Electronic Journal of Mathematical Analysis and Applications
Vol. 1(2) July 2013, pp. 138-148.
ISSN: 2090-792X (online)
http://ejmaa.6te.net/

MULTIPLE POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS WITH FRACTIONAL ORDER

MOUFFAK BENCHOHRA, BENAOUDA HEDIA

Abstract. In this paper we investigate the existence of multiple solutions for nonlinear boundary value problems with fractional order differential equations. We shall rely on the Leggett-Williams fixed point theorem.

1. Introduction

This paper is concerned with the existence of three nonnegative solutions for boundary value problems (BVP for short) of fractional order functional differential equations. We consider the BVP of the form :

$$^cD^\alpha y(t) + f(t, y(t)) = 0, \text{ for each, } t \in J = [0, T], \quad 1 < \alpha \le 2, \tag{1}$$

$$y(0) - y'(0) = \int_0^T g(s, y(s))ds, \tag{2}$$

$$y(T) + y'(T) = \int_0^T h(s, y(s))ds, \tag{3}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f, g, h : J \times \mathbb{R} \to [0, +\infty)$ are continuous functions.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic [6, 17, 27, 28]. The existence of positive solutions for kinds of boundary-value problems (BVPs) of fractional differential equations has been studied recently by many authors, and lot of excellent results have been obtained for both two-point BVPs and nonlocal BVPs by means of fixed point index theory, see [11, 12, 15, 17, 18, 30]. For three noteworthy papers dealing with the integral operator and the arbitrary fractional order differential operator,

2000 *Mathematics Subject Classification.* 26A33, 34B15.
Key words and phrases. Differential Equation, Caputo fractional derivative, fractional integral, existence, multiple solutions, fixed point.
Submitted December 18, 2012.

see [22]. There has been a significant development in fractional differential equations in recent years see the books by Abbas *et al.* [1], Lakshmikantham *et al.* [23], and the papers by [5, 24], and the references therein.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. These include two-point, three-point, multipoint and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamics [9] and cellular systems [2]. Moreover boundary value problems with integral boundary conditions have been studied by a number of authors such as, for instance Benchohra *et al.* [8, 7], Denche and Kourta [10], Infante [18], Jankowskii [19], Karakostas and Tsamatos [20], and Khan [21]. Agarwal and O'Regan [3] considered the existence of three nonnegative solutions to a class of impulsive differential equations and they established existence of three solutions to integral and discrete equations. Positive solutions of differential, difference and integral equations have considered in [4]. Some existence results were given for the problem (1)-(2) with $\alpha = 1$ by Tisdell in [29]. The existence of multiple solutions for differential, difference and integral equations has been investigated by several authors (see, for instance [3] and the references cited therein).

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we shall provide sufficient conditions ensuring the existence of three nonnegative solutions for problem $(1)-(3)$ via an application of the Leggett-Williams fixed point theorem in cones [25]. Finally in Section 4 we give an example to illustrate the theory presented in the previous sections.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

$L^\infty(J,\mathbb{R})$ denotes the Banach space of measurable and essentially bounded functions with norm

$$\|y\|_{L^\infty} = \inf\{d > 0 : |y(t)| \leq d, \text{ a.e. } t \in J\}.$$

Let $(E, \|.\|)$ be a Banach space and $C \subset E$ be a cone in E. by a concave, nonnegative and continuous functional ψ on C, we mean a continuous mapping

$$\psi : C \to [0,\infty)$$

with

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y) \quad \text{for all} \quad x, y \in C \quad \text{and} \quad \lambda \in [0,T]$$

For $K, L, r \geq 0$ constants with C and ψ as above, let

$$C_K = \{y \in C : \|y\| < K\}$$

and

$$C(\psi, L, K) = \{y \in C : \psi(y) \geq L \quad \text{and} \quad \|y\| \leq K\}.$$

Our consideration is based on the following fixed point theorem given by Leggett and Williams in 1979 [25] (see also Guo and Lakshmikantham [16]).

