SOME CLASSES OF ORDER $\alpha$ FOR SECOND-ORDER DIFFERENTIAL INEQUALITIES

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Abstract. For analytic functions $f(z)$ in the open unit disk $U$ with $f(0) = f'(0) = 1 = 0$, S. S. Miller and P. T. Mocanu (Integral Transform. Spec. Funct. 19(2008)) have considered some sufficient problems for starlikeness. The object of the present paper is to discuss some sufficient problems for $f(z)$ to be in some classes of order $\alpha$.

1. Introduction

Let $A_n$ denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$$

that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $A = A_1$. We denote by $S$ the subclass of $A_n$ consisting of univalent functions $f(z)$ in $U$. Let $S^*(\alpha)$ be defined by

$$S^*(\alpha) = \left\{ f(z) \in A_n : \Re \left( \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad 0 \leq \Re \alpha < 1 \right\}.$$

We denote by $S^* = S^*(0)$. Also, let $C(\alpha)$ be

$$C(\alpha) = \left\{ f(z) \in A_n : \Re f'(z) > \alpha, \quad 0 \leq \Re \alpha < 1 \right\}.$$

We also denote by $C = C(0)$. Also, let $K(\alpha)$ be defined by

$$K(\alpha) = \left\{ f(z) \in A_n : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad 0 \leq \Re \alpha < 1 \right\}.$$

We denote by $K = K(0)$. From the definitions for $S^*(\alpha)$ and $K(\alpha)$, we know that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$.

The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [3]).

Lemma 1 Let the function $w(z)$ defined by

$$w(z) = a_nz^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$$

(1) $n = 1, 2, 3, \ldots$
be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

and

$$\Re \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) + 1 \geq k.$$

2. Main results

Applying Lemma 1, we derive the following lemma.

**Lemma 2** If $f(z) \in \mathcal{A}_n$ satisfies

$$\left| zf''(z) - \beta \left( f'(z) - \frac{f(z)}{z} \right) \right| < \rho |n + 1 - \beta| \quad (z \in \mathbb{U})$$

for some real $\rho > 0$ and some complex $\beta$ with $\Re(\beta) < n + 1$, then

$$\left| f'(z) - \frac{f(z)}{z} \right| < \rho \quad (z \in \mathbb{U}).$$

**Proof.** Let us define $w(z)$ by

$$w(z) = f'(z) - \frac{f(z)}{z} \quad (1)$$

Then, clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differentiating both sides in (1), we obtain

$$zf''(z) = zw'(z) + w(z) \quad (z \in \mathbb{U}),$$

and therefore,

$$\left| zf''(z) - \beta \left( f'(z) - \frac{f(z)}{z} \right) \right| = \left| zw'(z) + (1 - \beta)w(z) \right|$$

$$= |w(z)| \left| \frac{zw'(z)}{w(z)} + 1 - \beta \right|$$

$$< \rho |n + 1 - \beta| \quad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0)$ ($k \geq n$). Thus we have

$$\left| z_0 f''(z_0) - \beta \left( f'(z_0) - \frac{f(z_0)}{z_0} \right) \right| = |w(z_0)| \left| \frac{zw'(z_0)}{w(z_0)} + 1 - \beta \right|$$

$$= \rho |k + 1 - \beta|$$

$$\geq \rho |n + 1 - \beta|.$$  

This contradicts our condition in the lemma. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$, that is, that

$$\left| f'(z) - \frac{f(z)}{z} \right| < \rho \quad (z \in \mathbb{U}).$$
Also applying Lemma 1, we have

**Lemma 3** If \( f(z) \in A_n \) satisfies

\[
\left| zf''(z) - \beta \left( f'(z) - \frac{f(z)}{z} \right) \right| < \rho n|n + 1 - \beta| \quad (z \in \mathbb{U})
\]

for some real \( \rho > 0 \) and some complex \( \beta \) with \( \Re(\beta) < n + 1 \), then

\[
\left| \frac{f(z)}{z} - 1 \right| < \rho \quad (z \in \mathbb{U}).
\]

