SECOND ORDER HANKEL DETERMINANTS FOR CLASS OF BOUNDED TURNING FUNCTIONS DEFINED BY SALAGEAN DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper, a brief study of certain properties of bounded turning functions is carried out. Furthermore, bound to the famous Fekete - Szego functional \( H_2(1) = |a_3 - ta_2^2| \), with \( t \) real and the Second Hankel Determinant \( H_2(2) = |a_2a_4 - a_3^2| \) for functions of bounded turning of order \( \beta \) are obtained using a succinct mathematical approach.

1. Introduction

Let \( D \subset \mathbb{C}, f : D \rightarrow \mathbb{C} \) holomorphic. Let \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk and \( A \) denote the class of holomorphic functions:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + a_2 z^2 + a_3 z^3 + \ldots,
\]

normalized by \( f(0) = 0 \) and \( f'(0) = 1 \).

Also, consider the class \( P \) of Caratheodory functions:

\[
p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k = 1 + c_1 z + c_2 z^2 + \ldots
\]
defined in $U = \{ z \in \mathbb{C} : |z| < 1 \}$ which are also holomorphic with $Re(\zeta) > 0$. The $q$ - th Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ is defined by
\[
H_q(n) = \begin{vmatrix}
    a_n & a_{n+1} & \cdots & a_{n+q-1} \\
    a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}, \quad (a_1 = 1)
\]

This determinant has been investigated by several authors in the literature ranging from rate of growth of $H_q(n)$ as $n \to \infty$ for the function $S$ with a bounded boundary [1], to the determination of precise bounds on $H_q(n)$ for specific $q$ and $n$ for some favored classes of functions [4, 5].

In this present work, we consider the 2nd - order Hankel determinant denoted by $H_2(n)$ in the cases of $q = 2$, $n = 1$ and $q = 2$, $n = 2$, given by
\[
H_2(1) = \begin{vmatrix}
    a_1 & a_2 \\
    a_2 & a_3
\end{vmatrix} = a_1a_3 - a_2^2
\]

For $f(z) \in A$ and choosing $a_1 = 1$ such that, the following is obtained
\[
H_2(1) = a_3 - a_2^2
\]

By applying triangle inequality, we arrived at
\[
|H_2(1)| \leq |a_3 - a_2^2| \tag{3}
\]

and
\[
H_2(2) = \begin{vmatrix}
    a_2 & a_3 \\
    a_3 & a_4
\end{vmatrix} = |a_2a_4 - a_3^2| \tag{4}
\]

We can observe that the right hand side of the inequality in (3) is the well - known Fekete-Szego functional for the second Hankel determinant $H_2(1) = |a_3 - a_2^2|$. Fekete-Szego further generalized the estimate $|a_3 - \mu a_2^2|$ with $\mu$ real and $f \in S$. For $R$ (the class of bounded turning functions), best possible (sharp) bound $2/3$ was given in [2] (with $R$ corresponding to $n = \alpha = 1, \beta = 0$ in the class of function $T_\alpha^n(\beta)$ studied) while for $S^*$ (Class of Starlike functions) and $C$ (Class of convex functions), best possible (sharp) bounds 1 and $1/3$, respectively, were reported in [6]. Further, best possible (sharp) bounds for the functional $|a_2a_4 - a_3^2|$ in (4) were report for various subclasses of univalent and multivalent holomorphic functions by several authors in the literature.

In this paper, our focus is on finding the bounds for the functional $|a_3 - ta_2^2|$, with $t$ real and $|a_2a_4 - a_3^2|$ for $BT(m, \beta)$, $\beta < 1, m \in \{0, 1, 2, \ldots\}$, the class of bounded turning functions of order $\beta$ defined by Salagean differential operator. It follows a method of classical analysis introduced by Libera and Zlotkiewicz [7, 8]. The same has been applied by many authors in the literature [4, 5].
In 1983, Salagean defined the following operator:

\[
D^0 f(z) = f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

\[
D^1 f(z) = z D^0 f(z) = z f'(z) = z + \sum_{k=2}^{\infty} k a_k z^k
\]

\[
D^2 f(z) = z D^1 f(z) = z f''(z) = z + \sum_{k=2}^{\infty} k^2 a_k z^k
\]

\[\vdots\]

\[
D^m f(z) = z D^{m-1} f(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k
\]

Obviously, from (5), we can deduce that

\[
[D^m f(z)]' = 1 + \sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1}
\]

Using (6), we define the following class of bounded turning functions of order \(\beta\), \(\beta < 1\), \(m \in \{0, 1, 2, \ldots\}\).

**Definition 1** The class \(BT(m, \beta)\) is said to be bounded turning of order \(\beta\) if

\[
\Re \left[ \frac{[D^m f(z)]'}{1 - \beta} - \beta \right] > 0
\]

That is,

\[
\Re \left[ \frac{1 + \sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1} - \beta}{1 - \beta} \right] > 0
\]

\[
\Re \left[ \frac{(1 - \beta) + \sum_{k=2}^{\infty} k^{m+1} a_k z^{k-1}}{1 - \beta} \right] > 0
\]

where \(f(z) \in A\), \(\beta < 1\), \(m \in \{0, 1, 2, \ldots\}\). Observe that with \(\beta = 0\) in (7), we obtain (6) whose anti-derivative was reported in [11]. We denote by \(BT(m, \beta)\) the class of functions in \(S\) which are bounded turning of order \(\beta\), in \(U = \{z \in C : |z| < 1\}\) which satisfy (7).