Theorem 2.1. *Let E be a Banach space, $C \subset E$ a cone in E and $R > 0$ a constant. Suppose there exists a concave nonnegative continuous functional on C with $\psi(y) \leq \|y\|$ for all $y \in \overline{C_R}$ and let $N : \overline{C_R} \to \overline{C_R}$ be a continuous compact map. Assume that there are numbers r, L and K with $0 < r < L < K \leq R$ such that*

- (A_1) *$\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \emptyset$ and $\psi(N(y)) > L$ for all $y \in C(\psi, L, K)$;*
- (A_2) *$\|N(y)\| < r$ for all $y \in \overline{C_r}$;*
- (A_3) *$\psi(N(y)) > L$ for all $y \in C(\psi, L, R)$ with $\|N(y)\| > K$.*

Then N has at least three fixed points y_1, y_2, y_3 in $\overline{C_R}$. Furthermore, we have

$$y_1 \in C_r, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\}$$

and

$$y_3 \in \overline{C_R} - \{C(\psi, L, R) \cup \overline{C_r}\}.$$

Definition 2.2. *([13]-[14]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.3. *([13]-[14]). For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional-order derivative of h, $\alpha \in (0, 1)$, is defined by*

$$\begin{aligned}
(D_{a+}^\alpha h)(t) &= \frac{d^\alpha h(t)}{dt^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} h(s) ds \\
&= \frac{d}{dt} I_a^{1-\alpha} h(t).
\end{aligned}$$

Definition 2.4. *For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, $\alpha \in (0, 1)$, is defined by*

$$({}^c D_{a+}^\alpha h)(t) = \big(D_{a+}^\alpha [h(x) - h(a)]\big)(t).$$

3. Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1)–(3).

Definition 3.1. *A function $y \in C^2(J, \mathbb{R})$ is said to be a solution of (1)-(3) if y satisfies the equation ${}^c D^\alpha y(t) = f(t, y(t))$ on J, and conditions $y(0) - y'(0) = \displaystyle\int_0^T g(s, y(s)) ds$ and $y(T) + y'(T) = \displaystyle\int_0^T h(s, y(s)) ds$.*

For the existence of solutions for the problem (1)-(3), we need the following auxiliary lemma:

Lemma 3.2. [30] *Let $\alpha > 0$, then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, c_i \in \mathbb{R},\ \ i = 0, 1, 2, \ldots, n-1,\ n = [\alpha] + 1$.

Lemma 3.3. [30] *Let $\alpha > 0$, then*

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R},\ \ i = 0, 1, 2, \ldots, n-1,\ n = [\alpha] + 1$.

Let σ be a continuous function and consider the linear problem

$$^cD^\alpha y(t) = \sigma(t),\ \ t \in J, \tag{4}$$

$$y(0) - y'(0) = \int_0^T \rho_1(s) ds \tag{5}$$

$$y(T) + y'(T) = \int_0^T \rho_2(s) ds, \tag{6}$$

then

Lemma 3.4. [7] *The problem (9)-(10) has a unique solution given by:*

$$y(t) = P(t) + \int_0^T G(t,s)\sigma(s) ds, \tag{7}$$

where

$$P(t) = \frac{(T+1-t)}{T+2} \int_0^T \rho_1(s) ds + \frac{(t+1)}{T+2} \int_0^T \rho_2(s) ds \tag{8}$$

and

$$G(t,s) = \begin{cases} \dfrac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} + \dfrac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \\ \dfrac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} + \dfrac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)}, & t \le s < T. \end{cases} \tag{9}$$

Let us now introduce additional conditions that will be used our existence result.

(H_1) There exist functions $\nu : [0,\infty) \to [0,\infty)$ continuous, nondecreasing and $q \in L^\infty(J, \mathbb{R}^+)$ such that

$$|f(t,u)| \le q(t)\nu(|u|) \text{ for each } t \in J \text{ and all } u \in \mathbb{R}.$$

(H_2) There exist functions $\mu,\ \mu_1,\ \mu_2 : [0,\infty) \to [0,\infty)$ continuous, nondecreasing and $p, p_1, p_2 \in L^\infty(J, \mathbb{R}^+)$ such that

$$f(t,u) \ge p(t)\mu(u) \text{ for each } t \in J \text{ and all } u \in \mathbb{R} \text{ with } \ u \ge 0,$$

$$g(t,u) \le p_1(t)\mu_1(u) \text{ for each } t \in J \text{ and all } u \in \mathbb{R} \text{ with } \ u \ge 0,$$

$$h(t,u) \le p_2(t)\mu_2(u) \text{ for each } t \in J \text{ and all } u \in \mathbb{R} \text{ with } \ u \ge 0.$$

(H_3) There exists a constant $r > 0$ such that

$$P_r + \mu(r)\|p\|_\infty \sup_{t\in J}\left\{\int_0^T G(t,s)ds\right\} < r,$$

where

$$P_r = \frac{(T+1)}{T+2}\mu_1(r)\|p_1\|_\infty + \frac{(T+1)}{T+2}\mu_2(r)\|p_2\|_\infty.$$