**Proof.** Let us define the function \( w(z) \) by

\[
w(z) = \frac{f(z)}{z} - 1 = a_n + 1 z^n + a_{n+2} z^{n+1} + \ldots \quad (z \in \mathbb{U}).
\]

Clearly, \( w(z) \) is analytic in \( \mathbb{U} \) and \( w(0) = 0 \). We want to prove that \( |w(z)| < \rho \) in \( \mathbb{U} \). Since

\[
zf''(z) = z^2 w''(z) + 2zw'(z) \quad (z \in \mathbb{U}),
\]

we see that

\[
\left| zf''(z) - \beta \left( f'(z) - \frac{f(z)}{z} \right) \right| = |z^2 w''(z) + (2 - \beta)zw'(z)|
< \rho n|n + 1 - \beta| \quad (z \in \mathbb{U}).
\]

If there exists a point \( z_0 \in \mathbb{U} \) such that

\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,
\]

then Lemma 1 gives us that \( w(z_0) = \rho e^{i\theta}, \ z_0 w'(z_0) = kw(z_0) (k \geq n) \) and

\[
\Re \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) + 1 \geq k.
\]

Thus we have

\[
\left| z_0 f''(z_0) - \beta \left( f'(z_0) - \frac{f(z_0)}{z_0} \right) \right| = |z_0^2 w''(z_0) + (2 - \beta)z_0 w'(z_0)|
= |z_0 w'(z_0)| \left| \frac{z_0 w''(z_0)}{w'(z_0)} + 2 - \beta \right|
= \rho k \left| \frac{z_0 w''(z_0)}{w'(z_0)} + 2 - \beta \right|
\geq \rho k \Re \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) + 2 - \beta
\geq \rho k |k + 1 - \beta|
\geq \rho n|n + 1 - \beta|.
\]

This contradicts the condition in the lemma. Therefore, there is no \( z_0 \in \mathbb{U} \) such that \( |w(z_0)| = \rho \). This means that \( |w(z)| < \rho \) for all \( z \in \mathbb{U} \).

From Lemma 2 and Lemma 3, we drive the following results for \( S^*(\alpha) \).

**Theorem 1** If \( f(z) \in A_n \) satisfies

\[
\left| zf''(z) - \beta \left( f'(z) - \frac{f(z)}{z} \right) \right| < \frac{(1 - \alpha)n|n + 1 - \beta|}{n + 1 - \alpha} \quad (z \in \mathbb{U})
\]
for some real $0 \leq \alpha < 1$ and some complex $\beta$ with $\Re(\beta) < n + 1$, then

$$\frac{|zf''(z)|}{f(z)} - 1 < 1 - \alpha \quad (z \in \mathbb{U}),$$

so that $f(z) \in \mathcal{S}^*(\alpha)$.

**Proof.** From Lemma 2 and Lemma 3, we have

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{n(1 - \alpha)}{n + 1 - \alpha} \quad (z \in \mathbb{U}) \quad (2)$$

and

$$\left| \frac{f(z)}{z} - 1 \right| < \frac{1 - \alpha}{n + 1 - \alpha} \quad (z \in \mathbb{U}). \quad (3)$$

From (2) and (3),

$$\frac{n(1 - \alpha)}{n + 1 - \alpha} > \left| f'(z) - \frac{f(z)}{z} \right| = \left| \frac{f(z)}{z} \right| \left| \frac{zf''(z)}{f(z)} - 1 \right| > \left| 1 - \frac{1 - \alpha}{n + 1 - \alpha} \right| \left| \frac{zf''(z)}{f(z)} - 1 \right| = \frac{n}{n + 1 - \alpha} \left| \frac{zf''(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).$$

So, we can get

$$\frac{n}{n + 1 - \alpha} \left| \frac{zf''(z)}{f(z)} - 1 \right| < \frac{n(1 - \alpha)}{n + 1 - \alpha} \quad (z \in \mathbb{U}),$$

which completes the proof of the theorem.