**Definition 2** A function \(f(z) \in S\) is said to be bounded turning of order \(\beta\) in \(U = \{z \in C : |z| < 1\}\), if and only if

\[
\Re \left\{ [D^m f(z)]' \right\} > \beta, z \in U
\]

for fixed \(\beta, \beta < 1\), \(m \in \{0, 1, 2, \ldots\}\).
2. PRELIMINARY RESULTS

Before we proceed into the main results, the following Lemmata shall be necessary.

Lemma 1[3] The power series for

\[ p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \]

converges in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) to a function in \( P \) if and only if the Toeplitz determinants

\[
D_n = \begin{vmatrix}
2 & c_1 & c_2 & \cdots & c_n \\
c_{-1} & 2 & c_1 & \cdots & c_{k-1} \\
c_{-2} & c_{-1} & 2 & \cdots & c_{k-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{-k} & c_{-k+1} & c_{-k+2} & \cdots & 2
\end{vmatrix}, \quad k = 1, 2, 3, \ldots
\]

and \( c_{-k} = \bar{c}_k \), are all non-negative. They are strictly positive except for

\[
p(z) = \sum_{k=1}^{m} \rho_k p_0 \left( e^{it_k z} \right)
\]

\( \rho_k > 0, t_k \) real and \( t_k \neq t_j \), for \( k \neq j \), where

\[
\rho_0(z) = \frac{1 + z}{1 - z}
\]

in this case \( D_n > 0 \) for \( n < (m - 1) \wedge D_n = 0 \) for \( n \geq m \).

The necessary and sufficient condition in Lemma 1 is due to Caratheodory and Toeplitz. It may be assumed without restriction that \( c_1 > 0 \).

Using Lemma 1, for \( n = 2 \) \wedge n = 3 \), we have the following:

\[
D_2 = \begin{vmatrix}
2 & c_1 & c_2 \\
c_1 & 2 & c_1 \\
c_2 & c_1 & 2
\end{vmatrix} = \left[ 8 + 2\Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2 \right] \geq 0,
\]

\[
= c_2 \begin{vmatrix}
\bar{c}_1 & 2 \\
\bar{c}_2 & c_1
\end{vmatrix} - c_1 \begin{vmatrix}
2 & c_1 \\
c_2 & \bar{c}_1
\end{vmatrix} + 2 \begin{vmatrix}
2 & c_1 \\
c_1 & \bar{c}_1
\end{vmatrix},
\]

\[
= c_2(\bar{c}_1^2 - 2\bar{c}_2) - c_1(2\bar{c}_1 - c_1 \bar{c}_2) + 2(4 - c_1 \bar{c}_1)
\]

Taking rid of all the negative signs on \( c's \) on the right hand-side of \( D_2 \) and expanding the brackets, we get

\[
= c_2(c_1^2 - 2c_2) - c_1(2c_1 - c_1 c_2) + 2(4 - c_1 c_1)
\]

\[
= c_2c_1^2 - 2c_2^2 - 2c_1^2 + c_1^2 c_2 + 8 - 2c_1^2
\]

Collecting like terms, we get

\[
= c_2c_1^2 + c_1^2 c_2 - 2c_2^2 - 2c_1^2 - 2c_1^2 + 8
\]

\[
= 2c_2c_1^2 - 2c_2^2 - 4c_1^2 + 8
\]

\[
= 8 + 2\Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2 \geq 0
\]
\[ D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = \left[ 8 + 2\text{Re}(c_1^2c_2) - 2|c_2|^2 - 4|c_1|^2 \right] \geq 0, \]
equivalent to
\[ 2c_2 = c_1^2 + x \left( 4 - c_1^2 \right), \text{ for some } x, |x| \leq 1. \] (9)
and
\[ D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_2 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} \geq 0, \]
equivalent to
\[ \left| (4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2 \right| \leq 2 \left( 4 - c_1^2 \right)^2 - 2 \left| (2c_2 - c_1^2) \right|^2. \] (10)
By simplifying (9) and (10), we obtained the following:
\[ 4c_3 = \left\{ c_1^3 + 2c_1 \left( 4 - c_1^2 \right) x - c_1 \left( 4 - c_1^2 \right) x^2 + 2 \left( 4 - c_1^2 \right) (1 - |x|^2) z \right\} \text{ for some } z, \text{ with } |z| \leq 1. \] (11)

**Lemma** (2([9],[10])) if \( p \in P \), then \( |p_k| \leq 2 \), for each \( k \geq 1 \) and the inequality is sharp for the function \( \frac{1+z}{1-z} \).

