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PROPERTIES OF P-VALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH LINEAR OPERATOR

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ABSTRACT. In this paper we introduce a class of p-valent meromorphic functions associated with an integral operator and obtain some properties for functions belonging to this class.

1. INTRODUCTION

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\}$.

For functions $f(z) \in \sum_p$, given by (1.1) and $g(z) \in \sum_p$ defined by

$$g(z) = z^{-p} +_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} +_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.3}$$

Using the operator $Q^{\alpha}_{\beta,p} : \sum_p \to \sum_p$ defined by Aqlan et al. [1], where:

$$Q^{\alpha}_{\beta,p} f(z) = \begin{cases} z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum\limits_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} a_{k-p} z^{k-p} & (\alpha > 0; \beta > -1; p \in \mathbb{N}; f \in \Sigma_p) \\ f(z) & (\alpha = 0\ ; \beta > -1; p \in \mathbb{N}; f \in \Sigma_p)\ . \end{cases} \tag{1.4}$$

Mostafa [5] defined the operator $H^{\alpha}_{p,\beta,\mu} : \Sigma_p \to \Sigma_p$ as follows:
For $Q^{\alpha}_{\beta,p}$, given by (1.4), let $G^{\alpha*}_{\beta,p,\mu}$ be defined by

$$Q^{\alpha}_{\beta,p}(z) * G^{\alpha*}_{\beta,p,\mu}(z) = \frac{1}{z^p(1-z)^{\mu}} \quad (\mu > 0; p \in \mathbb{N}). \tag{1.5}$$

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Then

$$H^{\alpha}_{p,\beta,\mu}f(z) = G^{\alpha *}_{\beta,p}(z) * f(z) \quad (f \in \Sigma_p)\,. \tag{1.6}$$

Using (1.4) – (1.6), we have

$$H^{\alpha}_{p,\beta,\mu}f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta+\alpha)(\mu)_k}{\Gamma(k+\beta)(1)_k}a_{k-p}z^{k-p}\,, \tag{1.7}$$

where $f \in \Sigma_p$ is in the form (1.1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N})\,. \end{cases}$$

It is readily verified from (1.7) that (see [5])

$$z(H^{\alpha}_{p,\beta,\mu}f(z))^{'} = (\alpha+\beta)H^{\alpha+1}_{p,\beta,\mu}f(z) - (\alpha+\beta+p)H^{\alpha}_{p,\beta,\mu}f(z) \tag{1.8}$$

and

$$z(H^{\alpha}_{p,\beta,\mu}f(z))^{'} = \mu H^{\alpha}_{p,\beta,\mu+1}f(z) - (\mu+p)H^{\alpha}_{p,\beta,\mu}f(z)\,. \tag{1.9}$$

It is noticed that, putting $\mu = 1$ in (1.7), we obtain the operator

$$H^{\alpha}_{p,\beta,1}f(z) = H^{\alpha}_{p,\beta}f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}a_{k-p}z^{k-p}, \tag{1.10}$$

and

$$H^{0}_{p,\beta,1}f(z) = f(z).$$

Recently, Frasin [3] (see also [2]) has obtained new properties of meromorphic p-valent functions.

In the present paper using the operator $H^{\alpha}_{p,\beta,\mu}$defined by (1.7), we investigate some new properties of meromorphic p-valent functions.

Definition 1. Let H be the set of complex valued functions $h(r,s,t) : \mathbb{C}^3 \to \mathbb{C}$ such is continuous in a domain $D \subset \mathbb{C}^3, (1,1,1) \in D$, $|h(1,1,1)| < 1$ and

$$\left| h\left(e^{i\theta}, \frac{(\alpha+\beta-1)e^{i\theta}+\delta+1}{\alpha+\beta}, \frac{(\alpha+\beta-1)^2e^{2i\theta}+3(\alpha+\beta-1)(\delta+1)e^{i\theta}+4\delta+\zeta+2}{(\alpha+\beta+1)[(\alpha+\beta-1)e^{i\theta}+\delta+1]}\right)\right| \geq 1, \tag{1.11}$$

whenever

$$\left(e^{i\theta}, \frac{(\alpha+\beta-1)e^{i\theta}+\delta+1}{\alpha+\beta}, \frac{(\alpha+\beta-1)^2e^{2i\theta}+3(\alpha+\beta-1)(\delta+1)e^{i\theta}+4\delta+\zeta+2}{(\alpha+\beta+1)[(\alpha+\beta-1)e^{i\theta}+\delta+1]}\right) \in D$$

with $Re\zeta \geq \delta(\delta-1)$, $\delta \geq 1, \alpha \geq 1, \beta > -1$ and θ real.

