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SOME SUBCLASSES OF P-VALENT FUNCTIONS INVOLVING THE EXTENDED MULTIPLIER TRANSFORMATIONS

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Abstract. New classes of p-valent analytic functions are introduced. Such results as inclusion relationships, integral representations, integral-preserving properties and convolution properties for these function classes are obtained.

1. Introduction

Let $A(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, ...\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows $f \prec g$ in U or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. Indeed it is known that $f(z) \prec g(z)$ $(z \in U) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$. Further, if the function $g(z)$ is univalent in U, then we have the following equivalent (cf., e.g., [11]; see also [12, p.4])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let P denote the class of functions of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic and convex in U and satisfies the following condition

$$Re\{p(z)\} > 0, \quad z \in U$$

For functions $f_j(z) \in A(p)$, given by

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{p+n,j} z^{p+n} \quad (j = 1, 2), \tag{1.2}$$

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we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p} = (f_2 * f_1)(z). \tag{1.3}$$

Catas [4] extended the multiplier transformation and defined the operator $I_p^m(\lambda;\ell)$ on $A(p)$ by the following infinite series

$$I_p^m(\lambda,\ell)f(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{p+\ell+\lambda n}{p+\ell}\right]^m a_{n+p} z^{n+p}$$
$$(\ell \geq 0; \lambda \geq 0; p \in \mathbb{N} \; and \; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).1.4 \tag{1}$$

We note that:

$I_p^0(1,0)f(z) = f(z) \; and \; I_p^1(1,0)f(z) = \dfrac{zf'(z)}{p}$.

By specializing the parameters m, λ, ℓ and p, we obtain the following operators studied by various authors:

(i)$I_p^m(1,\ell)f(z) = I_p(m,\ell)f(z)$ (see Kumar et al. [10] and Srivastava et al. [18]);
(ii)$I_p^m(1,0)f(z) = D_p^m f(z)$ (see, [3], [9] and [15]);
(iii)$I_1^m(1,\ell)f(z) = I_\ell^m f(z)$ (see Cho and Kim [5] and Cho and Srivastava [6]);
(iv)$I_1^m(1,0)f(z) = D^m f(z)$(see Salagean [17]);
(v)$I_1^m(\lambda,0)f(z) = D_\lambda^m f(z)$ (see Al-Oboudi [1]);
(vi)$I_1^m(1,1)f(z) = I^m f(z)$ (see Uralegaddi and Somanatha [19]);
(vii)$I_p^m(\lambda,0)f(z) = D_{\lambda,p}^m f(z)$, (see El-Ashwah and M. K. Aouf [8]).

Also we note that

$$\lambda z((I_p^m(\lambda,\ell)f(z))' = (p+\ell)I_p^{m+1}(\lambda,\ell)f(z) - [p(1-\lambda)+\ell]I_p^m(\lambda,\ell)f(z) \quad (\lambda > 0), \tag{1.5}$$

and

$$I_p^{m_1}(\lambda,\ell)(I_p^{m2}(\lambda,\ell)f(z)) = I_p^{m2}(\lambda,\ell)(I_p^{m_1}(\lambda,\ell)f(z)) = I_p^{m_1+m_2}(\lambda,\ell)f(z),$$

for all integers m_1 and m_2.

Also if f is given by (1.1), then we have

$$I_p^m(\lambda,\ell)f(z) = (\phi_{p,\lambda,\ell}^{m,n} * f)(z),$$

where

$$\phi_{p,\lambda,\ell}^m(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{p+\ell+\lambda n}{p+\ell}\right]^m z^{p+n}.$$

Throughout this paper, we assume that $p, k \in \mathbb{N}, m \in \mathbb{N}_0, \in_k = \exp(\dfrac{2\pi i}{k})$ and

$$f_{p,k}^m(\lambda,\ell;z) = \frac{1}{k}\sum_{j=0}^{k-1} \in_k^{-jp} (I_p^m(\lambda,\ell)f)(\in_k^j z) = z^p + ...(f \in A(p)). \tag{1.6}$$

Clearly, for $k = 1$, we have

$$f_{p,1}^m(\lambda,\ell;z) = I_p^m(\lambda,\ell)f(z).$$

Making use of the extended multiplier transformations $I_p^m(\lambda,\ell)$ and the above mentioned principle of subordination between analytic functions, we now introduce and investigate the following subclasses of the class $A(p)$ of p-valent analytic functions.

Definition 1. A function $f(z) \in A(p)$ is said to be in the class $S_{p,k}^{m}(\lambda,\ell;\varphi)$ if it satisfies the following subordination condition:

$$\frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{pf_{p,k}^m(\lambda,\ell;z)} \prec \varphi(z), \tag{1.7}$$

where $\varphi \in P$ and $f_{p,k}^m(\lambda,\ell;z) \neq 0$ $(z \in U^*)$ is defined by (1.6).

Remark 1. Putting $p=\lambda=1$ and $m=\ell=0$ in the class $S_{p,k}^{m}(\lambda,\ell;\varphi)$, we obtain the function class $S_s^{(k)}(\varphi)$ which introduced and studied by Wang et al. [20].

Definition 2. A function $f \in A(p)$ is said to be in the class $K_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$ if it satisfies the following subordination condition:

$$(1-\alpha)\frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{pf_{p,k}^m(\lambda,\ell;z)} + \alpha\frac{z\left(I_p^{m+1}(\lambda,\ell)f\right)'(z)}{pf_{p,k}^{m+1}(\lambda,\ell;z)} \prec \varphi(z)\ , \tag{1.8}$$

for some $\alpha(\alpha \geq 0)$, where $\varphi \in P$ and $f_{p,k}^m(\lambda,\ell;z)$ is defined by (1.6) and satisfying $f_{p,k}^{m+1}(\lambda,\ell;z) \neq 0$ $(z \in U^*)$.