3. **MAIN RESULTS**

**Theorem 1** If \( f(z) \in BT(m, \beta) \), the class of bounded turning functions of order \( \beta \), with \( \beta < 1 \), \( m \in \{ 0, 1, 2, \ldots \} \), then
\[ |a_3 - ta_2^2| \leq \frac{4(1 - \beta)}{2 \times 3^{m+1}}. \]

**Proof.** From (8), we say that
\[ [D^mf(z)]' - \beta > 0 \]
For the function \( f(z) \in BT(m, \beta), \exists \) a holomorphic function \( p \in P \) in \( U = \{ z \in C : |z| < 1 \} \) with \( p(0) = 1 \) and \( \text{Rep}(z) > 0 \) such that
\[ \frac{[D^mf(z)]'}{1 - \beta} = p(z) \iff [D^mf(z)]' - \beta = (1 - \beta)p(z) \] (12)
Using the series representation for \( [D^mf(z)]' \) and \( p(z) \) in (6) and (2), we have
\[ \left\{ 1 + \sum_{k=2}^{\infty} k^{m+1}a_kz^{k-1} \right\} - \beta = (1 - \beta) \left\{ 1 + \sum_{k=1}^{\infty} c_kz^k \right\} \]
\[ (1 - \beta) + \left\{ \sum_{k=2}^{\infty} k^{m+1}a_kz^{k-1} \right\} = (1 - \beta) \left\{ 1 + \sum_{k=1}^{\infty} c_kz^k \right\}. \]
Simplifying, we have

\[(1 - \beta) + 2^{m+1} a_2 z + 3^{m+1} a_3 z^2 + 4^{m+1} a_4 z^3 + 5^{m+1} a_5 z^4 + 6^{m+1} a_6 z^5 + \ldots = (1 - \beta)\]
\[\left(1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + \ldots\right)\]

\[(1 - \beta) + 2^{m+1} a_2 z + 3^{m+1} a_3 z^2 + 4^{m+1} a_4 z^3 + 5^{m+1} a_5 z^4 + 6^{m+1} a_6 z^5 + \ldots = (1 - \beta)\]
\[+ (1 - \beta)c_1 z + (1 - \beta)c_2 z^2 + (1 - \beta)c_3 z^3 + (1 - \beta)c_4 z^4 + (1 - \beta)c_5 z^5 + (1 - \beta)c_6 z^6\]

\[2^{m+1} a_2 z + 3^{m+1} a_3 z^2 + 4^{m+1} a_4 z^3 + 5^{m+1} a_5 z^4 + 6^{m+1} a_6 z^5 + \ldots = (1 - \beta)c_1 z\]
\[+ (1 - \beta)c_2 z^2 + (1 - \beta)c_3 z^3 + (1 - \beta)c_4 z^4 + (1 - \beta)c_5 z^5 + (1 - \beta)c_6 z^6 + \ldots\]  
(13)

By equating the coefficients of the like powers of z's in (13), we have

\[2^{m+1} a_2 z = (1 - \beta)c_1 \implies a_2 = \frac{(1 - \beta)c_1}{2^{m+1}} \]  
(14)

\[3^{m+1} a_3 = (1 - \beta)c_2 \implies a_3 = \frac{(1 - \beta)c_2}{3^{m+1}} \]  
(15)

\[4^{m+1} a_4 = (1 - \beta)c_3 \implies a_4 = \frac{(1 - \beta)c_3}{4^{m+1}} \]  
(16)

\[5^{m+1} a_5 = (1 - \beta)c_4 \implies a_5 = \frac{(1 - \beta)c_4}{5^{m+1}} \]  
(17)

\[6^{m+1} a_6 = (1 - \beta)c_5 \implies a_6 = \frac{(1 - \beta)c_5}{6^{m+1}} \]  
(18)

In general, we can see that

\[|a_k| \leq \frac{(1 - \beta)c_{k-1}}{k^{m+1}} \]  
(19)

Substituting the values of \(a_2\) and \(a_3\) from (14) and (15) in the functional \(|a_3 - ta_2^2|\) for the function \(BT(m, \beta)\), we have

\[|a_3 - ta_2^2| \leq \left| \frac{(1 - \beta)c_2}{3^{m+1}} - t \left( \frac{(1 - \beta)c_1}{2^{m+1}} \right)^2 \right|
\leq \left| \frac{(1 - \beta)c_2}{3^{m+1}} - t \left( \frac{(1 - \beta)^2 c_1^2}{2^{2(m+1)}} \right) \right|\]
Using Lemma 1, we know that

\[
|a_3 - ta_2^2| \leq \left| \frac{(1 - \beta)}{3^{m+1}} \left[ \frac{1}{2} \left( c_1^2 + x(4 - c_1^2) \right) \right] - t \left( \frac{(1 - \beta)^2 c_1^2}{2^{m+1}} \right) \right|
\]
\[
|a_3 - ta_2^2| \leq \left| \frac{(1 - \beta) \{ c_1^2 + x(4 - c_1^2) \}}{2 \times 3^{m+1}} - t \left( \frac{(1 - \beta)^2 c_1^2}{2^{m+1}} \right) \right|
\]
\[
|a_3 - ta_2^2| \leq \left| \frac{(1 - \beta) \{ c_1^2 + x(4 - c_1^2) \}}{2 \times 3^{m+1}} - t \left( \frac{(1 - \beta)(1 - \beta)c_1^2}{2^{m+1}} \right) \right|
\]
\[
|a_3 - ta_2^2| \leq \left| \frac{(1 - \beta)c_1^2 + x(4 - c_1^2)(1 - \beta)}{2 \times 3^{m+1}} - t \left( \frac{(1 - 2\beta)c_1^2}{2^{m+1}} \right) \right|
\]
\[
|a_3 - ta_2^2| \leq \left| \frac{c_1^2 - \beta c_1^2 + 4x - 4x\beta - xc_1^2 + xc_1^2\beta}{2 \times 3^{m+1}} - t \left( \frac{(c_1^2 - 2\beta c_1^2 + \beta^2 c_1^2)}{2^{m+1}} \right) \right|
\]
\[
|a_3 - ta_2^2| \leq \left| \frac{c_1^2 - \beta c_1^2 + 4x - 4x\beta - xc_1^2 + xc_1^2\beta}{2 \times 3^{m+1}} - t \left( \frac{(t c_1^2 - 2\beta c_1^2 + \beta^2 c_1^2)}{2^{m+1}} \right) \right|
\]
\[
|a_3 - ta_2^2| \leq \left| \frac{c_1^2 - \beta c_1^2 + 4x - 4x\beta - xc_1^2 + xc_1^2\beta}{2 \times 3^{m+1}} - t \left( \frac{tc_1^2 - 2t\beta c_1^2 + t\beta^2 c_1^2}{2^{m+1}} \right) \right|
\]