Definition 2. Let K be the set of complex valued functions $\phi(r,s,t) : \mathbb{C}^3 \to \mathbb{C}$ such is continuous in a domain $D \subset \mathbb{C}^3, (1,1,1) \in D$, $|\phi(1,1,1)| < 1$ and

$$\left|\phi\left(e^{i\theta}, \frac{1}{\mu}[(\mu-1)e^{i\theta}+\delta+1], \frac{(\mu-1)^2e^{2i\theta}+3(\mu-1)(\delta+1)e^{i\theta}+4\delta+\zeta+2}{(\mu+1)[(\mu-1)e^{i\theta}+\delta+1]}\right)\right| \geq 1, \tag{1.12}$$

whenever

$$\left(e^{i\theta}, \frac{1}{\mu}[(\mu-1)e^{i\theta}+\delta+1], \frac{(\mu-1)^2e^{2i\theta}+3(\mu-1)(\delta+1)e^{i\theta}+4\delta+\zeta+2}{(\mu+1)[(\mu-1)e^{i\theta}+\delta+1]}\right) \in D$$

with $Re\zeta \geq \delta(\delta-1)$, $\delta \geq 1, \mu > 1$ and θ real.

2. Main Results

Unless otherwise maintained we assume that $\zeta \geq \delta(\delta-1)$, $\delta \geq 1, \alpha \geq 1, \beta > -1, \mu > 1$ and θ real.

To prove our main results, we need the following lemma.

Lemma 1[4]. *Let $w(z) = c + c_k z^k + c_{k+1} z^{k+1} \ldots$ be analytic in U with with $w(z) \neq a$ and $k \geq 1$. If $z_0 = r_0 e^{i\theta} (0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$. Then*

$$z_0 w'(z_0) = \zeta w(z_0) \tag{it2.1}$$

and

$$Re\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq \delta, \tag{it2.2}$$

where ζ is a real number and

$$\zeta \geq k \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq k \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

Theorem 1. *Let the functions $h(r,s,t) \in H$ and let $f(z) \in \Sigma_p$ satisfy:*

$$\left(\frac{H^{\alpha}_{p,\beta,\mu} f(z)}{H^{\alpha-1}_{p,\beta,\mu} f(z)}, \frac{H^{\alpha+1}_{p,\beta,\mu} f(z)}{H^{\alpha}_{p,\beta,\mu} f(z)}, \frac{H^{\alpha+2}_{p,\beta,\mu} f(z)}{H^{\alpha+1}_{p,\beta,\mu} f(z)} \right) \in D \subset \mathbb{C}^3 \tag{it2.3}$$

and

$$\left| h\left(\frac{H^{\alpha}_{p,\beta,\mu} f(z)}{H^{\alpha-1}_{p,\beta,\mu} f(z)}, \frac{H^{\alpha+1}_{p,\beta,\mu} f(z)}{H^{\alpha}_{p,\beta,\mu} f(z)}, \frac{H^{\alpha+2}_{p,\beta,\mu} f(z)}{H^{\alpha+1}_{p,\beta,\mu} f(z)} \right) \right| < 1 \ (z \in U). \tag{it2.4}$$

Then, for $\alpha \geq 1$, we have

$$\left| \frac{H^{\alpha}_{p,\beta,\mu} f(z)}{H^{\alpha-1}_{p,\beta,\mu} f(z)} \right| < 1 \ (z \in U).$$

Proof. Let

$$w(z) = \frac{H^{\alpha}_{p,\beta,\mu} f(z)}{H^{\alpha-1}_{p,\beta,\mu} f(z)} \ (z \in U). \tag{2.5}$$

Then it follows that $w(z)$ is either analytic or meromorphic in U, $w(0) = 1$ and $w(z) \neq 1$. Differentiating (2.5) logarithmically with respect to z , and using (1.8) in the resulting equation, we have

$$\frac{H^{\alpha+1}_{p,\beta,\mu} f(z)}{H^{\alpha}_{p,\beta,\mu} f(z)} = \frac{1}{\alpha+\beta} \{(\alpha+\beta-1) w(z) + \frac{z w'(z)}{w(z)} + 1\}. \tag{2.6}$$

Differentiating (2.6) logarithmically with respect to z , we have

$$\begin{aligned} \frac{z\left(H^{\alpha+1}_{p,\beta,\mu} f(z)\right)'}{H^{\alpha+1}_{p,\beta,\mu} f(z)} - \frac{z\left(H^{\alpha}_{p,\beta,\mu} f(z)\right)'}{H^{\alpha}_{p,\beta,\mu} f(z)} &= \frac{z\left\{(\alpha+\beta-1) w(z) + \frac{z w\prime(z)}{w(z)} + 1\right\}'}{(\alpha+\beta-1) w(z) + \frac{z w\prime(z)}{w(z)} + 1} \\ &= \frac{(\alpha+\beta-1) z w'(z) + \frac{z w\prime(z)}{w(z)} + z^2 \frac{w''(z)}{w(z)} - (\frac{z w\prime(z)}{w(z)})^2}{(\alpha+\beta-1) w(z) + \frac{z w\prime(z)}{w(z)} + 1}. \end{aligned}$$

2.7 (1)