Remark 2. Putting $p=\lambda=1$ and $m=\ell=0$ in the class $K_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$, we obtain the function class $K_s^{(k)}(\alpha,\varphi)$ of functions which are $\alpha-$convex with respect to k-symmetric points (see Yuan and Liu [21]).

Definition 3. A function $f \in A(p)$ is said to be in the class $C_{p,k}^{m}(\lambda,\ell;\varphi)$ if it satisfies the following subordination condition :

$$\frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{pg_{p,k}^m(\lambda,\ell;z)} \prec \varphi(z) \quad (g \in S_{p,k}^m(\lambda,\ell;\varphi)), \tag{1.9}$$

where $\varphi \in P$ and $g_{p,k}^m(\lambda,\ell;z) \neq 0$ $(z \in U^*)$ is defined by (1.6).

Remark 3. Taking $\lambda=k=1, m=\ell=0$ and $\varphi(z)=\dfrac{1+z}{1-z}$ in the class $C_{p,k}^{m}(\lambda,\ell;\varphi)$, we obtain the class of p-valent close-to-convex functions (see Aouf [2]).

Definition 4. A function $f \in A(p)$ is said to be in the class $G_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$ if it satisfies the following subordination condition:

$$(1-\alpha)\frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{pg_{p,k}^m(\lambda,\ell;z)} + \alpha\frac{z\left(I_p^{m+1}(\lambda,\ell)f\right)'(z)}{pg_{p,k}^{m+1}(\lambda,\ell;z)} \prec \varphi(z) \quad (\alpha \geq 0; g \in S_{p,k}^m(\lambda,\ell;\varphi)), \tag{1.10}$$

where $\varphi \in P$, $g_{p,k}^m(\lambda,\ell;z)$ is defined by (1.6) and $g_{p,k}^{m+1}(\lambda,\ell;z) \neq 0$ $(z \in U^*)$.

Remark 4. **(i)** Putting $p=\lambda=1$ and $m=\ell=0$ in the class $G_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$, we obtain the class $QC_s^{(k)}(\alpha;\varphi)$ of functions which are $\alpha-$quasi-convex with respect to k-symmetric points (see Yuan and Liu [21]);

(ii) Taking $p=\lambda=k=\alpha=1, m=\ell=0$ and $\varphi(z)=\dfrac{1+z}{1-z}$ in the class $G_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$, we obtain the familiar class of quasi-convex functions (see Noor [14]).

In order to establish our main results, we shall use of the following lemmas.

Lemma 1 [7, 12]. *Let* $\beta,\gamma \in \mathbb{C}$. *Suppose also that* $\varphi(z)$ is convex and univalent in U with

$$\varphi(0)=1 \text{ and } Re\{\beta\varphi(z)+\gamma\} > 0 \quad (z \in U).$$

If $p(z)$ is analytic in U with $p(0) = 1$, then the following subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \varphi(z),$$

implies that

$$p(z) \prec \varphi(z).$$

Lemma 2 [16]. *Let* $\beta, \gamma \in \mathbb{C}$. *Suppose that* $\varphi(z)$ is convex and univalent in U with

$$\varphi(0) = 1 \text{ and } Re\{\beta\varphi(z) + \gamma\} > 0 \quad (z \in U).$$

Also let

$$q(z) \prec \varphi(z).$$

If $p(z) \in P$ and satisfies the following subordination:

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \varphi(z),$$

then

$$q(z) \prec \varphi(z).$$

Lemma 3. *Let* $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. Then

$$\frac{z(f^m_{p,k}(\lambda, \ell; z))'}{pf^m_{p,k}(\lambda, \ell; z)} \prec \varphi(z) \ . \tag{1.11}$$

Proof. In view of (1.6), we replace z by $\in^j_k z$ $(j = 0, 1, 2, .., k-1)$ in $f^m_{p,k}(\lambda, \ell; z)$. We thus obtain

$$\begin{aligned} f^m_{p,k}(\lambda, \ell; \in^j_k z) &= \frac{1}{k}\sum_{n=0}^{k-1} \in^{-np} (I^m_p(\lambda, \ell)f)(\in^{n+j}_k z) \\ &= \in^{jp} \frac{1}{k}\sum_{n=0}^{k-1} \in^{-(n+j)p} (I^m_p(\lambda, \ell)f)(\in^{n+j}_k z) \\ &= \in^{jp} f^m_{p,k}(\lambda, \ell; z). \end{aligned} \tag{1.12}$$

Differentiating both sides of (1.6) with respect to z, we obtain

$$(f^m_{p,k}(\lambda, \ell; z))' = \frac{1}{k}\sum_{j=0}^{k-1} \in^{-j(p-1)} (I^m_p(\lambda, \ell)f)'(\in^j_k z). \tag{1.13}$$

Therefore, from (1.12) and (1.13), we find that

$$\begin{aligned} \frac{z(f^m_{p,k}(\lambda, \ell; z))'}{pf^m_{p,k}(\lambda, \ell; z)} &= \frac{1}{k}\sum_{j=0}^{k-1} \frac{\in^{-j(p-1)} z(I^m_p(\lambda, \ell)f)'(\in^j_k z)}{pf^m_{p,k}(\lambda, \ell; z)} \\ &= \frac{1}{k}\sum_{j=0}^{k-1} \frac{\in^j_k z(I^m_p(\lambda, \ell)f)'(\in^j_k z)}{pf^m_{p,k}(\lambda, \ell; \in^j_k z)}. \end{aligned} \tag{1.14}$$