When we put $f(z)$ by $zf'(z)$ in Theorem 1, we have

**Corollary 1** If $f(z) \in \mathcal{A}_n$ satisfies

$$|zf''(z) + (2 - \beta)zf'''(z)| < \frac{(1 - \alpha)n|n + 1 - \beta|}{n + 1 - \alpha} \quad (z \in \mathbb{U})$$

for some real $0 \leq \alpha < 1$ and some complex $\beta$ with $\Re(\beta) < n + 1$, then

$$\left| 1 + \frac{zf'''(z)}{f'(z)} \right| - 1 < 1 - \alpha \quad (z \in \mathbb{U}),$$

so that $f(z) \in \mathcal{K}(\alpha)$.

**Example 1** For some real $0 \leq \alpha < 1$ and some complex $\beta$ with $\Re(\beta) < n + 1$, we consider the function $f(z)$ given by

$$f(z) = z + \frac{1 - \alpha}{n + 1 - \alpha} z^{n+1} \quad (z \in \mathbb{U}).$$

The function $f(z)$ satisfies Theorem 1.

Next, we consider $\mathcal{C}(\alpha)$.

**Theorem 2** If $f(z) \in \mathcal{A}_n$ satisfies

$$|zf''(z) - \beta(f'(z) - 1)| < (1 - \alpha)|n - \beta| \quad (z \in \mathbb{U})$$

for some real $0 \leq \alpha < 1$ and some complex $\beta$ with $\Re(\beta) < n$, then

$$|f'(z) - 1| < 1 - \alpha \quad (z \in \mathbb{U}).$$

This means that $f(z) \in \mathcal{C}(\alpha)$. 
Proof. Define \( w(z) \) in \( U \) by
\[
 w(z) = \frac{f'(z) - 1}{1 - \alpha} = \frac{(n+1)a_{n+1}z^n + (n+2)a_{n+2}z^{n+1} + \ldots}{1 - \alpha} \quad (z \in U).
\]
Evidently, \( w(z) \) analytic in \( U \) and \( w(0) = 0 \). We want to prove \( |w(z)| < 1 \). Differentiating (4) and simplifying, we obtain
\[
 zf''(z) = (1 - \alpha)zw'(z) \quad (z \in U).
\]
and, hence
\[
 |zf''(z) - \beta(f'(z) - 1)| = |(1 - \alpha)zw'(z) - \beta(1 - \alpha)w(z)|
\]
\[
 = (1 - \alpha)|w(z)| \left| \frac{zw'(z)}{w(z)} - \beta \right|
\]
\[
 < (1 - \alpha)|n - \beta| \quad (z \in U).
\]
If there exists a point \( z_0 \in U \) such that
\[
 \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,
\]
then Lemma 1 gives us that \( w(z_0) = e^{i\theta} \) and \( z_0w'(z_0) = kw(z_0) \) \((k \geq n)\). Thus we have
\[
 |z_0f''(z_0) - \beta(f'(z_0) - 1)| = (1 - \alpha)|w(z_0)| \left| \frac{z_0w'(z_0)}{w(z_0)} - \beta \right|
\]
\[
 = (1 - \alpha)|k - \beta|
\]
\[
 \geq (1 - \alpha)|n - \beta|.
\]
This contradicts our condition in the theorem. Therefore, there is no \( z_0 \in U \) such that \( w(z_0) = 1 \). This means that \( |w(z)| < 1 \) for all \( z \in U \).

Example 2 For some real \( 0 \leq \alpha < 1 \) and some complex \( \beta \) with \( \Re(\beta) < n \), we take
\[
 f(z) = z + \frac{1 - \alpha}{n+1}z^{n+1} \quad (z \in U).
\]
Then, \( f(z) \) satisfies Theorem 2.

We get the following lemma from Lemma 1.