Applying (1.8) again in (2.7), we have:

$$\begin{aligned}(\alpha+\beta+1)\frac{H^{\alpha+2}_{p,\beta,\mu}f(z)}{H^{\alpha+1}_{p,\beta,\mu}f(z)} &= (\alpha+\beta)\frac{H^{\alpha+1}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)}+1+\frac{(\alpha+\beta-1)zw'(z)+\frac{zw\prime(z)}{w(z)}+z^2\dfrac{w''(z)}{w(z)}-(\frac{zw\prime(z)}{w(z)})^2}{(\alpha+\beta-1)w(z)+\frac{zw\prime(z)}{w(z)}+1}\\ &= (\alpha+\beta-1)w(z)+\frac{zw\prime(z)}{w(z)}+2+\frac{(\alpha+\beta-1)zw'(z)+\frac{zw\prime(z)}{w(z)}+z^2\frac{w''(z)}{w(z)}-(\frac{zw\prime(z)}{w(z)})^2}{(\alpha+\beta-1)w(z)+\frac{zw\prime(z)}{w(z)}+1}.\end{aligned}$$

We claim that $|w(z)|<1$, $z\in U$. If it is not true, then there exists a point $z_0\in U$ such that $\underset{|z|\le r_0}{max}\ |w(z)|=|w(z)|=1$. Taking $w(z_0)=e^{i\theta}$ and applying Lemma 1 with $c=k=1$, we have

$$\begin{aligned}\frac{H^{\alpha}_{p,\beta,\mu}f(z)}{H^{\alpha-1}_{p,\beta,\mu}f(z)} &= e^{i\theta},\\ \frac{H^{\alpha+1}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)} &= \frac{1}{\alpha+\beta}[(\alpha+\beta-1)e^{i\theta}+\delta+1]\end{aligned}$$

and

$$\frac{H^{\alpha+2}_{p,\beta,\mu}f(z)}{H^{\alpha+1}_{p,\beta,\mu}f(z)}=\frac{(\alpha+\beta-1)^2e^{2i\theta}+3(\alpha+\beta-1)(\delta+1)e^{i\theta}+4\delta+\zeta+2}{(\alpha+\beta+1)[(\alpha+\beta-1)e^{i\theta}+\delta+1]},$$

where

$$\zeta=\frac{z^2w''(z_0)}{w(z_0)}\quad\text{and } \zeta\ge 1.$$

Applying (2.2), we have $Re\zeta\ge\delta(\delta-1)$.

Since $h(r,s,t)\in H$, we have

$$\begin{aligned}&\left|h\left(\frac{H^{\alpha}_{p,\beta,\mu}f(z_0)}{H^{\alpha-1}_{p,\beta,\mu}f(z_0)},\frac{H^{\alpha+1}_{p,\beta,\mu}f(z_0)}{H^{\alpha}_{p,\beta,\mu}f(z_0)},\frac{H^{\alpha+2}_{p,\beta,\mu}f(z_0)}{H^{\alpha+1}_{p,\beta,\mu}f(z_0)}\right)\right|\\ =&\left|h\left(e^{i\theta},\frac{(\alpha+\beta-1)e^{i\theta}+\delta+1}{\alpha+\beta},\frac{(\alpha+\beta-1)^2e^{2i\theta}+3(\alpha+\beta-1)(\delta+1)e^{i\theta}+4\delta+\zeta+2}{(\alpha+\beta+1)[(\alpha+\beta-1)e^{i\theta}+\delta+1]}\right)\right|\ge 1.\end{aligned}$$

This contradicts the condition (2.4) of The theorem. Therefore, we conclude that

$$\left|\frac{H^{\alpha}_{p,\beta,\mu}f(z)}{H^{\alpha-1}_{p,\beta,\mu}f(z)}\right|<1\quad(z\in U).$$

This completes the proof of Theorem 1.

Theorem 2. *Let the functions $\phi(r,s,t)\in K$ and let $f(z)\in\Sigma_p$ satisfy*:

$$\left(\frac{H^{\alpha}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu-1}f(z)},\frac{H^{\alpha}_{p,\beta,\mu+1}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)},\frac{H^{\alpha}_{p,\beta,\mu+2}f(z)}{H^{\alpha}_{p,\beta,\mu+1}f(z)}\right)\in D\subset\mathbb{C}^3 \qquad \text{(it2.3)}$$

and

$$\left|\phi\left(\frac{H^{\alpha}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu-1}f(z)},\frac{H^{\alpha}_{p,\beta,\mu+1}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)},\frac{H^{\alpha}_{p,\beta,\mu+2}f(z)}{H^{\alpha}_{p,\beta,\mu+1}f(z)}\right)\right|<1\ (z\in U). \qquad \text{(it2.4)}$$

Then, for $\mu\ge 1$, we have

$$\left|\frac{H^{\alpha}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu-1}f(z)}\right|<1\ (z\in U).$$

Proof. The proof follows by applying the same steps used in the proof of Theorem 1 and using the identity (1.9) instead of (1.8).

Remark. *Putting* $\mu = 1$*, in Theorem 1, we obtain results corresponding to the operator* $H_{p,\beta}^{\alpha}$.

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