SOME SUBCLASSES OF P-VALENT FUNCTIONS INVOLVING
THE EXTENDED MULTIPLIER TRANSFORMATIONS

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Abstract. New classes of \( p \)-valent analytic functions are introduced. Such results as inclusion relationships, integral representations, integral-preserving properties and convolution properties for these function classes are obtained.

1. Introduction

Let \( A(p) \) denote the class of functions \( f(z) \) of the form:

\[
f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \ldots\})
\]

which are analytic and \( p \)-valent in the open unit disk \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). If \( f(z) \) and \( g(z) \) are analytic in \( U \), we say that \( f(z) \) is subordinate to \( g(z) \) written symbolically as follows \( f \prec g \) in \( U \) or \( f(z) \prec g(z) \) (\( z \in U \)), if there exists a Schwarz function \( w(z) \), which (by definition) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in U \)), such that \( f(z) = g(w(z)) \) (\( z \in U \)). Indeed it is known that \( f(z) \prec g(z) \) (\( z \in U \)) \( \Rightarrow \) \( f(0) = g(0) \) and \( f(U) \subset g(U) \). Further, if the function \( g(z) \) is univalent in \( U \), then we have the following equivalent (cf., e.g., [11]; see also [12, p.4])

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

Let \( P \) denote the class of functions of the form:

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,
\]

which are analytic and convex in \( U \) and satisfies the following condition

\[
\text{Re}\{p(z)\} > 0, \quad z \in U
\]

For functions \( f_j(z) \in A(p) \), given by

\[
f_j(z) = z^p + \sum_{n=1}^{\infty} a_{p+n,j}z^{p+n} \quad (j = 1, 2),
\]

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we define the Hadamard product (or convolution) of \( f_1(z) \) and \( f_2(z) \) by

\[
(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p} = (f_2 * f_1)(z).
\]  

(1.3)

Catas [4] extended the multiplier transformation and defined the operator \( I^m_p(\lambda; \ell) \) on \( A(p) \) by the following infinite series

\[
I^m_p(\lambda; \ell)f(z) = z^p + \sum_{n=1}^{\infty} \left[ \frac{p + \ell + \lambda n}{p + \ell} \right]^m a_{n+p,2} z^{n+p}
\]

(\( \ell \geq 0; \lambda \geq 0; p \in \mathbb{N} \) and \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \)).

(1)

We note that:

\[
I^0_p(1,0)f(z) = f(z) \quad \text{and} \quad I^1_p(1,0)f(z) = \frac{zf'(z)}{p}.
\]

By specializing the parameters \( m, \lambda, \ell \) and \( p \), we obtain the following operators studied by various authors:

(i) \( I^m_p(1, \ell)f(z) = I_p(m, \ell)f(z) \) (see Kumar et al. [10] and Srivastava et al. [18]);

(ii) \( I^m_p(1,0)f(z) = D^m_p f(z) \) (see [3], [9] and [15]);

(iii) \( I^m_p(1, \ell)f(z) = I^{m_{1, \ell}}_p f(z) \) (see Cho and Kim [5] and Cho and Srivastava [6]);

(iv) \( I^m_p(1,0)f(z) = D^m_p f(z) \) (see Salagean [17]);

(v) \( I^m_p(\lambda,0)f(z) = D^m_{\lambda p} f(z) \) (see Al-Oboudi [1]);

(vi) \( I^m_p(1,1)f(z) = I^m f(z) \) (see Uralegaddi and Somanatha [19]);

(vii) \( I^m_p(\lambda,0)f(z) = D^m_{\lambda ,\ell p} f(z) \), (see El-Ashwah and M. K. Aouf [8]).

Also we note that

\[
\lambda (I^m_p(\lambda, \ell)f(z))^{1} = (p + \ell)I^{m+1}_p(\lambda, \ell)f(z) - [p(1 - \lambda) + \ell]I^m_p(\lambda, \ell)f(z) \quad (\lambda > 0),
\]

(1.5)

and

\[
I^{m_1}_p(\lambda, \ell)(I^{m_2}_p(\lambda, \ell)f(z)) = I^{m_2}_p(\lambda, \ell)(I^{m_1}_p(\lambda, \ell)f(z)) = I^{m_1 + m_2}_p(\lambda, \ell)f(z),
\]

for all integers \( m_1 \) and \( m_2 \).