(H_4) There exists a constant $L > r$ and an interval $[a,b] \subset (0,T)$ such that

$$\mu(L)\min_{t\in[a,b]}\left(\int_0^T G(t,s)p(s)ds\right) \geq L.$$

(H_5) There exist constants c_0, c_1 such that

$$P(t) \geq c_0 P_R,\ \ G(t,s) \geq c_1 G(s,s).$$

(H_6) There exist, $0 < r < L < K < R$ with $M^{-1}L \leq K$ such that

$$P_R + \mu(R)\|p\|_\infty \sup\left\{\int_0^T G(t,s)ds,\ t\in J\right\} \leq R,$$

where

$$M = \min(c_0,\ c_1).$$

Theorem 3.5. *Suppose that hypotheses $(H_1)-(H_6)$ are satisfied. Then the boundary value problem (1)–(3) has at least three positive solutions.*

Proof. Our result is based on Legett-Williams fixed point theorem. Transform the problem (1)-(3) into a fixed point problem. Consider the operator

$$N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$$

defined by

$$(Ny)(t) = P(t) + \int_0^T G(t,s)f(s,y(s))ds,$$

where

$$P(t) = \frac{(T+1-t)}{T+2}\int_0^T g(s,y(s))ds + \frac{(t+1)}{T+2}\int_0^T h(s,y(s))ds,$$

and the function $G(t,s)$ is given by (9). Clearly, the fixed points of the operator N are solution of the problem (1)-(3). We shall show that N satisfies the assumptions of Leggett-Williams fixed point theorem. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,\mathbb{R})$ Then for each $t \in [0,T]$ we have

$$\begin{aligned}
|N(y_n)(t) - N(y)(t)| \leq\ & \frac{T+1}{T+2}\int_0^T |g(s,y_n(s)) - g(s,y(s))|ds \\
+\ & \frac{T+1}{T+2}\int_0^T |h(s,y_n(s)) - h(s,y(s))|ds \\
+\ & \int_0^T G(s,t)|f(s,y_n(s)) - f(s,y(s))|ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{T(T+1)}{T+2}\|g(\cdot,y_n(\cdot))-g(\cdot,y(\cdot))\|_\infty \\
&\quad +\frac{T(T+1)}{T+2}\|h(\cdot,y_n(\cdot))-h(\cdot,y(\cdot))\|_\infty \\
&\quad + T\tilde{G}\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty,
\end{aligned}$$

where

$$\tilde{G}=\sup\left\{\int_0^T |G(t,s)|ds,\ t\in J\right\}.$$

Since f,h,g are continuous functions, we have

$$\|N(y_n)-N(y)\|_\infty \to 0 \quad as\ n\to\infty.$$

Step 2: *N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.*

Indeed, it is enough to show that for any $\eta^*>0$, there exists a positive constant ℓ such that for each $y\in B_{\eta^*}=\{y\in C(J,\mathbb{R}),\|y\|_\infty\leq\eta^*\}$, we have $\|N(y)\|_\infty\leq\ell$. By (H_1) we have for each $t\in[0,T]$,

$$\begin{aligned}
|N(y)(t)| &\leq \frac{T+1}{T+2}\int_0^T|g(s,y(s))|ds+\frac{T+1}{T+2}\int_0^T|h(s,y(s))|ds \\
&+ \int_0^T|G(t,s)||f(s,y(s))|ds \\
&\leq \frac{(T+1)}{T+2}\mu_1(\|y\|_\infty)\int_0^T p_1(s)ds \\
&\quad +\frac{(T+1)}{T+2}\mu_2(\|y\|_\infty)\int_0^T p_2(s)ds+\nu(\|y\|_\infty)\int_0^T G(t,s)q(s)ds.
\end{aligned}$$

Thus for every $t\in J$, we have

$$\|y\|_\infty\leq P_{\eta^*}+\nu(\eta^*)\tilde{G}\|q\|_\infty:=\ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

