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SOME CLASSES OF ORDER α FOR SECOND-ORDER DIFFERENTIAL INEQUALITIES

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Abstract. For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ with $f(0) = f'(0) - 1 = 0$, S. S. Miller and P. T. Mocanu (Integral Transform. Spec. Funct. **19**(2008)) have considered some sufficient ploblems for starlikeness. The object of the present paper is to discuss some sufficient problems for $f(z)$ to be in some classes of order α.

1. Introduction

Let $\mathcal{A}_n$ denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \qquad (n = 1, 2, 3, \dots)$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A} = \mathcal{A}_1$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}_n$ consisting of univalent functions $f(z)$ in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ be defined by

$$\mathcal{S}^*(\alpha) = \left\{ f(z) \in \mathcal{A}_n : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leqq {}^{\exists}\alpha < 1 \right\}.$$

We denote by $\mathcal{S}^* = \mathcal{S}^*(0)$. Also, let $\mathcal{C}(\alpha)$ be

$$\mathcal{C}(\alpha) = \left\{ f(z) \in \mathcal{A}_n : \Re f'(z) > \alpha,\ 0 \leqq {}^{\exists}\alpha < 1 \right\}.$$

We also denote by $\mathcal{C} = \mathcal{C}(0)$. Also, let $\mathcal{K}(\alpha)$ be defined by

$$\mathcal{K}(\alpha) = \left\{ f(z) \in \mathcal{A}_n : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \leqq {}^{\exists}\alpha < 1 \right\}.$$

We denote by $\mathcal{K} = \mathcal{K}(0)$. From the definitions for $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, we know that $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$.
The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [3]).
Lemma 1 Let the function $w(z)$ defined by

$$w(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \qquad (n = 1, 2, 3, \dots)$$

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be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geqq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

and

$$\Re\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) + 1 \geqq k.$$

2. Main results

Applying Lemma 1, we derive the following lemma.

Lemma 2 If $f(z) \in \mathcal{A}_n$ satisfies

$$\left| z f''(z) - \beta\left(f'(z) - \frac{f(z)}{z}\right)\right| < \rho|n + 1 - \beta| \qquad (z \in \mathbb{U})$$

for some real $\rho > 0$ and some complex β with $\Re(\beta) < n + 1$, then

$$\left| f'(z) - \frac{f(z)}{z}\right| < \rho \qquad (z \in \mathbb{U}).$$

Proof. Let us define $w(z)$ by

$$\begin{aligned} w(z) &= f'(z) - \frac{f(z)}{z} \qquad (1)\\ &= n a_{n+1} z^n + (n+1) a_{n+2} z^{n+1} + \dots \qquad (z \in \mathbb{U}). \end{aligned}$$

Then, clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differentiating both sides in (1), we obtain

$$z f''(z) = z w'(z) + w(z) \qquad (z \in \mathbb{U}),$$

and therefore,

$$\begin{aligned} \left| z f''(z) - \beta\left(f'(z) - \frac{f(z)}{z}\right)\right| &= |z w'(z) + (1 - \beta) w(z)| \\ &= |w(z)| \left| \frac{z w'(z)}{w(z)} + 1 - \beta \right| \\ &< \rho |n + 1 - \beta| \qquad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0)$ $(k \geqq n)$. Thus we have

$$\begin{aligned} \left| z_0 f''(z_0) - \beta\left(f'(z_0) - \frac{f(z_0)}{z_0}\right)\right| &= |w(z_0)| \left| \frac{z_0 w'(z_0)}{w(z_0)} + 1 - \beta \right| \\ &= \rho |k + 1 - \beta| \\ &\geqq \rho |n + 1 - \beta|. \end{aligned}$$

This contradicts our condition in the lemma. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$, that is, that

$$\left| f'(z) - \frac{f(z)}{z}\right| < \rho \qquad (z \in \mathbb{U}).$$

Also applying Lemma 1, we have
Lemma 3 If $f(z) \in \mathcal{A}_n$ satisfies

$$\left| zf''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| < \rho n |n + 1 - \beta| \qquad (z \in \mathbb{U})$$

for some real $\rho > 0$ and some complex β with $\Re(\beta) < n + 1$, then

$$\left| \frac{f(z)}{z} - 1 \right| < \rho \qquad (z \in \mathbb{U}).$$

Proof. Let us define the function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{f(z)}{z} - 1 \\ &= a_{n+1} z^n + a_{n+2} z^{n+1} + \dots \qquad (z \in \mathbb{U}). \end{aligned}$$

Clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. We want to prove that $|w(z)| < \rho$ in $\mathbb{U}$. Since

$$zf''(z) = z^2 w''(z) + 2zw'(z) \qquad (z \in \mathbb{U}),$$

we see that

$$\begin{aligned} \left| zf''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| &= |z^2 w''(z) + (2 - \beta) z w'(z)| \\ &< \rho n |n + 1 - \beta| \qquad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$, $z_0 w'(z_0) = k w(z_0)$ $(k \geqq n)$ and

$$\Re \left(\frac{z_0 w''(z_0)}{w'(z_0)} \right) + 1 \geqq k.$$

Thus we have

$$\begin{aligned} \left| z_0 f''(z_0) - \beta \left(f'(z_0) - \frac{f(z_0)}{z_0} \right) \right| &= |z_0^2 w''(z_0) + (2 - \beta) z_0 w'(z_0)| \\ &= |z_0 w'(z_0)| \left| \frac{z_0 w''(z_0)}{w'(z_0)} + 2 - \beta \right| \\ &= \rho k \left| \frac{z_0 w''(z_0)}{w'(z_0)} + 2 - \beta \right| \\ &\geqq \rho k \left| \Re \left(\frac{z_0 w''(z_0)}{w'(z_0)} \right) + 2 - \beta \right| \\ &\geqq \rho k |k + 1 - \beta| \\ &\geqq \rho n |n + 1 - \beta|. \end{aligned}$$

This contradicts the condition in the lemma. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$.

From Lemma 2 and Lemma 3, we drive the following results for $\mathcal{S}^*(\alpha)$.
Theorem 1 If $f(z) \in \mathcal{A}_n$ satisfies

$$\left| zf''(z) - \beta \left(f'(z) - \frac{f(z)}{z} \right) \right| < \frac{(1 - \alpha) n |n + 1 - \beta|}{n + 1 - \alpha} \qquad (z \in \mathbb{U})$$

for some real $0 \leqq \alpha < 1$ and some complex β with $\Re(\beta) < n+1$, then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \qquad (z \in \mathbb{U}),$$

so that $f(z) \in \mathcal{S}^*(\alpha)$.

Proof. From Lemma 2 and Lemma 3, we have

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{n(1-\alpha)}{n+1-\alpha} \qquad (z \in \mathbb{U}) \tag{2}$$

and

$$\left|\frac{f(z)}{z} - 1\right| < \frac{1-\alpha}{n+1-\alpha} \qquad (z \in \mathbb{U}). \tag{3}$$

From (2) and (3),

$$\begin{aligned}\frac{n(1-\alpha)}{n+1-\alpha} &> \left|f'(z) - \frac{f(z)}{z}\right| \\ &= \left|\frac{f(z)}{z}\right| \left|\frac{zf'(z)}{f(z)} - 1\right| \\ &> \left(1 - \frac{1-\alpha}{n+1-\alpha}\right) \left|\frac{zf'(z)}{f(z)} - 1\right| \\ &= \frac{n}{n+1-\alpha} \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U}).\end{aligned}$$

So, we can get

$$\frac{n}{n+1-\alpha} \left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{n(1-\alpha)}{n+1-\alpha} \qquad (z \in \mathbb{U}).$$

which completes the proof of the theorem.

When we put $f(z)$ by $zf'(z)$ in Theorem 1, we have

Corollary 1 If $f(z) \in \mathcal{A}_n$ satisfies

$$|z^2 f'''(z) + (2-\beta) z f''(z)| < \frac{(1-\alpha)n|n+1-\beta|}{n+1-\alpha} \qquad (z \in \mathbb{U})$$

for some real $0 \leqq \alpha < 1$ and some complex β with $\Re(\beta) < n+1$, then

$$\left|\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| < 1 - \alpha \qquad (z \in \mathbb{U}),$$

so that $f(z) \in \mathcal{K}(\alpha)$.

Example 1 For some real $0 \leqq \alpha < 1$ and some complex β with $\Re(\beta) < n+1$, we consider the function $f(z)$ given by

$$f(z) = z + \frac{1-\alpha}{n+1-\alpha} z^{n+1} \qquad (z \in \mathbb{U}).$$

The function $f(z)$ satisfies Theorem 1.

Next, we consider $\mathcal{C}(\alpha)$.

Theorem 2 If $f(z) \in \mathcal{A}_n$ satisfies

$$|zf''(z) - \beta(f'(z) - 1)| < (1-\alpha)|n-\beta| \qquad (z \in \mathbb{U})$$

for some real $0 \leqq \alpha < 1$ and some complex β with $\Re(\beta) < n$, then

$$|f'(z) - 1| < 1 - \alpha \qquad (z \in \mathbb{U}).$$

This means that $f(z) \in \mathcal{C}(\alpha)$.