Moreover, since $f \in S^m_{p,k}(\lambda, \ell; \varphi)$, it follows that

$$\frac{\in^j_k z(I^m_p(\lambda, \ell)f)'(\in^j_k z)}{pf^m_{p,k}(\lambda, \ell; \in^j_k z)} \prec \varphi(z) \quad (j = 0, 1, .., k-1). \tag{1.15}$$

Finally, by noting that $\varphi(z)$ is convex and univalent in U, from (1.14) and (1.5), we conclude that the assertion (1.11) of Lemma 3 holds true.

Similarly, for the class $K_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$, we can prove the following result.

Lemma 4. *Let* $f\in K_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$. Then

$$(1-\alpha)\frac{z(f_{p,k}^{m}(\lambda,\ell;z))'}{pf_{p,k}^{m}(\lambda,\ell;z)}+\alpha\frac{z(f_{p,k}^{m+1}(\lambda,\ell;z))'}{pf_{p,k}^{m+1}(\lambda,\ell;z)}\prec\varphi(z)\ . \tag{1.16}$$

In the present paper, we obtain some inclusion relationships, integral representation, convolution properties and integral-preserving properties for each of the function classes $S_{p,k}^{m}(\lambda,\ell;\varphi);K_{p,k}^{m}(\lambda,\ell;\alpha;\varphi),C_{p,k}^{m}(\lambda,\ell;\varphi)$ and $G_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$.

2. A set of inclusion relationships

In this section, we obtain some inclusion relationships for the function classes $S_{p,k}^{m}(\lambda,\ell;\varphi),K_{p,k}^{m}(\lambda,\ell;\alpha;\varphi),C_{p,k}^{m}(\lambda,\ell;\varphi)$ and $G_{p,k}^{m}(\lambda,\ell;\alpha;\varphi)$.

Unless otherwise mentioned we shall assume throughout the paper that $\lambda>0;\ell\geq 0;p,k\in\mathbb{N}$ and $m\in\mathbb{N}_0$.

Theorem 1. *Let* $\varphi\in P$ with

$$Re\left\{p\varphi(z)+\frac{p(1-\lambda)+\ell}{\lambda}\right\}>0\quad(z\in U),$$

then

$$S_{p,k}^{m+1}(\lambda,\ell;\varphi)\subset S_{p,k}^{m}(\lambda,\ell;\varphi).$$

Proof. Making use of the relationships in equations (1.5) and (1.6), we know that

$$z\left(f_{p,k}^{m}(\lambda,\ell;\varphi)\right)'+\left[\frac{p(1-\lambda)+\ell}{\lambda}\right]f_{p,k}^{m}(\lambda,\ell;z)$$

$$=\frac{p+\ell}{\lambda k}\sum_{j=0}^{k-1}\in_k^{-jp}\left(I_p^{m+1}(\lambda,\ell)f(\in_k^{j}z)\right)=\frac{p+\ell}{\lambda}f_{p,k}^{m+1}(\lambda,\ell;z). \tag{2.1}$$

Let $f\in S_{p,k}^{m+1}(\lambda,\ell;\varphi)$ and suppose that

$$w(z)=\frac{z\left(f_{p,k}^{m}(\lambda,\ell;z)\right)'}{pf_{p,k}^{m}(\lambda,\ell;z)}\qquad(z\in U). \tag{2.2}$$

Then $w(z)$ is analytic in U and $w(0)=1$. It follows from (2.1) and (2.2) that

$$pw(z)+\frac{p(1-\lambda)+\ell}{\lambda}=\frac{p+\ell}{\lambda}\frac{f_{p,k}^{m+1}(\lambda,\ell;z)}{f_{p,k}^{m}(\lambda,\ell;z)}. \tag{2.3}$$

Differentiating both sides of (2.3) logarithmically with respect to z and using (2.2), we obtain

$$w(z)+\frac{zw'(z)}{pw(z)+\frac{p(1-\lambda)+\ell}{\lambda}}=\frac{z\left(f_{p,k}^{m+1}(\lambda,\ell;z)\right)'}{pf_{p,k}^{m+1}(\lambda,\ell;z)}. \tag{2.4}$$

From (2.4) and Lemma 3 (with m replaced by $(m+1)$), we can see that

$$w(z)+\frac{zw'(z)}{pw(z)+\frac{p(1-\lambda)+\ell}{\lambda}}\prec\varphi(z). \tag{2.5}$$

Since $Re\left\{p\varphi(z)+\frac{p(1-\lambda)+\ell}{\lambda}\right\}>0 \ \ (z\in U)$, by Lemma 1, we have

$$w(z)=\frac{z\left(f_{p,k}^{m}(\lambda,\ell;z)\right)^{'}}{pf_{p,k}^{m}(\lambda,\ell;z)}\prec\varphi(z)\ . \tag{2.6}$$

By setting

$$q(z)=\frac{z\left(I_{p}^{m}(\lambda,\ell)f\right)^{'}(z)}{pf_{p,k}^{m}(\lambda,\ell;z)}\quad (z\in U), \tag{2.7}$$

we observe that $q(z)$ is analytic in U and $q(0)=1$. It follows from (1.5) and (2.7) that

$$q(z)f_{p,k}^{m}(\lambda,\ell;z)=\frac{(p+\ell)}{\lambda p}I_{p}^{m+1}(\lambda,\ell)f(z)-\frac{[p(1-\lambda)+\ell]}{\lambda p}I_{p}^{m}(\lambda,\ell)f(z). \tag{2.8}$$