Also if \( f \) is given by (1.1), then we have

\[
I^m_p(\lambda, \ell)f(z) = (\phi_{m, \lambda, \ell}^m f)(z),
\]

where

\[
\phi_{m, \lambda, \ell}(z) = z^p + \sum_{n=1}^{\infty} \left[ \frac{p + \ell + \lambda n}{p + \ell} \right]^m z^{p+n}.
\]

Throughout this paper, we assume that \( p, k \in \mathbb{N}, m \in \mathbb{N}_0, \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right) \) and

\[
f^{m}_{p,k}(\lambda, \ell; z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{-j \ell} (I^m_p(\lambda, \ell)f)(\varepsilon_k^j z) = z^p + ... (f \in A(p)).
\]

(1.6)

Clearly, for \( k = 1 \), we have

\[
f^{m}_{p,1}(\lambda, \ell; z) = I^m_p(\lambda, \ell)f(z).
\]

Making use of the extended multiplier transformations \( I^m_p(\lambda, \ell) \) and the above mentioned principle of subordination between analytic functions, we now introduce and investigate the following subclasses of the class \( A(p) \) of \( p \)-valent analytic functions.
Definition 1. A function \( f(z) \in A(p) \) is said to be in the class \( S_{p,k}^{m}(\lambda, \ell; \varphi) \) if it satisfies the following subordination condition:

\[
\frac{z(I_{p}^{m}(\lambda, \ell)f)'(z)}{pf_{p,k}^{m}(\lambda, \ell; z)} < \varphi(z),
\]

where \( \varphi \in P \) and \( f_{p,k}^{m}(\lambda, \ell; z) \neq 0 \) \((z \in U^*)\) is defined by (1.6).

Remark 1. Putting \( p = \lambda = 1 \) and \( m = \ell = 0 \) in the class \( S_{p,k}^{m}(\lambda, \ell; \varphi) \), we obtain the function class \( S_{k}^{(\ell)}(\varphi) \) which introduced and studied by Wang et al. [20].

Definition 2. A function \( f \in A(p) \) is said to be in the class \( K_{p,k}^{m}(\lambda, \ell; \alpha; \varphi) \) if it satisfies the following subordination condition:

\[
(1 - \alpha)\frac{z(I_{p}^{m}(\lambda, \ell)f)'(z)}{pf_{p,k}^{m}(\lambda, \ell; z)} + \alpha\frac{z(I_{p}^{m+1}(\lambda, \ell)f)'(z)}{pf_{p,k}^{m+1}(\lambda, \ell; z)} < \varphi(z),
\]

for some \( \alpha(\alpha \geq 0) \), where \( \varphi \in P \) and \( f_{p,k}^{m+1}(\lambda, \ell; z) \) is defined by (1.6) and satisfying \( f_{p,k}^{m+1}(\lambda, \ell; z) \neq 0 \) \((z \in U^*)\).

Remark 2. Putting \( p = \lambda = 1 \) and \( m = \ell = 0 \) in the class \( K_{p,k}^{m}(\lambda, \ell; \alpha; \varphi) \), we obtain the function class \( K_{k}^{(\ell)}(\alpha, \varphi) \) of functions which are \( \alpha \)-convex with respect to \( k \)-symmetric points (see Yuan and Liu [21]).

Definition 3. A function \( f \in A(p) \) is said to be in the class \( C_{p,k}^{m}(\lambda, \ell; \varphi) \) if it satisfies the following subordination condition:

\[
\frac{z(I_{p}^{m}(\lambda, \ell)f)'(z)}{pg_{p,k}^{m}(\lambda, \ell; z)} < \varphi(z), \quad (g \in S_{p,k}^{m}(\lambda, \ell; \varphi)),
\]

where \( \varphi \in P \) and \( g_{p,k}^{m}(\lambda, \ell; z) \neq 0 \) \((z \in U^*)\) is defined by (1.6).

Remark 3. Taking \( \lambda = k = 1, m = \ell = 0 \) and \( \varphi(z) = \frac{1 + z}{1 - z} \) in the class \( C_{p,k}^{m}(\lambda, \ell; \varphi) \), we obtain the class of p-valent close-to-convex functions (see Aouf [2]).