Let \( |x| = t \in [0,1], c_1 = c \in [0,2] \) and applying the triangle inequality, above equation reduce to

\[
|a_3 - ta_2^2| \leq \left| \frac{4x}{2 \times 3^{m+1}} + \frac{4x\beta}{2 \times 3^{m+1}} + \frac{c_1^2}{2 \times 3^{m+1}} + \frac{tc_1^2}{2 \times 3^{m+1}} + \frac{tc_1^2\beta}{2 \times 3^{m+1}} + \frac{tc_1^2\beta^2}{2 \times 3^{m+1}} \right|
\]
\[
|a_3 - ta_2^2| \leq \left| \frac{4t}{2 \times 3^{m+1}} + \frac{4t\beta}{2 \times 3^{m+1}} + \frac{tc_1^2}{2 \times 3^{m+1}} + \frac{tc_1^2\beta}{2 \times 3^{m+1}} + \frac{tc_1^2\beta^2}{2 \times 3^{m+1}} \right|
\]

Suppose that

\[
F(c, t) := \frac{1}{2 \times 3^{m+1}} [4t + 4t\beta + tc_1^2 + t\beta c_1^2 + \beta c_1^2 + c_1^2] + \frac{1}{2^{m+2}} [tc_1^2 + 2t\beta c_1^2 + t\beta^2 c_1^2]
\]

Then we get

\[
\frac{\partial F}{\partial t} = \frac{1}{2 \times 3^{m+1}} [4 + 4\beta + c_1^2 + \beta c_1^2] + \frac{1}{2^{m+2}} [c_1^2 + 2\beta c_1^2 + \beta^2 c_1^2] \geq 0
\]
This shows that $F(c, t)$ is an increasing function on the closed interval $[0, 1]$ about $t$. Therefore, the function $F(c, t)$ can get the maximum value at $t = 1$, that is,

$$
max F(c, t) = F(c, 1) = \frac{1}{2 \times 3^{m+1}} [4 + 4\beta + c^2 + \beta c^2 + c^3] + \frac{1}{2^{2m+2}} [c^2 + 2\beta c^2 + \beta^2 c^2]
$$

Next, let

$$
G(c) := \frac{1}{2 \times 3^{m+1}} [4 + 4\beta + 2c^2 + 2\beta c^2] + \frac{1}{2^{2m+2}} [c^2 + 2\beta c^2 + \beta^2 c^2]
$$

Then we easily see that the function $G(c)$ have maximum value at $c = 0$, also which

$$
|a_3 - ta_2^2| \leq G(c) := \frac{4 + 4\beta}{2 \times 3^{m+1}} + \left[ \frac{1}{2^{2m+2}} + \frac{2\beta}{2 \times 3^{m+1}} + \frac{2\beta}{2 \times 3^{m+1}} + \frac{2\beta}{2^{2m+2}} + \frac{\beta^2}{2^{2m+2}} \right] \times 0^2
$$

$$
= \frac{4 + 4\beta}{2 \times 3^{m+1}} + 0
$$

$$
= \frac{4 + 4\beta}{2 \times 3^{m+1}}
$$

$$
|a_3 - ta_2^2| \leq \frac{4(1 - \beta)}{2 \times 3^{m+1}}
$$

\[\square\]

**Theorem 2** Let $f(z) \in BT(m, \beta)$, with $\beta < 1$, $m \in \{0, 1, 2, \ldots\}$. Then

$$
|a_2a_4 - a_3^2| \leq \frac{4(1 - \beta)^2}{3^{2m+2}}
$$

**Proof.** Recall from (14), (15) and (16) we have that

$$
a_2 = \frac{(1 - \beta)c_1}{2^{m+1}}, \quad a_3 = \frac{(1 - \beta)c_2}{3^{m+1}}, \quad a_4 = \frac{(1 - \beta)c_3}{4^{m+1}}
$$
Substituting the values of $a_2$, $a_3$ and $a_4$ from (20) in the functional $|a_2a_4 - a_3^2|$ for the function $f(z) \in BT(m, \beta)$, and simplifying, we get

$$|a_2a_4 - a_3^2| = \left| \left( \frac{(1-\beta)c_1}{2^{m+1}} \right) \left( \frac{(1-\beta)c_3}{4^{m+1}} \right) - \left( \frac{(1-\beta)c_2}{3^{m+1}} \right)^2 \right|$$

$$= \left| \left( \frac{(1-\beta)^2c_1c_3}{2^{m+1} 	imes 2^{2(m+1)}} \right) - \left( \frac{(1-\beta)c_2}{3^{m+1}} \right)^2 \right|$$