Differentiating both sides of (2.8) with respect to z and using (2.7), we obtain

$$zq^{'}(z)+\left(\frac{[p(1-\lambda)+\ell]}{\lambda}+\frac{z\left(f_{p,k}^{m}(\lambda,\ell;z)\right)^{'}}{f_{p,k}^{m}(\lambda,\ell;z)}\right)q(z)=\frac{(p+\ell)}{\lambda p}.\frac{z\left(I_{p}^{m+1}(\lambda,\ell)f\right)^{'}(z)}{f_{p,k}^{m}(\lambda,\ell;z)}. \tag{2.9}$$

From $(2.2),(2.3)$ and (2.9), we can obtain

$$q(z)+\frac{zq^{'}(z)}{\frac{[p(1-\lambda)+\ell]}{\lambda}+pw(z)}=\frac{z\left(I_{p}^{m+1}(\lambda,\ell)f\right)^{'}(z)}{pf_{p,k}^{m+1}(\lambda,\ell;z)}\prec\varphi(z)\ .$$

Since

$$w(z)\prec\varphi(z)$$

and

$$Re\left\{p\varphi(z)+\frac{[p(1-\lambda)+\ell]}{\lambda}\right\}>0 \ \ (z\in U),$$

it follows from (2.9) and Lemma 2 that

$$q(z)\prec\varphi(z)\ ,$$

that is, that $f\in S_{p,k}^{m}(\lambda,\ell;\varphi)$. This implies that

$$S_{p,k}^{m+1}(\lambda,\ell;\varphi)\subset S_{p,k}^{m}(\lambda,\ell;\varphi).$$

Hence the proof of Theorem 1 is completed.

Theorem 2. Let $\varphi\in P$ with

$$Re\left\{p\varphi(z)+\frac{[p(1-\lambda)+\ell]}{\lambda}\right\}>0 \quad (z\in U).$$

Then

$$C_{p,k}^{m+1}(\lambda,\ell;\varphi)\subset C_{p,k}^{m}(\lambda,\ell;\varphi).$$

Proof. Suppose that $f\in C_{p,k}^{m+1}(\lambda,\ell;\varphi)$. Then we have

$$\frac{z\left(I_{p}^{m+1}(\lambda,\ell)f\right)^{'}(z)}{pg_{p,k}^{m+1}(\lambda,\ell;z)}\prec\varphi(z), \tag{2.10}$$

with $g \in S_{p,k}^{m+1}(\lambda, \ell; \varphi)$. Furthermore, it follows from Theorem 1 that $g \in S_{p,k}^{m}(\lambda, \ell; \varphi)$, and Lemma 3 yields

$$\psi(z) = \frac{z\left(g_{p,k}^{m}(\lambda, \ell; z)\right)^{'}}{pg_{p,k}^{m}(\lambda, \ell; z)} \prec \varphi(z)\ ,. \tag{2.11}$$

We now set

$$q(z) = \frac{z\left(I_p^m(\lambda, \ell)f\right)^{'}(z)}{pg_{p,k}^{m}(\lambda, \ell; z)} \quad (z \in U). \tag{2.12}$$

Then $q(z)$ is analytic in U and $q(0) = 1$. It follows from (1.5) and (2.12) that

$$q(z)g_{p,k}^{m}(\lambda, \ell; z) = \frac{(p+\ell)}{\lambda p} I_p^{m+1}(\lambda, \ell)f(z) - \frac{[p(1-\lambda)+\ell]}{\lambda p} I_p^m(\lambda, \ell)f(z). \tag{2.13}$$

Differentiating both sides of (2.13) with respect to z and using (2.1) (with f replaced by g), we have

$$zq^{'}(z) + \left(\frac{p(1-\lambda)+\ell}{\lambda} + \frac{z\left(g_{p,k}^{m}(\lambda, \ell; z)\right)^{'}}{g_{p,k}^{m}(\lambda, \ell; z)}\right) q(z) = \frac{(p+\ell)}{\lambda p} . \frac{z\left(I_p^{m+1}(\lambda, \ell)f\right)^{'}(z)}{g_{p,k}^{m}(\lambda, \ell; z)}. \tag{2.14}$$

From (2.10), (2.11) and (2.14), we can obtain

$$q(z) + \frac{zq^{'}(z)}{\frac{[p(1-\lambda)+\ell]}{\lambda} + p\psi(z)} = \frac{z\left(I_p^{m+1}(\lambda, \ell)f\right)^{'}(z)}{pg_{p,k}^{m+1}(\lambda, \ell; z)} \prec \varphi(z). \tag{2.15}$$

Since

$$\psi(z) \prec \varphi(z),$$

and

$$Re\left\{p\varphi(z) + \frac{p(1-\lambda)+\ell}{\lambda}\right\} > 0 \quad (z \in U),$$

it follows from (2.15) and Lemma 2 that

$$q(z) \prec \varphi(z),$$

that is, that $f \in C_{p,k}^{m}(\lambda, \ell; \varphi)$. This implies that

$$C_{p,k}^{m+1}(\lambda, \ell; \varphi) \subset C_{p,k}^{m}(\lambda, \ell; \varphi).$$

The proof of Theorem 2 is thus completed.