Let $t_1,t_2\in[0,T],\ t_1<t_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$ as in Step 2, and let $y\in B_{\eta^*}$. Then by (H_1) we have :

$$\begin{aligned}
|N(y)(t_2)-N(y)(t_1)| &\leq \Bigg|P(t_2)+\int_0^T G(t_2,s)f(s,y(s))ds \\
&\quad -P(t_1)-\int_0^T G(t_1,s)f(s,y(s))ds\Bigg| \\
&\leq \left|\frac{(T+1-t_2)}{T+2}\int_0^T g(s,y(s))ds-\frac{(T+1-t_1)}{T+2}\int_0^T g(s,y(s))ds\right| \\
&\quad +\left|\frac{(t_2+1)}{T+2}\int_0^T h(s,y(s))ds-\frac{(t_1+1)}{T+2}\int_0^T h(s,y(s))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^T G(t_2,s)f(s,y(s))ds-\int_0^T G(t_1,s)f(s,y(s))ds\right|\\
\leq\ &\frac{T(t_1-t_2)}{T+2}\mu_1(\eta^*)\|p_1\|_\infty+\frac{T(t_2-t_1)}{T+2}\mu_2(\eta^*)\|p_2\|_\infty\\
&+\nu(\eta^*)\|q\|_\infty\int_0^T|G(t_2,s)-G(t_1,s)|ds.
\end{aligned}
$$

The right-hand side of the above inequality tends to zero, as $t_2\to t_1$ and this proves that: $N(B(0,\eta^*))$ is equicontinuous in $C(J,\mathbb{R})$. As a consequence of the steps 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that the operator $N\ :\ C(J,\mathbb{R})\to C(J,\mathbb{R})$ is continuous and completely continuous.

Let

$$\mathcal{C}=\{y\in C(J,\mathbb{R}):y(t)\geq 0\ \text{ for }\ t\in J\}$$

be a cone in $C(J,\mathbb{R})$. Using the hypotheses (H_2) and (H_6), we prove that $N(\mathcal{C})\subset\mathcal{C}$ and $N:\overline{C}_R\to\overline{C}_R$ is completely continuous. Let $\psi:\mathcal{C}\to[0,\infty)$ be defined by :

$$\psi(y)=\min_{t\in[a,b]}y(t).$$

It is clear that ψ is a nonnegative concave continuous functional and

$$\psi(y)\leq\|y\|_\infty\ \text{ for }\ y\in\overline{C}_R.$$

Now it remains to show that the hypotheses of Theorem 2.1 are satisfied. First notice from (H_3) that condition (A_2) of Theorem 2.1 holds since for $y\in\overline{C}_r$, one need only to see others conditions. Next, let

$$y(t)=\frac{L+K}{2}\ \text{ for }\ t\in[0,T].$$

By the definition of $C(\psi,L,K)$, y belongs to $C(\psi,L,K)$. Also for arbitrary $y\in C(\psi,L,K)$. From (H_2) and (H_4), one has

$$
\begin{aligned}
\psi(N(y)) &= \min_{t\in[a,b]}\left(P(t)+\int_0^T G(t,s)f(s,y(s))ds\right)\\
&\geq\min_{t\in[a,b]}\left(\int_0^T G(t,s)f(s,y(s))ds\right)\\
&\geq\min_{t\in[a,b]}\left(\int_0^T G(t,s)p(s)\mu(\|y\|)ds\right)\\
&\geq\mu(L)\min_{t\in[a,b]}\left(\int_0^T G(t,s)p(s)ds\right)\geq L,
\end{aligned}
$$

which establishes condition (A_1) of Theorem 2.1.

Finally, we will show that (A_3) of Theorem 2.1 holds. To that end, let $y\in C(\psi,L,R)$ with $\|N(y)\|_\infty>K$. Then by (H_5) and (H_6), we have

$$\psi(N(y))\qquad\geq\min_{t\in[a,b]}\left(P(t)+\int_0^T G(t,s)f(s,y(s))ds\right)$$

$$\geq \min_{t\in[a,b]} \left(c_0 P_R + c_1 \int_0^T G(s,s) f(s,y(s))ds \right)$$
$$\geq M\|Ny\|_\infty \geq MK \geq L.$$

Thus, condition (A_3) of Theorem 2.1 holds.