Definition 4. A function \( f \in A(p) \) is said to be in the class \( G_{p,k}^{m}(\lambda, \ell; \alpha; \varphi) \) if it satisfies the following subordination condition:

\[
(1 - \alpha)\frac{z(I_{p}^{m}(\lambda, \ell)f)'(z)}{pg_{p,k}^{m}(\lambda, \ell; z)} + \alpha\frac{z(I_{p}^{m+1}(\lambda, \ell)f)'(z)}{pg_{p,k}^{m+1}(\lambda, \ell; z)} < \varphi(z), \quad (\alpha \geq 0; g \in S_{p,k}^{m}(\lambda, \ell; \varphi)),
\]

where \( \varphi \in P \), \( g_{p,k}^{m}(\lambda, \ell; z) \) is defined by (1.6) and \( g_{p,k}^{m+1}(\lambda, \ell; z) \neq 0 \) \((z \in U^*)\).

Remark 4. (i) Putting \( p = \lambda = 1 \) and \( m = \ell = 0 \) in the class \( G_{p,k}^{m}(\lambda, \ell; \alpha; \varphi) \), we obtain the class \( QC_{k}^{(\ell)}(\alpha; \varphi) \) of functions which are \( \alpha \)-quasi-convex with respect to \( k \)-symmetric points (see Yuan and Liu [21]);

(ii) Taking \( p = \lambda = k = \alpha = 1, m = \ell = 0 \) and \( \varphi(z) = \frac{1 + z}{1 - z} \) in the class \( G_{p,k}^{m}(\lambda, \ell; \alpha; \varphi) \), we obtain the familiar class of quasi-convex functions (see Noor [14]).

In order to establish our main results, we shall use of the following lemmas.

Lemma 1 [7, 12]. Let \( \beta, \gamma \in \mathbb{C} \). Suppose also that \( \varphi(z) \) is convex and univalent in \( U \) with

\[
\varphi(0) = 1 \quad \text{and} \quad \text{Re}\{\beta\varphi(z) + \gamma\} > 0, \quad (z \in U).
\]
If \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \), then the following subordination:

\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < \varphi(z),
\]

implies that

\[
p(z) < \varphi(z).
\]

**Lemma 2 [16].** Let \( \beta, \gamma \in \mathbb{C} \). Suppose that \( \varphi(z) \) is convex and univalent in \( U \) with

\[
\varphi(0) = 1 \text{ and } \Re\{\beta \varphi(z) + \gamma\} > 0 \quad (z \in U).
\]

Also let

\[
q(z) < \varphi(z).
\]

If \( p(z) \in P \) and satisfies the following subordination:

\[
p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} < \varphi(z),
\]

then

\[
q(z) < \varphi(z).
\]

**Lemma 3.** Let \( f \in S^m_{p,k}(\lambda, \ell; \varphi) \). Then

\[
\frac{z(f^m_{p,k}(\lambda, \ell; z))'}{pf^m_{p,k}(\lambda, \ell; z)} < \varphi(z). \quad (1.11)
\]

**Proof.** In view of (1.6), we replace \( z \) by \( \varepsilon^j_k z \) \( (j = 0, 1, 2, \ldots, k - 1) \) in \( f^m_{p,k}(\lambda, \ell; z) \).

We thus obtain

\[
f^m_{p,k}(\lambda, \ell; \varepsilon^j_k z) = \frac{1}{k} \sum_{n=0}^{k-1} \varepsilon^{-np} (I_p^m(\lambda, \ell)f)(\varepsilon^{n+j}_k z)
\]

\[
= \varepsilon^{jp} \frac{1}{k} \sum_{n=0}^{k-1} \varepsilon^{-(n+j)p} (I_p^m(\lambda, \ell)f)(\varepsilon^{n+j}_k z)
\]

\[
= \varepsilon^{jp} f^m_{p,k}(\lambda, \ell; z). \quad (1.12)
\]

Differentiating both sides of (1.6) with respect to \( z \), we obtain

\[
(f^m_{p,k}(\lambda, \ell; z))' = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon^{-j(p-1)} (I_p^m(\lambda, \ell)f)'(\varepsilon^j_k z). \quad (1.13)
\]