$$= \left| \frac{(1-\beta)^2}{2^{m+1}(m+1)(2m+2)} \right| c_1 c_3 - \frac{1}{3^{2(m+1)}c_2^2}$$

$$= (1-\beta)^2 \left| \frac{c_1 c_3}{2^{3(m+3)}} - \frac{c_2^2}{3^{2(m+1)}} \right|$$

$$= (1-\beta)^2 \left| \frac{1}{2^{3(m+1)+2}} c_1 c_3 - \frac{1}{3^{2(m+1)}c_2^2} \right|$$

$$|a_2a_4 - a_3^2| = \left( 1-\beta \right)^2 |w_1 c_1 c_3 + w_2 c_2^{2}|$$

(21)

where

$$w_1 = \frac{1}{2^{3(m+1)}}, \quad w_2 = -\frac{1}{3^{2(m+1)}}$$

(22)

Substituting the values of $c_2$ and $c_3$ from (9) and (11) respectively from Lemma 1 on the right-hand side of (21),

$$|w_1 c_1 c_3 + w_2 c_2^{2}| = \left| w_1 c_1 \times \frac{1}{4} \left( c_1^3 + 2c_1 (4-c_1^2) x - c_1 (4-c_1^2) x^2 + 2 (4-c_1^2) (1-|x|^2) z \right) \right|$$

$$+ w_1 \times \left[ \frac{1}{2} \left( c_1^2 + x (4-c_1^2) \right) \right]^2$$

$$|w_1 c_1 c_3 + w_2 c_2^{2}| = \left| \frac{1}{4} \left( w_1 c_1^3 + 2w_1 c_1^2 (4-c_1^2) x - w_1 c_1 (4-c_1^2) x^2 + 2w_1 c_1 (4-c_1^2) (1-|x|^2) z \right) \right|$$

$$+ \frac{1}{4} \left[ w_2 c_2^4 + w_2 (4-c_1^2)^2 x^2 \right]$$
\[ |w_1 c_1 c_3 + w_2 c_2^2| = \frac{1}{4} \{ w_1 c_1 c_3 + 2 w_1 c_1^2 (4 - c_1^2) x - w_1 c_1^2 (4 - c_1^2) x^2 + 2 w_1 c_1 (4 - c_1^2) (1 - |x|^2) z \}
\]
\[ + \frac{1}{4} \left[ w_2 c_4^4 + w_2 (4 - c_1^2)^2 x^2 \right] \]
\[ = \frac{1}{4} \left\{ w_1 c_1 c_3 + 2 w_1 c_1^2 (4 - c_1^2) x - w_1 c_1^2 (4 - c_1^2) x^2 + 2 w_1 c_1 (4 - c_1^2) (1 - |x|^2) z \right\} \]
\[ + w_2 c_4^4 + w_2 (4 - c_1^2)^2 x^2 \]
\[ = \frac{1}{4} \left\{ w_1 c_1 c_3 + 2 w_1 c_1^2 (4 - c_1^2) x - w_1 c_1^2 (4 - c_1^2) x^2 + 2 w_1 c_1 (4 - c_1^2) (1 - |x|^2) z \right\} \]
\[ + w_2 c_4^4 + w_2 (4 - c_1^2)^2 x^2 \]
\[ = \frac{1}{4} \left\{ w_1 c_1 c_3 + 2 w_1 c_1^2 (4 - c_1^2) x - w_1 c_1^2 (4 - c_1^2) x^2 + 2 w_1 c_1 (4 - c_1^2) (1 - |x|^2) z \right\} \]
\[ + w_2 c_4^4 + w_2 (4 - c_1^2)^2 x^2 \]
\[ 4 |w_1 c_1 c_3 + w_2 c_2^2| = \left| w_1 c_1 c_3 + 2 w_1 c_1^2 (4 - c_1^2) x - w_1 c_1^2 (4 - c_1^2) x^2 + 2 w_1 c_1 (4 - c_1^2) (1 - |x|^2) z \right| \]
\[ + w_2 c_4^4 + w_2 (4 - c_1^2)^2 x^2 \]
\[ = \left| w_1 c_1 c_3 + 2 w_1 c_1^2 (4 - c_1^2) x - w_1 c_1^2 (4 - c_1^2) x^2 + 2 w_1 c_1 (4 - c_1^2) (1 - |x|^2) z \right| \]
\[ - 2 w_1 c_1 (4 - c_1^2) |x|^2 z + w_2 (4 - c_1^2)^2 x^2 \]
\[ = \left| (w_1 + w_2) c_1^4 + 2 w_1 c_1 (4 - c_1^2) + 2 w_1 c_1^2 (4 - c_1^2) x - w_1 c_1^2 (4 - c_1^2) x^2 \right| \]
\[ - 2 w_1 c_1 (4 - c_1^2) |x|^2 z + w_2 (4 - c_1^2)^2 x^2 \]