Then, the Leggett-Williams fixed point theorem implies that N has at least three fixed points y_1, y_2 and y_3 which are fixed points N solutions to the problem (1)-(3). Furthermore, we have :

$$y_1 \in C_r, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\},$$
$$y_3 \in \overline{C_R} - \{C(\psi, L, R) \cup \overline{C_r}\}.$$

4. Example

We consider the following fractional boundary value problem,

$$^cD^{\frac{3}{2}} y(t) = \frac{12|y(t)|}{(1+t)e^{-t}}, \quad t \in J := [0,1], \tag{10}$$

$$y(0) - y'(0) = \int_0^1 \frac{e^s|y(s)|}{4(1+e^s)(1+|y(s)|)} ds, \tag{11}$$

$$y(1) + y'(1) = \int_0^1 \frac{e^s|y(s)|}{8(1+|y(s)|)} ds. \tag{12}$$

Set

$$f(t,u) = \frac{12u}{(t+1)e^{-t}}, \quad (t,u) \in J \times [0,\infty),$$

$$g(s,u) = \frac{e^s|u|}{4(1+e^s)(1+|u|)},$$

$$h(s,u) = \frac{e^s|u|}{8(1+|u|)},$$

$$\mu(u) = 12u, \quad u \geq 0, \quad p(t) = \frac{1}{t+1}, \quad t \in [0,1],$$

$$\nu(u) = 12u, \quad u \geq 0, \quad q(t) = \frac{1}{e^{-t}}, \quad t \in [0,1].$$

We have

$$|f(t,u)| \leq q(t)\nu(u), \quad f(t,u) \geq p(t)\mu(u), \quad (t,u) \in [0,1] \times [0,\infty),$$

$$\mu_1(u) = \frac{3u}{8(e-1)}, \ u \geq 0, \quad p_1(t) = e^t, \ t \in [0,1],$$

$$\mu_2(u) = \frac{3u}{8(e-1)}, \ u \geq 0, \quad p_2(t) = e^t, \quad t \in [0,1],$$

$$P_r = \frac{2}{3}\mu_1(r) \int_0^1 p_1(s)ds + \frac{2}{3}\mu_2(r) \int_0^1 p_2(s)ds = \frac{1}{2}r.$$

G is given by :

$$G(t,s)=\begin{cases} \dfrac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)}+\dfrac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)}-\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\\ \dfrac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)}+\dfrac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)}, & t\le s\ <1. \end{cases} \tag{13}$$

Because $\alpha=\frac{3}{2},\ \Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2},\ \Gamma(\frac{1}{2})=\sqrt{\pi}$ then (13) becomes :

$$G(t,s)=\frac{1}{3\sqrt{\pi}}\begin{cases} 2(1+t)\sqrt{1-s}+\dfrac{1+t}{\sqrt{1-s}}-3\sqrt{t-s}, & 0\le s\le t\le 1\\ 2(1+t)\sqrt{1-s}+\dfrac{1+t}{\sqrt{1-s}}, & 0\le t\le s<1. \end{cases} \tag{14}$$

An easy computation shows that :

$$\sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)p(s)ds\right\}=\frac{1+\alpha}{3\alpha\Gamma(\alpha)},$$

$$\min_{t\in[0,1]}\left(\int_0^1 G(t,s)p(s)ds\right)=\frac{\alpha-\frac{1}{2}}{3\alpha\Gamma(\alpha)}.$$

Because $\alpha=\frac{3}{2}$, then we have :

$$\sup_{t\in[0,1]}\left\{\int_0^1 G(t,s)p(s)ds\right\}=\frac{10}{9\sqrt{\pi}},$$

and

$$\min_{t\in[0,1]}\left(\int_0^1 G(t,s)p(s)ds\right)=\frac{2}{9\sqrt{\pi}}.$$

From this we have :

$$P_r+\mu(r)\sup_{t\in J}\left\{\int_0^1 G(t,s)p(s)ds\right\}<r,$$

that is

$$\frac{1}{2}r+\frac{10}{9\sqrt{\pi}}r-r<0,$$

thus

$$\frac{-1}{2}r+\frac{10}{9\sqrt{\pi}}r<0,$$

and then (H_3) holds since r a positive constant. Also, we have :

$$\mu(L)\min_{t\in[a,b]}\left(\int_0^1 G(t,s)p(s)ds\right)\ge L,$$

and so

$$\frac{12L(\alpha-\frac{1}{2})}{3\alpha\Gamma(\alpha)}\ge L,$$

Putting $\alpha=\frac{3}{2}$ we have then :

$$\frac{24}{9\sqrt{\pi}}-1\ge 1,$$

which implies that (H_4) holds since L is a positive constant.

Now

$$P_R + \mu(R) \sup \left\{ \int_0^1 G(t,s)p(s)ds,\ t \in J \right\} \leq R,$$

and so

$$\frac{1}{2}R + \frac{10}{9\sqrt{\pi}}R - R < 0,$$

which yields

$$\frac{-1}{2}R + \frac{10}{9\sqrt{\pi}}R < 0,$$

and then(H_6) holds since R is a positive constant.