Theorem 3. Let $\varphi \in P$ with

$$Re\left\{p\varphi(z) + \frac{p(1-\lambda)+\ell}{\lambda}\right\} > 0 \quad (z \in U),$$

then

$$G_{p,k}^{m}(\lambda, \ell; \alpha_2; \varphi) \subset G_{p,k}^{m}(\lambda, \ell; \alpha_1; \varphi)\ (\alpha_2 > \alpha_1 \geq 0).$$

Proof. Suppose that $f \in G_{p,k}^{m}(\lambda, \ell, \alpha_2; \varphi)$. Then we have

$$(1-\alpha_2)\frac{z\left(I_p^m(\lambda, \ell)f\right)^{'}(z)}{pg_{p,k}^{m}(\lambda, \ell; z)} + \alpha_2 \frac{z\left(I_p^{m+1}(\lambda, \ell)f\right)^{'}(z)}{pg_{p,k}^{m+1}(\lambda, \ell; z)} \prec \varphi(z)\ . \tag{2.16}$$

Since $g \in S_{p,k}^{m}(\lambda, \ell; \varphi)$, it follows from (2.11) to (2.16) that

$$q(z) + \frac{\alpha_2 z q'(z)}{\frac{p(1-\lambda)+\ell}{\lambda} + p\psi(z)} = (1-\alpha_2)\frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{p g_{p,k}^m(\lambda,\ell;z)} +$$

$$\alpha_2 \frac{z\left(I_p^{m+1}(\lambda,\ell)f(z)\right)'}{p g_{p,k}^{m+1}(\lambda,\ell;z)} \prec \varphi(z) \ . \tag{2.17}$$

Since

$$\psi(z) \prec \varphi(z),$$

and

$$\frac{1}{\alpha_2} Re\left\{p\varphi(z) + \frac{p(1-\lambda)+\ell}{\lambda}\right\} > 0 \qquad (z \in U),$$

it follows from (2.17) and Lemma 2 that

$$q(z) = \frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{p g_{p,k}^m(\lambda,\ell;z)} \prec \varphi(z) \ . \tag{2.18}$$

Moreover, since $0 \le \dfrac{\alpha_1}{\alpha_2} < 1$ and the function $\varphi(z)$ is convex and univalent in U, we deduce from (2.17) and (2.18) that

$$(1-\alpha_1)\frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{p g_{p,k}^m(\lambda,\ell;z)} + \alpha_1 \frac{z\left(I_p^{m+1}(\lambda,\ell)f\right)'(z)}{p g_{p,k}^{m+1}(\lambda,\ell;z)}$$

$$\begin{aligned} &= \frac{\alpha_1}{\alpha_2}\left[(1-\alpha_2)\frac{z\left(I_p^m(\lambda,\ell)f\right)'(z)}{p g_{p,k}^m(\lambda,\ell;z)} + \alpha_2 \frac{z\left(I_p^{m+1}(\lambda,\ell)f\right)'(z)}{p g_{p,k}^{m+1}(\lambda,\ell;z)}\right] + \left(1-\frac{\alpha_1}{\alpha_2}\right) q(z) \\ &\prec \varphi(z), \end{aligned}$$

which implies that $f \in G_{p,k}^{m}(\lambda, \ell; \alpha_1; \varphi)$. Hence the proof of Theorem 3, is completed .

By applying the same method of Theorem 3, we can easily get the following inclusion relationship.

Corollary 1. *Let* $\varphi \in P$ *with*

$$Re\left\{p\varphi(z) + \frac{p(1-\lambda)+\ell}{\lambda}\right\} > 0 \quad (z \in U).$$

Then $K_{p,k}^{m}(\lambda, \ell; \alpha_2; \varphi) \subset K_{p,k}^{m}(\lambda, \ell; \alpha_1; \varphi)$ $(\alpha_2 > \alpha_1 \ge 0)$.

In view of Theorem 3, we can also easily get the following inclusion relationships. In particular, a direct proof of Corollary 2 would require use of Lemma 4.

Corollary 2. *Let* $\alpha \ge 0$ and $\varphi \in P$. *Then*

$$G_{p,k}^{m}(\ell; \alpha; \varphi) \subset C_{p,k}^{m}(\lambda, \ell; \varphi).$$

Corollary 3. *Let* $\alpha \ge 0$ and $\varphi(z) \in P$. *Then*

$$K_{p,k}^{m}(\lambda, \ell; \alpha; \varphi) \subset S_{p,k}^{m}(\lambda, \ell; \varphi).$$

3. Integral representation

In this section, we obtain a number of integral representations associated with the function class $S_{p,k}^{m}(\lambda,\ell;\varphi)$.

Theorem 4. *Let* $f\in S_{p,k}^{m}(\lambda,\ell;\varphi)$. *Then*

$$f_{p,k}^{m}(\lambda,\ell;z)=z^{p}\exp\left\{\frac{p}{k}\sum_{j=0}^{k-1}\int_{0}^{z}\frac{\varphi(w(\in_{k}^{j}\xi))-1}{\xi}d\xi\right\}, \tag{3.1}$$

where $f_{p,k}^{m}(\lambda,\ell;z)$ is defined by (1.6), $w(z)$ is analytic in U and satisfy $w(0)=1$ and $|w(z)|<1$ $(z\in U)$.