Lemma 4 If \( f(z) \in A_n \) satisfies
\[
 |zf''(z) - \beta(f'(z) - 1)| < \rho|n - \beta| \quad (z \in U)
\]
for some real \( \rho > 0 \) and some complex \( \beta \) with \( \Re(\beta) < n \), then
\[
 |f'(z) - 1| < \rho \quad (z \in U).
\]
Proof. Letting
\[
 w(z) = \frac{f'(z) - 1}{1 - \alpha} = \frac{(n+1)a_{n+1}z^n + (n+2)a_{n+2}z^{n+1} + \ldots}{1 - \alpha} \quad (z \in U),
\]
we see that \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). Noting that
\[
 zf''(z) = zw'(z) \quad (z \in U),
\]
\[ |zf''(z) - \beta(f'(z) - 1)| = |zw'(z) - \beta w(z)| \]
\[ = |w(z)| \left| \frac{zw'(z)}{w(z)} - \beta \right| \]
\[ < \rho |n - \beta| \quad (z \in \mathbb{U}). \]

If there exists a point \( z_0 \in \mathbb{U} \) such that
\[ \max_{|z| = |z_0|} |w(z)| = |w(z_0)| = \rho, \]
then Lemma 1 gives us that \( w(z_0) = \rho e^{i\theta} \) and \( z_0 w'(z_0) = kw(z_0) (k \geq n) \). Thus we have
\[ |z_0f''(z_0) - \beta(f'(z_0) - 1)| = |w(z_0)| \left| \frac{z_0w'(z_0)}{w(z_0)} - \beta \right| \]
\[ = \rho |k - \beta| \]
\[ \geq \rho |n - \beta| \]
which contradicts our condition in the lemma. Therefore, there is no \( z_0 \in \mathbb{U} \) such that \( |w(z_0)| = \rho \). This means that \( |w(z)| < \rho \) for all \( z \in \mathbb{U} \).

Using Lemma 4, we have next theorem.

**Theorem 3** If \( f(z) \in A_n \) satisfies
\[ |zf''(z) - \beta(f'(z) - 1)| < \alpha |n - \beta| \quad (z \in \mathbb{U}) \]
for some real \( 0 < \alpha \leq \frac{1}{2} \) and some complex \( \beta \) with \( \Re(\beta) < \alpha \), or
\[ |zf''(z) - \beta(f'(z) - 1)| < (1 - \alpha) |n - \beta| \quad (z \in \mathbb{U}) \]
for some real \( \frac{1}{2} \leq \alpha < 1 \) and some complex \( \beta \) with \( \Re(\beta) < \alpha \), then
\[ \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}), \]
which implies that \( f(z) \in C(\alpha) \).

**Proof.** We can get
\[ |f'(z) - 1| < \rho \quad (z \in \mathbb{U}). \quad (5) \]
for \( 0 < \alpha \leq \frac{1}{2} \) and \( \rho = \alpha \), or \( \frac{1}{2} \leq \alpha < 1 \) and \( \rho = 1 - \alpha \) from Lemma 4. Using (5), we have
\[ |f'(z) - 2\alpha| < S < |f'(z)| \]
for \( 0 < \alpha \leq \frac{1}{2} \) and \( S = 1 - \alpha \), or \( \frac{1}{2} \leq \alpha < 1 \) and \( S = \alpha \). Thus we get
\[ S \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < |f'(z)| \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| \]
\[ = \left| 1 - \frac{f'(z)}{2\alpha} \right| \]
\[ = \frac{1}{2\alpha} |f'(z) - 2\alpha| \]
\[ < \frac{S}{2\alpha} \quad (z \in \mathbb{U}). \]
So we obtain
\[ S \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \frac{S}{2\alpha} \quad (z \in \mathbb{U}). \]

**Example 3** For some real \( 0 < \alpha \leq \frac{1}{2} \) and some complex \( \beta \) with \( \Re(\beta) < n \), we consider the function \( f(z) \) given by
\[ f(z) = z + \frac{\alpha}{n+1} z^{n+1} \quad (z \in \mathbb{U}). \]

The function \( f(z) \) satisfies Theorem 3.

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