Using the triangle inequality and the fact that \(|z| < 1\), and simplifying, we get

\[ 4 |w_1 c_1 c_3 + w_2 c_2^2| \leq \left| (w_1 + w_2) c_1^4 + 2 w_1 c_1 (4 - c_1^2) + 2 w_1 c_1^2 (4 - c_1^2) |x| \right| \]
\[ - w_1 c_1^2 (4 - c_1^2) |x|^2 - 2 w_1 c_1 (4 - c_1^2) |x|^2 + w_2 (4 - c_1^2)^2 |x|^2 \]

\[ 4 |w_1 c_1 c_3 + w_2 c_2^2| \leq \left| (w_1 + w_2) c_1^4 + 2 w_1 c_1 (4 - c_1^2) + 2 w_1 c_1^2 (4 - c_1^2) |x| \right| \]
\[ - w_1 c_1 (c_1 - 2) (4 - c_1^2) |x|^2 + w_2 (4 - c_1^2) (4 - c_1^2) |x|^2 \]

\[ (23) \]
Using the values of $w_1, w_2$ given in (22), we can write

$$w_1 + w_2 = \frac{1}{2^{3(m+1)}} + \left( -\frac{1}{3^2(m+1)} \right)$$

$$= \frac{1}{2^{3(m+1)}} - \frac{1}{3^2(m+1)}$$

$$= \frac{1}{2^{3(m+1)}} - \frac{3^2(m+1)}{3^2(m+1)}$$

$$= \frac{9^{m+1} - 8^{m+1}}{2^{3(m+1)} \times 3^2(m+1)}$$

and

$$2w_1 = 2 \left( \frac{1}{2^{3(m+1)}} \right)$$

$$= \frac{2}{2^{3(m+1)}}$$

$$= 2 \times 2^{-3(m+1)}$$

$$= 2 \times 2^{-3m-3}$$

$$= 2^{-3m-3+1}$$

$$= 2^{-3m-2}$$

$$= 2^{-(3m+2)}$$

$$2w_1 = \frac{1}{2^{3m+2}}$$

Therefore,

$$w_1 + w_2 = \frac{9^{m+1} - 8^{m+1}}{2^{3(m+1)} \times 3^2(m+1)}; \quad 2w_1 = \frac{1}{2^{3m+2}} \quad (24)$$

Substituting the values from (24) on the right–hand side of (23), we have

$$4 \left| w_1 c_1 c_3 + w_2 c_2^2 \right| \leq \left| \frac{(9^{m+1} - 8^{m+1})}{2^{3(m+1)} \times 3^2(m+1)} c_1^4 + \frac{1}{2(3m+2)} c_1(4 - c_1^3) + \frac{1}{2(3m+2)} c_1^2(4 - c_1^3) \right| |x| \frac{1}{2^{3(m+1)}} c_1^2(4 - c_1^2) \ |x| \ |

- \frac{1}{2(3m+2)} c_1(4 - c_1^3) \ |x| \ |x| \frac{1}{3^2(m+1)} (4 - c_1^2) \ (4 - c_1^2) \ |x| \ |

\leq \left| \frac{(9^{m+1} - 8^{m+1})}{2^{3(m+1)} \times 3^2(m+1)} c_1^4 + \frac{1}{2(3m+2)} c_1(4 - c_1^3) + \frac{1}{2(3m+2)} c_1^2(4 - c_1^3) \ |x| \ |

- \frac{1}{2(3m+2)} c_1(4 - c_1^3) \ |x| \ |x| \frac{1}{3^2(m+1)} (4 - c_1^2) \ (4 - c_1^2) \ |x| \ |
Since \( c_1 = c \in [0, 2] \), noting that \( c_1 + a = c_1 - a \), where \( a \geq 0 \) and replacing \(|x|\) by \( \mu \) on the right-hand side of the above inequality, we have

\[
4 |w_1 c_1 + w_2 c_2|^2 \leq \left[ \frac{(9^{m+1} - 8^{m+1})}{2^{3(m+1)} \times 3^{2m+2}} c^4 + \frac{1}{2^{3m+2}} (4 - c^2) c + \frac{1}{2^{3m+2}} c^2 (4 - c^2) \mu \right] \nonumber
\]

- \( \frac{1}{2^{3(m+1)}} c + 2 c (4 - c^2) \mu^2 - \frac{1}{2^{3(m+1)}} (4 - c^2)(4 - c^2) \mu^2 \) 
- \( \frac{9^{m+1} - 8^{m+1}}{2^{3m+3} \times 3^{2m+2}} c^4 + \frac{1}{2^{3m+2}} (4 - c^2) c + \frac{1}{2^{3m+2}} c^2 (4 - c^2) \mu 
+ \frac{1}{2^{3m+2}} (c + 2) c (4 - c^2) \mu^2 + \frac{1}{3^{2m+2}} (4 - c^2)(4 - c^2) \mu^2 \n
= F(c, \mu), 0 \leq \mu = |x| \leq 1 \quad \text{and} \quad 0 \leq c \leq 2 \quad (25)

where

\[
F(c, \mu) = \left[ \frac{(9^{m+1} - 8^{m+1})}{2^{3m+3} \times 3^{2m+2}} c^4 + \frac{1}{2^{3m+2}} (4 - c^2) c + \frac{1}{2^{3m+2}} c^2 (4 - c^2) \mu \n+ \frac{1}{3^{2m+2}} (c + 2) c (4 - c^2) \mu^2 + \frac{1}{3^{2m+2}} (4 - c^2)(4 - c^2) \mu^2 \right] \quad (26)
\]

