+\hat{\sigma}(t)$$

with the domain

$$D(A)=\{z(\cdot)\in\mathbb{X}\ :\ z,z^{'}\ \text{absolutely continuous, } z^{''}\in\mathbb{X}, z(0)=z(\pi)=0\}.$$

Then it is not difficult to verify that $A(t)$ generates an evolution operator $S(t,s)$

$$S(t,s)=T(t-s)e^{-\int_s^t\hat{\sigma}(\tau)d\tau},$$

where $T(t)$ is the compact analytic semigroup generated by the operator $\hat{A}$. We define the operators $B(t,s)\ :\ D(A)\subset\mathbb{X}\to\mathbb{X}$ as follows:

$$B(t,s)z=\gamma e^{-\lambda(t-s)}\hat{A}z,\quad \text{for}\quad 0\leq s\leq t.$$

Furthermore it is not difficult to see that conditions $(C_1)-(C_4)$ are satisfied. Then system (10), has a resolvent operator $R(t,s)$ which can be extracted from the evolution system $S(t,s)$(see [17]). Since $S(t,s)$ is a compact operator for every $t>s$, the resolvent operator $R(t,s)$ is a compact operator for every $t>s$. Further, in virtue of [17, Theorem 4.1], we have that $\|R(t,s)\|\leq e^{-\lambda(t-s)}$. Hence, (A_1) is verified with $\alpha=\lambda$ and $M=1$.

For the phase space, we choose $\mathcal{B}=C_\tau\times L^2(g,\mathbb{X})$ endowed with the following norm

$$\|\varphi\|_{\mathcal{B}}=\sup_{\theta\in[-\tau,0]}\|\varphi\|+\Big(\int_{-\infty}^{-\tau}g(\theta)\|\varphi(\theta)\|^pd\theta\Big)^{1/p}.$$

Thus, for $\tau=0$ and $p=2$, this space coincides $C_0\times L^2(g,\mathcal{H})$, $K=1$, $K_1(\omega)=\Lambda(-\omega)^{1/2}$, $K_2(\omega)=1+\Big(\int_{-\tau}^{0}g(\sigma)d\sigma\Big)^{1/2}$.

For $\varphi\in\mathcal{B}$ and $\nu\in[0,\pi]$, we put

$$z(t)(\nu)=y(t,\nu),\ t\geq 0,\ \nu\in[0,\pi],$$
$$\varphi(\theta)(\nu)=\varphi(\theta,\nu),\theta\leq 0,\ \nu\in[0,\pi].$$

To represent system (10) in the abstract form, we define the following functions:

$$h(t,\vartheta)(\nu)=\frac{e^{-2t}}{1+t^2}\int_{-\infty}^{0}e^{2s}\vartheta(\tau)(\nu)ds,\ t\geq 0,\ \nu\in[0,\pi],$$
$$\sigma(s,\vartheta)=s-\sigma_1(s)\sigma_2\Big(\int_0^{\pi}a(\xi)|\vartheta(0,\xi)|^2d\xi\Big).$$

Then, system (10) can be rewritten in the abstract form given by system (1).

Verification of (A_2): For $t \geq 0$, we have

$$
\begin{aligned}
|h(t,\vartheta)(\nu)| &= \Big(\frac{e^{-2t}}{1+t^2}\int_0^{\pi}\Big|\int_{-\infty}^{0} e^{2s}\vartheta(\tau)(\nu)ds\Big|^2 d\nu\Big)^{1/2}\\
&\leq \frac{e^{-2t}}{1+t^2}\Big(\int_0^{\pi}\Big(\int_{-\infty}^{0} e^{2s}\|\vartheta(\tau)(\nu)\|ds\Big)^2 d\nu\Big)^{1/2}\\
&\leq \frac{e^{-2t}}{1+t^2}\Big(\int_0^{\pi}\Big(\int_{-\infty}^{0} e^{2s}\sup\|\vartheta(\tau)(\nu)\|ds\Big)^2 d\nu\Big)^{1/2}\\
&\leq \frac{e^{-2t}}{1+t^2}\Big(\int_0^{\pi}\|\vartheta\|_{\mathcal{B}}^2 d\nu\Big)^{1/2}\\
&\leq \mu(t)\Theta(\|\vartheta\|_{\mathcal{B}}),
\end{aligned}
$$

where $\mu(t)=\frac{e^{-2t}\pi}{1+t^2}$ and $\Theta(t)=t$. Hence, (A_2) is satisfied.

Moreover, we have

$$
\int_0^t e^{-2(t-s)}\frac{e^{-2s}\pi}{1+s^2}ds = e^{-2t}\pi\int_0^t \frac{1}{1+s^2}ds = e^{-2t}\pi\arctan(t)\to 0 \text{ as } t\to\infty,
$$

and therefore (A_3) is fulfilled. Assuming that (A_4) holds, we conclude that all solutions of Eq (10) are locally attractive.

6. Conclusion

We studied mild solutions' global existence and attractivity for some functional integrodifferential equations with state-dependent delay in Banach spaces. We apply the Schauder fixed point theorem and Grimmer's resolvent operator to prove the outcomes. Finally, we stated an example to validate the abstract results. However, we can extend this work to a stochastic case. In the future, we will analyze the stability of impulsive stochastic integrodifferential evolutions with state-dependent delay.

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