Therefore, from (1.12) and (1.13), we find that

\[
\frac{z(f^m_{p,k}(\lambda, \ell; z))'}{pf^m_{p,k}(\lambda, \ell; z)} = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon^{-j(p-1)} z(I_p^m(\lambda, \ell)f)'(\varepsilon^j_k z)
\]

\[
= \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon^j_k z(I_p^m(\lambda, \ell)f)'(\varepsilon^j_k z). \quad (1.14)
\]

Moreover, since \( f \in S^m_{p,k}(\lambda, \ell; \varphi) \), it follows that

\[
\frac{\varepsilon^j_k z(I_p^m(\lambda, \ell)f)'(\varepsilon^j_k z)}{pf^m_{p,k}(\lambda, \ell; \varepsilon^j_k z)} < \varphi(z) \quad (j = 0, 1, \ldots, k - 1). \quad (1.15)
\]
Finally, by noting that \( \varphi(z) \) is convex and univalent in \( U \), from (1.14) and (1.5), we conclude that the assertion (1.11) of Lemma 3 holds true.

Similarly, for the class \( K_{p,k}^m(\lambda, \ell; \alpha; \varphi) \), we can prove the following result.

**Lemma 4.** Let \( f \in K_{p,k}^m(\lambda, \ell; \alpha; \varphi) \). Then

\[
(1 - \alpha) \frac{z(f_{p,k}^m(\lambda, \ell; z))'}{p f_{p,k}^m(\lambda, \ell; z)} + \alpha \frac{z(f_{p,k}^{m+1}(\lambda, \ell; z))'}{p f_{p,k}^{m+1}(\lambda, \ell; z)} < \varphi(z). \quad (1.16)
\]

In the present paper, we obtain some inclusion relationships, integral representation, convolution properties and integral-preserving properties for each of the function classes \( S_{p,k}^m(\lambda, \ell; \varphi) \), \( K_{p,k}^m(\lambda, \ell; \alpha; \varphi) \), \( C_{p,k}^m(\lambda, \ell; \varphi) \) and \( G_{p,k}^m(\lambda, \ell; \alpha; \varphi) \).

### 2. A set of inclusion relationships

In this section, we obtain some inclusion relationships for the function classes \( S_{p,k}^m(\lambda, \ell; \varphi) \), \( K_{p,k}^m(\lambda, \ell; \alpha; \varphi) \), \( C_{p,k}^m(\lambda, \ell; \varphi) \) and \( G_{p,k}^m(\lambda, \ell; \alpha; \varphi) \).

Unless otherwise mentioned we shall assume throughout the paper that \( \lambda > 0; \ell \geq 0; p, k \in \mathbb{N} \) and \( m \in \mathbb{N}_0 \).

**Theorem 1.** Let \( \varphi \in P \) with

\[
\text{Re} \left\{ p \varphi(z) + \frac{p(1 - \lambda) + \ell}{\lambda} \right\} > 0 \quad (z \in U),
\]

then

\[
S_{p,k}^{m+1}(\lambda, \ell; \varphi) \subset S_{p,k}^m(\lambda, \ell; \varphi).
\]

**Proof.** Making use of the relationships in equations (1.5) and (1.6), we know that

\[
z \left( f_{p,k}^m(\lambda, \ell; \varphi) \right)' + \left[ \frac{p(1 - \lambda) + \ell}{\lambda} \right] f_{p,k}^m(\lambda, \ell; z)
= \frac{p + \ell}{\lambda} \sum_{j=0}^{k-1} \varepsilon_k^j p \left( f_{p,k}^{m+1}(\lambda, \ell)f(\varepsilon_k^j z) \right) = \frac{p + \ell}{\lambda} f_{p,k}^{m+1}(\lambda, \ell; z). \quad (2.1)
\]

Let \( f \in S_{p,k}^{m+1}(\lambda, \ell; \varphi) \) and suppose that

\[
w(z) = \frac{z \left( f_{p,k}^m(\lambda, \ell; z) \right)'}{p f_{p,k}^m(\lambda, \ell; z)} \quad (z \in U). \quad (2.2)
\]

Then \( w(z) \) is analytic in \( U \) and \( w(0) = 1 \). It follows from (2.1) and (2.2) that

\[
pw(z) + \frac{p(1 - \lambda) + \ell}{\lambda} = \frac{p + \ell}{\lambda} f_{p,k}^{m+1}(\lambda, \ell; z). \quad (2.3)
\]