Proof. Suppose that $f\in S_{p,k}^{m}(\lambda,\ell;\varphi)$. Then condition (1.7) can be written as follows:

$$\frac{z\left(I_{p}^{m}(\lambda,\ell)f(z)\right)'}{pf_{p,k}^{m}(\lambda,\ell;z)}=\varphi(w(z))\ (z\in U), \tag{3.2}$$

where $w(z)$ is analytic in U and satisfy $w(0)=1$ and $|w(z)|<1$ $(z\in U)$. Replacing z by $\in_{k}^{j}z$ $(j=0,1,...,k-1)$ in (3.2), we observe that (3.2) becomes

$$\frac{\in_{k}^{j}z\left(I_{p}^{m}(\lambda,\ell)f\right)'(\in_{k}^{j}z)}{pf_{p,k}^{m}(\lambda,\ell;\in_{k}^{j}z)}=\varphi(w(\in_{k}^{j}z))\ (z\in U). \tag{3.3}$$

We note that

$$f_{p,k}^{m}(\lambda,\ell;\in_{k}^{j}z)=\in_{k}^{jp}f_{p,k}^{m}(\lambda,\ell;z)\ (z\in U).$$

Thus, by letting $j=0,1,....,k-1$ in (3.3), successively, and summing the resulting equations, we have

$$\frac{z\left(f_{p,k}^{m}(\lambda,\ell;z)\right)'}{pf_{p,k}^{m}(\lambda,\ell;z)}=\frac{1}{k}\sum_{j=0}^{k-1}\varphi(w(\in_{k}^{j}z))\ \ (z\in U). \tag{3.4}$$

From (3.4), we get

$$\frac{\left(f_{p,k}^{m}(\lambda,\ell;z)\right)'}{f_{p,k}^{m}(\lambda,\ell;z)}-\frac{p}{z}=\frac{p}{k}\sum_{j=0}^{k-1}\left[\frac{\varphi(w(\in_{k}^{j}z))-1}{z}\right](z\in U), \tag{3.5}$$

which , upon integration, yields

$$\log\left(\frac{f_{p,k}^{m}(\lambda,\ell;z)}{z^{p}}\right)=\frac{p}{k}\sum_{j=0}^{k-1}\int_{0}^{z}\frac{\varphi(w(\in_{k}^{j}\xi))-1}{\xi}d\xi. \tag{3.6}$$

Then, the assertion (3.1) of Theorem 4 can now easily obtained from (3.6).

Theorem 5. *Let* $f\in S_{p,k}^{m}(\lambda,\ell;\varphi)$. *Then*

$$I_{p}^{m}(\lambda,\ell)f(z)=p\int_{0}^{z}\zeta^{p-1}\varphi(w(\zeta)).\exp\left(\frac{p}{k}\sum_{j=0}^{k-1}\int_{0}^{\zeta}\frac{\varphi(w(\in_{k}^{j}\xi))-1}{\xi}d\xi\right)d\zeta, \tag{3.7}$$

where $w(z)$ is analytic in U and satisfy $w(0)=1$ and $|w(z)|<1$ $(z\in U)$.

Proof. Suppose that $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. Then, from (3.1) and (3.2), we have

$$\left(I^m_p(\lambda, \ell) f(z)\right)' = \frac{p f^m_{p,k}(\lambda, \ell; z)}{z} \varphi(w(z))$$
$$= p z^{p-1} \varphi(w(z)). \exp\left(\frac{p}{k} \sum_{j=0}^{k-1} \int_0^z \frac{\varphi(w(\in_k^j \xi)) - 1}{\xi} d\xi\right), \tag{3.8}$$

which, upon integration, leads us easily to the assertion (3.7) of Theorem 5.

Remark 5. Putting $p = \lambda = 1$ and $\ell = m = 0$ in Theorem 5, we obtain the result obtained by Wang et al. [20,Theorem 6].

Moreover, in view of Lemma 3 and Theorem 1, we can get integral representation for the function class $S^m_{p,k}(\lambda, \ell; \varphi)$.

Theorem 6. *Let* $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. *Then*

$$I^m_p(\lambda, \ell) f(z) = p \int_0^z \zeta^{p-1} \varphi(w_2(\zeta)). \exp\left(\int_0^\xi \frac{p[\varphi(w_1(\xi)) - 1]}{\xi} d\xi\right) d\xi, \tag{3.9}$$

where $w_j(z)(j = 1, 2)$ are analytic in U with $w_j(0) = 0$ and $|w_j(z)| < 1(z \in U; j = 1, 2)$.

Proof. Suppose that $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. We then find from (1.11) that

$$\frac{z\left(f^m_{p,k}(\lambda, \ell; z)\right)'}{p f^m_{p,k}(\lambda, \ell; z)} = \varphi(w_1(z)) \quad (z \in U), \tag{3.10}$$

where $w_1(z)$ is analytic in U with $w_1(0) = 1$. Thus, by similarly applying the method of proof of Theorem 4, we find that

$$f^m_{p,k}(\lambda, \ell; z) = z^p. \exp\left(\int_0^z \frac{p[\varphi(w_1(\xi)) - 1]}{\xi} d\xi\right). \tag{3.11}$$

It now follows from (3.2) and (3.11) that

$$\left(I^m_p(\lambda, \ell) f(z)\right)' = \frac{p f^m_{p,k}(\lambda, \ell; z)}{z} .\varphi(w_2(z))$$
$$= p z^{p-1} \varphi(w_2(z)). \exp\left(\int_0^z \frac{p[\varphi(w_1(\xi)) - 1]}{\xi} d\xi\right), \tag{3.12}$$

where $w_j(z)(j = 1, 2)$ are analytic in U with $w_j(0) = 0$ and $|w_j(z)| < 1(z \in U; j = 1, 2)$. Integrating both sides of (3.12), we will obtain the assertion (3.9) of Theorem 6.