Assume there exist a constant $c_0,\ c_1, K > 0$, such that

$$P(t) \geq c_0 P_R,\ G(t,s) \geq c_1 G(s,s),$$

set $M = \min\{c_0, c_1\}$, and chose $r, L, K, R,$ such that $0 < r < L < K < R$, with $M^{-1}L \leq K$, then problem $(10) - (12)$ has at least three positive solutions y_1, y_2, y_3 in $\overline{C_R}$. Furthermore, we have

$$y_1 \in C_r, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\},$$

and

$$y_3 \in \overline{C_R} - \{C(\psi, L, R) \cup \overline{C_r}\}.$$

$\psi,\ C_r,\ C(\psi, L, R)$ are defined as above.

References

[1] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[2] G. Adomian and G. E. Adomian, Cellular systems and aging models, *Comput. Math. Appl.* **11** (1985), 283-291.

[3] R.P. Agarwal and D. O'Regan, Existence of three solutions to integral and discrete equations via the Leggett-Williams fixed point theorem, *Rocky Mountain J. Math.* **31** (2001), 23-35.

[4] R.P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[5] Z. Bai, On positive solutions of a nonlocal fractional boundary-value problem, *Nonlinear Analysis*, **72**, (2010), 916-924 .

[6] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, New York, 2012.

[7] M. Benchohra, F. Berhoun and J. Henderson, Multiple positive solutions for impulsive boundary value problem with integral boundary conditions, *Math. Sci. Res. J.* **11** (12) (2007), 614-626.

[8] M. Benchohra, S. Hamani and J. Henderson, Functional differential inclusions with integral boundary conditions, *Eletron. J. Qual. The. Differ. Equa.* **15** (2007), 1-13.

[9] K. W. Blayneh, Analysis of age structured host-parasitoid model, *Far East J. Dyn. Syst.* **4** (2002), 125-145.

[10] M. Denche and A. Kourta, Boundary value problem for second-order differential operators with integral conditions, *Appl. Anal.* **84** (2005), 1247-1266.

[11] P. W. Eloe and B. Ahmad, Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Appl. Math. Lett.* (18) **5** (2005), 521-527.

[12] M. El-Shahed, Positive solutions for boundary value problem of nonlinear fractional dierential equation, *Abstr. Appl. Anal.*, **ID 10368** (2007) 1-8.

[13] A. M. A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, *Nonlinear Anal.* **33** (1998), 181-186.

[14] A. M. A. El-Sayed and A. G. Ibrahim, Multivalued fractional differential equations, *Appl. Math. Comput.* **68** (1995), 15-25.

[15] M. Feng and W. Ge, Positive solutions for a class of m-point singular boundary value problems. *Math. Comput. Model.* **46** (2007), 375-383.

[16] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.
[17] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[18] G. Infante, Eigenvalues and positive solutions of ODEs involving integral boundary conditions, *Discrete Contin. Dyn. Syst.* (2005), 436-442.
[19] T. Jankowski, Differential equation with integral boundary conditions, *J. Comput. Appl. Math.* **147** (2002), 1-8.
[20] G. L. Karakostas and P. Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, *Electron. J. Differential Equations* **30** (2002), 1-17.
[21] R. A. Khan, The generalized method of quasilinearization and nonlinear boundary value problems with integral boundary conditions, *Electron. J. Qual. Theory Differ. Equ* **10** (2003), 1-15.
[22] A.A. Kilbas, H.M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
[23] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.
[24] C. F. Li, Xiannan Luo, Yong Zhou: Existence of positive solutions of the boundary-value problem for nonlinear fractional differential equations, *Comput. Math. Appl.*, **59** (2010), 1363-1375.
[25] R.W. Leggett and L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.* **28** (1979), 673-688.
[26] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity. An introduction to mathematical models.* Imperial College Press, London, 2010.
[27] I. Podlubny, *Fractional Differential Equation*, Academic Press, San Diego, 1999.
[28] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.
[29] C. Tisdell, On the solvability of nonlinear first-order boundary-value problems. *Electron. J. Differential Equations* **80** (2006), 1-8.
[30] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, *Electron. J. Differential Equations* **36** (2006), 1-12.

Mouffak Benchohra
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès,
B.P. 89, 22000, Sidi Bel-Abbès, Algérie,
E-mail address: benchohra@univ-sba.dz

Benaouda Hedia
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès,
B.P. 89, 22000, Sidi Bel-Abbès, Algérie,
E-mail address: nilpot_hedia@yahoo.fr