Differentiating both sides of (2.3) logarithmically with respect to \( z \) and using (2.2), we obtain

\[
w(z) + \frac{zw'(z)}{pw(z) + \frac{p(1 - \lambda) + \ell}{\lambda}} = \frac{z \left( f_{p,k}^{m+1}(\lambda, \ell; z) \right)'}{p f_{p,k}^{m+1}(\lambda, \ell; z)}. \quad (2.4)
\]

From (2.4) and Lemma 3 (with \( m \) replaced by \( (m + 1) \)), we can see that

\[
w(z) + \frac{zw'(z)}{pw(z) + \frac{p(1 - \lambda) + \ell}{\lambda}} < \varphi(z). \quad (2.5)
\]
Since $\text{Re} \left\{ p\varphi(z) + \frac{p(1 - \lambda) + \ell}{\lambda} \right\} > 0 \ (z \in U)$, by Lemma 1, we have

$$w(z) = \frac{z \left( f_{p,k}^m(\lambda, \ell; z) \right)'}{pf_{p,k}^m(\lambda, \ell; z)} \prec \varphi(z). \quad (2.6)$$

By setting

$$q(z) = \frac{z \left( I_p^m(\lambda, \ell) f \right)'}{pf_{p,k}^m(\lambda, \ell; z)} (z \in U), \quad (2.7)$$

we observe that $q(z)$ is analytic in $U$ and $q(0) = 1$. It follows from (1.5) and (2.7) that

$$q(z)f_{p,k}^m(\lambda, \ell; z) = \frac{(p + \ell)}{\lambda p} I_p^{m+1}(\lambda, \ell)f(z) - \frac{[p(1 - \lambda) + \ell]}{\lambda p} I_p^m(\lambda, \ell)f(z). \quad (2.8)$$

Differentiating both sides of (2.8) with respect to $z$ and using (2.7), we obtain

$$zq'(z) + \left( \frac{p(1 - \lambda) + \ell}{\lambda} + \frac{z \left( f_{p,k}^m(\lambda, \ell; z) \right)'}{f_{p,k}^m(\lambda, \ell; z)} \right) q(z) = \frac{(p + \ell)}{\lambda p} \frac{z \left( I_p^{m+1}(\lambda, \ell) f \right)'}{f_{p,k}^m(\lambda, \ell; z)} (z). \quad (2.9)$$

From (2.2), (2.3) and (2.9), we can obtain

$$q(z) + \frac{zq'(z)}{\frac{p(1 - \lambda) + \ell}{\lambda} + pw(z)} = \frac{z \left( I_p^{m+1}(\lambda, \ell) f \right)'}{pf_{p,k}^m(\lambda, \ell; z)} \prec \varphi(z).$$

Since

$$w(z) \prec \varphi(z)$$

and

$$\text{Re} \left\{ p\varphi(z) + \frac{p(1 - \lambda) + \ell}{\lambda} \right\} > 0 \ (z \in U),$$

it follows from (2.9) and Lemma 2 that

$$q(z) \prec \varphi(z),$$

that is, that $f \in S_{p,k}^m(\lambda, \ell; \varphi)$. This implies that

$$S_{p,k}^{m+1}(\lambda, \ell; \varphi) \subset S_{p,k}^m(\lambda, \ell; \varphi).$$

Hence the proof of Theorem 1 is completed.

**Theorem 2.** Let $\varphi \in P$ with

$$\text{Re} \left\{ p\varphi(z) + \frac{p(1 - \lambda) + \ell}{\lambda} \right\} > 0 \ (z \in U).$$

Then

$$C_{p,k}^{m+1}(\lambda, \ell; \varphi) \subset C_{p,k}^m(\lambda, \ell; \varphi).$$

**Proof.** Suppose that $f \in C_{p,k}^{m+1}(\lambda, \ell; \varphi)$. Then we have

$$\frac{z \left( I_p^{m+1}(\lambda, \ell) f \right)'}{pf_{p,k}^{m+1}(\lambda, \ell; z)} \prec \varphi(z),$$