4. **Convolution properties**

In this section, we derive some convolution properties for the class $S^m_{p,k}(\lambda, \ell; \varphi)$.

Theorem 7. *Let* $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. *Then*

$$f(z) = \left[p \int_0^z \zeta^{p-1} \varphi(w(\zeta)). \exp\left(\frac{p}{k} \sum_{j=0}^{k-1} \int_0^\zeta \frac{\varphi(w(\in_k^j \xi)) - 1}{\xi} d\xi\right) d\zeta\right] *$$
$$* \left(\sum_{n=0}^{\infty} \left(\frac{p + \ell}{p + \ell + \lambda n}\right)^m z^{n+p}\right), \tag{4.1}$$

where $w(z)$ is analytic in U with $w(0) = 1$ and $|w(z)| < 1 \quad (z \in U)$.

Proof. In view of (1.4) and (3.7), we know that

$$p_0^z\zeta^{p-1}\varphi(w(\zeta)).\exp\left(\frac{p}{k}\sum_{j=0}^{k-1}{}_0^\zeta\frac{\varphi(w(\in_k^j\xi))-1}{\xi}d\xi\right)d\zeta$$

$$=\left(z^p+\sum_{n=1}^{\infty}\left(\frac{p+\ell+\lambda n}{p+\ell}\right)^m z^{n+p}\right)*f(z)=\phi_{p,\lambda,\ell}^m(z)*f(z). \tag{4.2}$$

Thus, from (4.2), we can easily get the assertion (4.1) of Theorem 7.

Theorem 8. *Let* $f\in S_{p,k}^m(\lambda,\ell;\varphi)$. *Then*

$$f(z)=\left[p_0^z\zeta^{p-1}\varphi(w_2(\zeta)).\exp\left({}_0^\zeta\frac{p[\varphi(w_1(\xi))-1]}{\xi}d\xi\right)d\zeta\right]*$$

$$*\left(\sum_{n=0}^{\infty}\left(\frac{p+\ell}{p+\ell+\lambda n}\right)^m z^{n+p}\right), \tag{4.3}$$

where $w_j(z)(j=1,2)$ are analytic in U with $w_j(0)=0$ and $|w_j(z)|<1(z\in U;j=1,2)$.

Proof. In view of (1.4) and (3.9), we know that

$$p_0^z\zeta^{p-1}\varphi(w_2(\zeta)).\exp\left({}_0^\zeta\frac{p[\varphi(w_1(\xi))-1}{\xi}d\xi\right)d\zeta$$

$$=\left(z^p+\sum_{n=1}^{\infty}\left(\frac{p+\ell+\lambda n}{p+\ell}\right)^m z^{n+p}\right)*f(z)=\phi_{p,\lambda,\ell}^m(z)*f(z). \tag{4.4}$$

Thus, from (4.4), we easily obtain (4.3).

Theorem 9. *Let* $f\in A(p)$ *and* $\varphi\in P$. Then $f\in S_{p,k}^m(\lambda,\ell;\varphi)$ if and only if

$$\frac{1}{z}\left\{f*\left[\left(pz^p+\sum_{n=1}^{\infty}\left(\frac{p+\ell+\lambda n}{p+\ell}\right)^m(n+p)z^{n+p}\right)\right.\right.$$

$$\left.\left.-p\varphi(e^{i\theta})\left(z^p+\sum_{n=1}^{\infty}\left(\frac{p+\ell+\lambda n}{p+\ell}\right)^m z^{n+p}\right)*\left(\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{z^p}{1-\in^\nu z}\right)\right]\right\}\neq 0$$

$$(z\in U;\ 0\le\theta<2\pi). \tag{4.5}$$

Proof. Suppose that $f\in S_{p,k}^m(\lambda,\ell;\varphi)$. Since

$$\frac{z(I_p^m(\lambda,\ell)f(z))'}{pf_{p,k}^m(\lambda,\ell;z)}\prec\varphi(z)\ ,$$

is equivalent to

$$\frac{z(I_p^m(\lambda,\ell)f)'(z)}{pf_{p,k}^m(\lambda,\ell;z)}\neq\varphi(e^{i\theta})\quad(z\in U;0\le\theta<2\pi), \tag{4.6}$$

it is easy to see that the condition (4.6) can be written as follows:

$$\frac{1}{z}\left[z\left(I_p^m(\lambda,\ell)f\right)'(z)-pf_{p,k}^m(\lambda,\ell;z)\varphi(e^{i\theta})\right]\neq 0\quad(z\in U;0\le\theta<2\pi). \tag{4.7}$$

On the other hand, we know from (1.4) that

$$z\left(I_p^m(\lambda,\ell)f\right)'(z)=\left(pz^p+\sum_{n=1}^{\infty}\left(\frac{p+\ell+\lambda n}{p+\ell}\right)^m(n+p)z^{n+p}\right)*f(z). \tag{4.8}$$

Also, from the definition of $f_{p,k}^m(\lambda,\ell;z)$, we have

$$f_{p,k}^m(\lambda,\ell;z)=I_p^m(\lambda,\ell)f(z)*\left(\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{z^p}{1-\in^\nu z}\right)$$

$$=\left(z^p+\sum_{n=1}^{\infty}\left(\frac{p+\ell+\lambda n}{p+\ell}\right)^m z^{n+p}\right)*\left(\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{z^p}{1-\in^\nu z}\right)*f(z). \tag{4.9}$$

Upon substituting from (4.8) and (4.9) in (4.7), we can easily obtain the convolution property (4.5) asserted by Theorem 9.