Proof. Define $w(z)$ in $\mathbb{U}$ by

$$w(z) = \frac{f'(z)-1}{1-\alpha} \tag{4}$$
$$= \frac{(n+1)a_{n+1}}{1-\alpha}z^n + \frac{(n+2)a_{n+2}}{1-\alpha}z^{n+1} + \dots \qquad (z \in \mathbb{U}).$$

Evidently, $w(z)$ analytic in $\mathbb{U}$ and $w(0) = 0$. We want to prove $|w(z)| < 1$. Differentiating (4) and simplifiying, we obtain

$$zf''(z) = (1-\alpha)zw'(z) \qquad (z \in \mathbb{U}).$$

and, hence

$$\begin{aligned}|zf''(z) - \beta(f'(z)-1)| &= |(1-\alpha)zw'(z) - \beta(1-\alpha)w(z)| \\ &= (1-\alpha)|w(z)|\left|\frac{zw'(z)}{w(z)} - \beta\right| \\ &< (1-\alpha)|n-\beta| \qquad (z \in \mathbb{U}).\end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ $(k \geqq n)$. Thus we have

$$\begin{aligned}|z_0 f''(z_0) - \beta(f'(z_0)-1)| &= (1-\alpha)|w(z_0)|\left|\frac{z_0 w'(z_0)}{w(z_0)} - \beta\right| \\ &= (1-\alpha)|k-\beta| \\ &\geqq (1-\alpha)|n-\beta|.\end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $w(z_0) = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$.

Example 2 For some real $0 \leqq \alpha < 1$ and some complex β with $\Re(\beta) < n$, we take

$$f(z) = z + \frac{1-\alpha}{n+1}z^{n+1} \qquad (z \in \mathbb{U}).$$

Then, $f(z)$ satisfies Theorem 2.

We get the following lemma from Lemma 1.

Lemma 4 If $f(z) \in \mathcal{A}_n$ satisfies

$$|zf''(z) - \beta(f'(z)-1)| < \rho|n-\beta| \qquad (z \in \mathbb{U})$$

for some real $\rho > 0$ and some complex β with $\Re(\beta) < n$, then

$$|f'(z) - 1| < \rho \qquad (z \in \mathbb{U}).$$

Proof. Letting

$$\begin{aligned}w(z) &= f'(z) - 1 \\ &= (n+1)a_{n+1}z^n + (n+2)a_{n+2}z^{n+1} + \dots \qquad (z \in \mathbb{U}),\end{aligned}$$

we see that $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Noting that

$$zf''(z) = zw'(z) \qquad (z \in \mathbb{U}),$$

we have

$$\begin{aligned} |zf''(z) - \beta(f'(z) - 1)| &= |zw'(z) - \beta w(z)| \\ &= |w(z)| \left| \frac{zw'(z)}{w(z)} - \beta \right| \\ &< \rho|n - \beta| \qquad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ $(k \geqq n)$. Thus we have

$$\begin{aligned} |z_0 f''(z_0) - \beta(f'(z_0) - 1)| &= |w(z_0)| \left| \frac{z_0 w'(z_0)}{w(z_0)} - \beta \right| \\ &= \rho|k - \beta| \\ &\geqq \rho|n - \beta| \end{aligned}$$

which contradicts our condition in the lemma. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$.

Using Lemma 4, we have next theorem.

Theorem 3 If $f(z) \in \mathcal{A}_n$ satisfies

$$|zf''(z) - \beta(f'(z) - 1)| < \alpha|n - \beta| \qquad (z \in \mathbb{U})$$

for some real $0 < \alpha \leqq \frac{1}{2}$ and some complex β with $\Re(\beta) < n$, or

$$|zf''(z) - \beta(f'(z) - 1)| < (1 - \alpha)|n - \beta| \qquad (z \in \mathbb{U})$$

for some real $\frac{1}{2} \leqq \alpha < 1$ and some complex β with $\Re(\beta) < n$, then

$$\left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \qquad (z \in \mathbb{U}),$$

which implies that $f(z) \in \mathcal{C}(\alpha)$.

Proof. We can get

$$|f'(z) - 1| < \rho \qquad (z \in \mathbb{U}). \tag{5}$$

for $0 < \alpha \leqq \frac{1}{2}$ and $\rho = \alpha$, or $\frac{1}{2} \leqq \alpha < 1$ and $\rho = 1 - \alpha$ from Lemma 4. Using (5), we have

$$|f'(z) - 2\alpha| < S < |f'(z)|$$

for $0 < \alpha \leqq \frac{1}{2}$ and $S = 1 - \alpha$, or $\frac{1}{2} \leqq \alpha < 1$ and $S = \alpha$. Thus we get

$$\begin{aligned} S \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| &< |f'(z)| \left| \frac{1}{f'(z)} - \frac{1}{2\alpha} \right| \\ &= \left| 1 - \frac{f'(z)}{2\alpha} \right| \\ &= \frac{1}{2\alpha} |f'(z) - 2\alpha| \\ &< \frac{S}{2\alpha} \qquad (z \in \mathbb{U}). \end{aligned}$$

So we obtain

$$S\left|\frac{1}{f'(z)}-\frac{1}{2\alpha}\right|<\frac{S}{2\alpha} \qquad (z\in\mathbb{U}).$$

Example 3 For some real $0<\alpha\leqq\frac{1}{2}$ and some complex β with $\Re(\beta)<n$, we consider the function $f(z)$ given by

$$f(z)=z+\frac{\alpha}{n+1}z^{n+1} \qquad (z\in\mathbb{U}).$$

The function $f(z)$ satisfies Theorem 3.

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