(2.10)
with \( g \in S_{p,k}^{m+1}(\lambda, \ell; \varphi) \). Furthermore, it follows from Theorem 1 that \( g \in S_{p,k}^m(\lambda, \ell; \varphi) \), and Lemma 3 yields

\[
\psi(z) = \frac{z \left( g_{p,k}^m(\lambda, \ell; z) \right)'}{pg_{p,k}^m(\lambda, \ell; z)} < \varphi(z), \quad (2.11)
\]

We now set

\[
q(z) = \frac{z \left( I_p^m(\lambda, \ell) f \right)'}{pg_{p,k}^m(\lambda, \ell; z)} (z \in U).
\]

Then \( q(z) \) is analytic in \( U \) and \( q(0) = 1 \). It follows from (1.5) and (2.12) that

\[
q(z)g_{p,k}^m(\lambda, \ell; z) = \frac{(p + \ell) I_p^{m+1}(\lambda, \ell) f(z)}{\lambda p} - \frac{[p(1 - \lambda) + \ell]}{\lambda p} I_p^m(\lambda, \ell) f(z).
\]

(2.13)

Differentiating both sides of (2.13) with respect to \( z \) and using (2.1) (with \( f \) replaced by \( g \)), we have

\[
zq'(z) + \left( \frac{p(1 - \lambda) + \ell}{\lambda} + z \left( g_{p,k}^m(\lambda, \ell; z) \right)' \right) q(z) = \frac{(p + \ell) z \left( I_p^{m+1}(\lambda, \ell) f \right)'}{\lambda p} \frac{g_{p,k}^m(\lambda, \ell; z)}{g_{p,k}^m(\lambda, \ell; z)}.
\]

(2.14)

From (2.10), (2.11) and (2.14), we can obtain

\[
q(z) + \frac{zq(z)}{\frac{p(1 - \lambda) + \ell}{\lambda} + p\psi(z)} = \frac{z \left( I_p^{m+1}(\lambda, \ell) f \right)'}{pg_{p,k}^{m+1}(\lambda, \ell; z)} < \varphi(z).
\]

(2.15)

Since

\[
\psi(z) < \varphi(z),
\]

and

\[
Re \left\{ p\varphi(z) + \frac{p(1 - \lambda) + \ell}{\lambda} \right\} > 0 \quad (z \in U),
\]

it follows from (2.15) and Lemma 2 that

\[
q(z) < \varphi(z),
\]

that is, that \( f \in C_{p,k}^m(\lambda, \ell; \varphi) \). This implies that

\[
C_{p,k}^{m+1}(\lambda, \ell; \varphi) \subset C_{p,k}^m(\lambda, \ell; \varphi).
\]

The proof of Theorem 2 is thus completed.

**Theorem 3.** Let \( \varphi \in P \) with

\[
Re \left\{ p\varphi(z) + \frac{p(1 - \lambda) + \ell}{\lambda} \right\} > 0 \quad (z \in U),
\]

then

\[
G_{p,k}^m(\lambda, \ell; \alpha_2; \varphi) \subset G_{p,k}^m(\lambda, \ell; \alpha_1; \varphi) \quad (\alpha_2 > \alpha_1 \geq 0).
\]

**Proof.** Suppose that \( f \in G_{p,k}^m(\lambda, \ell, \alpha_2; \varphi) \). Then we have

\[
(1 - \alpha_2) \frac{z \left( I_p^m(\lambda, \ell) f \right)'}{pg_{p,k}^m(\lambda, \ell; z)} + \alpha_2 \frac{z \left( I_p^{m+1}(\lambda, \ell) f \right)'}{pg_{p,k}^{m+1}(\lambda, \ell; z)} < \varphi(z).
\]

(2.16)
Since \( g \in S_{p,k}^m(\lambda, \ell; \varphi) \), it follows from (2.11) to (2.16) that

\[
q(z) + \frac{\alpha_2 z q'(z)}{p(1-\lambda) + \ell} + p\varphi(z) = (1 - \alpha_2) \frac{z (I_p^m(\lambda, \ell)f)'(z)}{pg_{p,k}(\lambda, \ell; z)} +
\]

\[
\alpha_2 \frac{z (I_p^{m+1}(\lambda, \ell)f(z))'}{pg_{p,k}^{m+1}(\lambda, \ell; z)} \prec \varphi(z). \tag{2.17}
\]