5. Integral-preserving properties

In this section, we prove some integral - preserving properties for the class $S_{p,k}^m(\lambda,\ell;\varphi)$.

Theorem 10. *Let* $\varphi\in P$ *and*

$$Re\{p\varphi(z)+\mu\}>0 \quad (z\in U).$$

If $f\in S_{p,k}^m(\lambda,\ell;\varphi)$, then the function $F(z)\in A(p)$ defined by

$$F(z)=\frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1}f(t)dt \quad (\mu>-p; z\in U) \tag{5.1}$$

belongs to the class $S_{p,k}^m(\lambda,\ell;\varphi)$.

Proof. Let $f\in S_{p,k}^m(\lambda,\ell;\varphi)$. Then, from (5.1), we find that

$$z\left(I_p^m(\lambda,\ell)\,F(z)\right)'+\mu I_p^m(\lambda,\ell)F(z)=(\mu+p)I_p^m(\lambda,\ell)f(z). \tag{5.2}$$

Thus, in view of (1.6) and (5.1), we have

$$z\left(F_{p,k}^m(\lambda,\ell;z)\right)'+\mu F_{p,k}^m(\lambda,\ell;z)=(\mu+p)f_{p,k}^m(\lambda,\ell;z). \tag{5.3}$$

We now put

$$H(z)=\frac{z\left(F_{p,k}^m(\lambda,\ell;z)\right)'}{pF_{p,k}^m(\lambda,\ell;z)} \quad (z\in U). \tag{5.4}$$

Then $H(z)$ is analytic in U and $H(0)=1$. It follows from (5.3) and (5.4) that

$$\mu+pH(z)=(\mu+p)\frac{f_{p,k}^m(\lambda,\ell;z)}{F_{p,k}^m(\lambda,\ell;z)}. \tag{5.5}$$

Differentiating both sides of (5.5) logarithmically with respect to z and using Lemma 3, we obtain

$$H(z)+\frac{zH'(z)}{\mu+pH(z)}=\frac{z(f_{p,k}^m(\lambda,\ell;z))'}{pf_{p,k}^m(\lambda,\ell;z)}\prec\varphi(z)\,. \tag{5.6}$$

Since $Re\{p\varphi(z)+\mu\}>0$ $(z\in U)$, it follows from (5.6) and Lemma 1 that $H(z)\prec\varphi(z)$ $(z\in U)$. Furthermore, we suppose that

$$G(z) = \frac{z(I_p(\lambda,\ell)F(z))^{'}}{pF_{p,k}^m(\lambda,\ell;z)} \quad (z \in U).$$

The remainder of the proof of Theorem 10 is similar to that of Theorem 1. We, therefore, choose to omit the analogous details involved. We thus find that

$$G(z) \prec \varphi(z) \ ,$$

which implies that $F(z) \in S_{p,k}^m(\lambda,\ell;\varphi)$. This completes the proof of Theorem 10.

Theorem 11. *Let* $\varphi \in P$ *and*

$$Re\{p\beta\varphi(z)+\mu\} > 0 \ \ (z \in U).$$

If $f \in S_{p,1}^m(\lambda,\ell;\varphi)$, then the function $R(z) \in A(p)$ defined by

$$I_p^m(\lambda,\ell)R(z) = \left\{ \frac{\mu+p\beta}{z^\mu} \int_0^z t^{\mu-1}(I_p^m(\lambda,\ell)f(t))^\beta dt \right\}^{\frac{1}{\beta}} \ (z \in U) \tag{5.7}$$

belongs to the class $S_{p,1}^m(\lambda,\ell;\varphi)$.

Proof. Suppose that $f \in S_{p,1}^m(\lambda,\ell;\varphi)$. Then, by Definition 1, we have

$$\frac{z\left(I_p^m(\lambda,\ell)\ f\right)^{'}(z)}{pI_p^m(\lambda,\ell)f(z)} \prec \varphi(z) \ . \tag{5.8}$$

We now set

$$D(z) = \frac{z\left(I_p^m(\lambda,\ell)\ R\right)^{'}(z)}{pI_p^m(\lambda,\ell)R(z)} \ . \tag{5.9}$$

From (5.7), (5.8) and (5.9), we have

$$\mu + p\beta D(z) = (\mu+p\beta)\left(\frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)R(z)}\right)^\beta . \tag{5.10}$$

Using (5.7), (5.8) and (5.9), we can get

$$D(z) + \frac{zD^{'}(z)}{\mu+p\beta D(z)} = \frac{z(I_p^m(\lambda,\ell)f)'(z)}{pI_p^m(\lambda,\ell)f(z)} \prec \varphi(z) \ . \tag{5.11}$$

Since

$$Re\{p\beta\varphi(z)+\mu\} > 0 \quad (z \in U),$$

it follows from (5.11) and Lemma 1 that

$$D(z) \prec \varphi(z) \ ,$$

that is, that $R(z) \in S_{p,1}^m(\lambda,\ell;\varphi)$. This completes the proof of Theorem 11.

Remark 6 (i) Putting $\lambda = 1$ and $\ell = 0$ in the above results, we obtain corresponding results for the operator D_p^m;

(ii) Putting $\ell = 0$ in the above results, we obtain corresponding results for the operator $D_{\lambda,p}^m$;

(iii) Putting $\lambda = 1$ in the above results, we obtain corresponding results for the operator $I_p(m,\ell)$.

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