Further, we maximize the function \( F(c, \mu) \) on the region \([0, 2] \times [0, 1] \). Differentiating \( F(c, \mu) \) given in (26) partially with respect to \( \mu \), we get

\[
\frac{\partial F}{\partial \mu} = \frac{1}{2^{3m+2}} (4 - c^2) c^2 + \frac{2}{2^{3m+3}} c (4 - c^2) \mu + \frac{2}{3^{2m+2}} (4 - c^2)(4 - c^2) \mu 
\frac{\partial F}{\partial \mu} = \frac{1}{2^{3m+2}} (4 - c^2) c^2 + 2 \times 2^{-3m+3} c (4 - c^2) \mu + \frac{2}{3^{2m+2}} (4 - c^2)(4 - c^2) \mu 
\frac{\partial F}{\partial \mu} = \frac{1}{2^{3m+2}} (4 - c^2) c^2 + 2 \times 3^{-3m+3} c (4 - c^2) \mu + \frac{2}{3^{2m+2}} (4 - c^2)(4 - c^2) \mu 
\frac{\partial F}{\partial \mu} = \frac{1}{2^{3m+2}} (4 - c^2) c^2 + \frac{1}{2^{3m+2}} c (4 - c^2) \mu + \frac{2}{3^{2m+2}} (4 - c^2)(4 - c^2) \mu 
\frac{\partial F}{\partial \mu} = \left[ \frac{c^2}{2^{3m+2}} + \frac{(c + 2) \mu}{2^{3m+2}} + \frac{(4 - c^2) \mu}{3^{2m+2}} \right] \times (4 - c^2) 
\frac{\partial F}{\partial \mu} = \left[ \frac{c^2 + (c + 2) \mu}{2^{3m+2}} + \frac{2 (4 - c^2) \mu}{3^{2m+2}} \right] \times (4 - c^2) \quad (27)
\]

For \( 0 < \mu < 1 \), for fixed \( c \) with \( 0 < c < 2 \) and for \( \beta, m \) with \( \beta < 1, m \in \{0, 1, 2, \ldots\} \), from (27), we observe that \( \frac{\partial F}{\partial \mu} > 0 \). Therefore, \( F(c, \mu) \) becomes an increasing function of \( \mu \) and hence it cannot have a maximum at any point in the interior of the closed region \([0, 2] \times [0, 1] \). Moreover, for fixed \( c \in [0, 2] \), we have

\[
\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c) \quad (28)
\]

In view of (28), simplifying the relation (26), we have
Collecting like terms, we

\[
G(c) = \frac{(9m+1 - 8m+1)}{2^{3m+3} \times 3^{2m+2}} c^4 + \frac{1}{2^{3m+2}} (4 - c^2) c + \frac{1}{2^{3m+2}} c^2 (4 - c^2) \mu \\
+ \frac{1}{2^{3m+3}} (c + 2) c (4 - c^2) \mu^2 + \frac{1}{3^{2m+2}} (4 - c^2) (4 - c^2) \mu^2 \\
G(c) = \frac{(9m+1 - 8m+1)}{2^{3m+3} \times 3^{2m+2}} c^4 + \frac{1}{2^{3m+2}} (4 - c^2) c + \frac{1}{2^{3m+2}} c^2 (4 - c^2) \times 1 \\
+ \frac{1}{2^{3m+3}} (c + 2) c (4 - c^2) \times 1^2 + \frac{1}{3^{2m+2}} (4 - c^2) (4 - c^2) \times 1^2 \\
G(c) = \frac{(9m+1 - 8m+1)}{2^{3m+3} \times 3^{2m+2}} c^4 + \frac{1}{2^{3m+2}} (4 - c^2) c + \frac{1}{2^{3m+2}} c^2 (4 - c^2) \\
+ \frac{1}{2^{3m+3}} (c + 2) c (4 - c^2) + \frac{1}{3^{2m+2}} (4 - c^2) (4 - c^2) \\
G(c) = \left( \frac{1}{2^{3m+3}} - \frac{1}{3^{2m+2}} \right) c^4 + \frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{1}{2^{3m+3}} \left[ (4c^2 - c^4 + 8c - 2c^3) \right] \\
+ \frac{1}{3^{2m+2}} \left[ c^4 - 8c^2 + 16 \right] \\
G(c) = \frac{c^4}{2^{3m+3}} - \frac{c^4}{3^{2m+2}} + \frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{1}{2^{3m+3}} \left[ (4c^2 - c^4 + 8c - 2c^3) \right] \\
+ \frac{1}{3^{2m+2}} \left[ c^4 - 8c^2 + 16 \right] \\
G(c) = \frac{c^4}{2^{3m+3}} - \frac{c^4}{3^{2m+2}} + \frac{4c}{2^{3m+2}} - \frac{c^3}{2^{3m+2}} + \frac{4c^2}{2^{3m+2}} - \frac{c^4}{2^{3m+2}} + \frac{1}{2^{3m+3}} \left[ (4c^2 - c^4 + 8c - 2c^3) \right] \\
+ \frac{1}{3^{2m+2}} \left[ c^4 - 8c^2 + 16 \right] \\
\]
\[ G(c) = \frac{1}{2m+3} \left[ 8c + 4c^2 - 2c^3 \right] + \frac{1}{2m+3} \left[ 4c + 4c^2 - c^3 - c^4 \right] + \frac{1}{2m+2} \left[ 16 - 8c^2 \right] \]

\[ G(c) = \frac{8c}{2m+3} + \frac{4c^2}{2m+3} - \frac{2c^3}{2m+3} + \frac{4c^2}{2m+3} + \frac{4c^2}{2m+3} - \frac{c^3}{2m+2} - \frac{c^4}{2m+2} - \frac{8c^2}{2m+2} + \frac{16}{2m+2} \]

\[ G(c) = \left(2^3 \times 2^{-3(3m+3)}\right) c + \left(2^2 \times 2^{-(3m+3)}\right) c^2 - \frac{2c^3}{2m+3} + \left(2^2 \times 2^{-(3m+2)}\right) c + \left(2^2 \times 2^{-(3m+2)}\right) c^2 - \frac{c^3}{2m+2} - \frac{c^4}{2m+2} - \frac{8c^2}{2m+2} + \frac{16}{2m+2} \]

\[ G(c) = \frac{c}{2m} + \frac{c^2}{2m+1} - \frac{2c^3}{2m+3} + \frac{c^2}{2m+1} - \frac{c^3}{2m+2} - \frac{c^4}{2m+2} + \frac{8c^2}{2m+2} + \frac{16}{2m+2} \]

\[ G(c) = \frac{c}{2m} + \frac{c^2}{2m} + \frac{c^2}{2m+1} - \frac{8c^2}{2m+2} - \frac{c^3}{2m+3} - \frac{c^4}{2m+3} + \frac{16}{2m+3} \]

\[ G(c) = \frac{16}{2m+2} + \frac{2c}{2m} + \left[ \frac{1}{2m} + \frac{1}{2m+1} - \frac{8}{2m+2} \right] c^2 - \left[ \frac{2}{2m+3} + \frac{1}{2m+2} \right] c^3 - \frac{c^4}{2m+2} \tag{29} \]

\[ G'(c) = \frac{2}{2m} + 2 \left[ \frac{1}{2m} + \frac{1}{2m+1} - \frac{8}{2m+2} \right] c - 3 \left[ \frac{2}{2m+3} + \frac{1}{2m+2} \right] c^2 - \frac{4c^3}{2m+2} \]

\[ G'(c) = \frac{2}{2m} + \frac{2}{2m+1} - \frac{16}{2m+2} \right] c - \left[ \frac{6}{2m+3} + \frac{3}{2m+2} \right] c^2 - \frac{4c^3}{2m+2} \]

\[ G'(c) = \frac{2}{2m} + \frac{3}{2m+1} - \frac{16}{2m+2} \right] c - \left[ \frac{6}{2m+3} + \frac{3}{2m+2} \right] c^2 - \frac{4c^3}{2m+2} \]

\[ G'(c) = \left[ \frac{2}{2m} + \frac{1}{2m} - \frac{16}{2m+2} \right] c - \left[ \frac{6}{2m+3} + \frac{3}{2m+2} \right] c^2 - \frac{12c^2}{2m+2} \]

\[ G''(c) = \left[ \frac{2}{2m} + \frac{1}{2m} - \frac{16}{2m+2} \right] c - \left[ \frac{12}{2m+3} + \frac{6}{2m+2} \right] c^2 - \frac{12c^2}{2m+2} \]

\[ G''(c) = \frac{3}{2m} - \frac{16}{2m+2} - \frac{12c}{2m+3} - \frac{6c^2}{2m+2} - \frac{12c^2}{2m+2} \tag{31} \]

For optimum value of \( G(c) \), consider \( G'(c) = 0 \), then the root is \( c = 0 \), with \( m \in \{0, 1, 2, \ldots\} \). After a simple calculation, we can deduce that \( G''(c) > 0 \), which means
that the function $G(c)$ can take the minimum value at $c = 0$, $m \in \{0, 1, 2, \ldots \}$, also which is

$$|a_2a_4 - a_3^2| \leq G(0) = \frac{16}{3^{2m+2}} + \frac{2 \times 0}{2^{3m}} + \left[ \frac{1}{2^{3m}} + \frac{1}{2^{3m+1}} - \frac{8}{3^{2m+2}} \right] \times 0^2 - \left[ \frac{2}{2^{3m+3}} + \frac{1}{2^{3m+2}} \right] \times 0^3 \leq \frac{16}{3^{2m+2}}$$

Simplifying the expression (25) and (32), we get

$$|w_1c_1c_3 + w_2c_2^2| \leq \frac{1}{4} \times \left( \frac{16}{3^{2m+2}} \right)$$

$$|w_1c_1c_3 + w_2c_2^2| \leq \left( \frac{16}{3^{2m+2}} \right) \div \frac{4}{1}$$

$$|w_1c_1c_3 + w_2c_2^2| \leq \left( \frac{16}{3^{2m+2}} \right) \times \frac{1}{4}$$

$$|w_1c_1c_3 + w_2c_2^2| \leq \frac{4}{3^{2m+2}}$$

From the relation (21) and (33), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \leq (1 - \beta)^2 \times \frac{4}{3^{2m+2}}$$

$$|a_2a_4 - a_3^2| \leq \frac{4(1 - \beta)^2}{3^{2m+2}}$$

**Conclusion:** In this paper, we mainly determined the bound for the well-known Fekete-Szego functional $|a_3 - ta_2^2|$, with $t$ real and the Second Hankel Determinant $|a_2a_4 - a_3^2|$ for functions of bounded turning of order $\beta$ associated with Salagean differential operator.

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