Since

\[
\psi(z) \prec \varphi(z),
\]

and

\[
\frac{1}{\alpha_2} \text{Re} \left\{ p\varphi(z) + \frac{p(1-\lambda) + \ell}{\lambda} \right\} > 0 \quad (z \in U),
\]

it follows from (2.17) and Lemma 2 that

\[
q(z) = \frac{z (I_p^m(\lambda, \ell)f)'(z)}{pg_{p,k}^m(\lambda, \ell; z)} \prec \varphi(z). \tag{2.18}
\]

Moreover, since \( 0 \leq \frac{\alpha_1}{\alpha_2} < 1 \) and the function \( \varphi(z) \) is convex and univalent in \( U \), we deduce from (2.17) and (2.18) that

\[
(1 - \alpha_1) \frac{z (I_p^m(\lambda, \ell)f)'(z)}{pg_{p,k}^m(\lambda, \ell; z)} + \alpha_1 \frac{z (I_p^{m+1}(\lambda, \ell)f)'(z)}{pg_{p,k}^{m+1}(\lambda, \ell; z)}
\]

\[
= \frac{\alpha_1}{\alpha_2} \left[ (1 - \alpha_2) \frac{z (I_p^m(\lambda, \ell)f)'(z)}{pg_{p,k}^m(\lambda, \ell; z)} + \alpha_2 \frac{z (I_p^{m+1}(\lambda, \ell)f)'(z)}{pg_{p,k}^{m+1}(\lambda, \ell; z)} \right] + \left( 1 - \frac{\alpha_1}{\alpha_2} \right) q(z)
\]

\[
\prec \varphi(z),
\]

which implies that \( f \in C_{p,k}^m(\lambda, \ell; \alpha_2; \varphi) \). Hence the proof of Theorem 3, is completed.

By applying the same method of Theorem 3, we can easily get the following inclusion relationship.

**Corollary 1.** Let \( \varphi \in P \) with

\[
\text{Re} \left\{ p\varphi(z) + \frac{p(1-\lambda) + \ell}{\lambda} \right\} > 0 \quad (z \in U).
\]

Then \( K_{p,k}^m(\lambda, \ell; \alpha_2; \varphi) \subset K_{p,k}^m(\lambda, \ell; \alpha_1; \varphi) \) \( (\alpha_2 > \alpha_1 \geq 0) \).

In view of Theorem 3, we can also easily get the following inclusion relationships.

In particular, a direct proof of Corollary 2 would require use of Lemma 4.

**Corollary 2.** Let \( \alpha \geq 0 \) and \( \varphi \in P \). Then

\[
G_{p,k}^m(\ell; \alpha; \varphi) \subset C_{p,k}^m(\lambda, \ell; \varphi).
\]

**Corollary 3.** Let \( \alpha \geq 0 \) and \( \varphi(z) \in P \). Then

\[
K_{p,k}^m(\lambda, \ell; \alpha; \varphi) \subset S_{p,k}^m(\lambda, \ell; \varphi).
\]
3. INTEGRAL REPRESENTATION

In this section, we obtain a number of integral representations associated with the function class $S_{p,k}^m(\lambda, \ell; \varphi)$.

**Theorem 4.** Let $f \in S_{p,k}^m(\lambda, \ell; \varphi)$. Then

$$f_{p,k}^m(\lambda, \ell; z) = z^p \exp \left\{ \frac{p}{k} \sum_{j=0}^{k-1} \int_0^z \varphi(w(\xi)) - 1 d\xi \right\},$$

(3.1)

where $f_{p,k}^m(\lambda, \ell; z)$ is defined by (1.6), $w(z)$ is analytic in $U$ and satisfy $w(0) = 1$ and $|w(z)| < 1$ ($z \in U$).

**Proof.** Suppose that $f \in S_{p,k}^m(\lambda, \ell; \varphi)$. Then condition (1.7) can be written as follows:

$$\frac{z (I_p^m(\lambda, \ell)f(z))'}{p f_{p,k}^m(\lambda, \ell; z)} = \varphi(w(z)) \ (z \in U),$$

(3.2)

where $w(z)$ is analytic in $U$ and satisfy $w(0) = 1$ and $|w(z)| < 1$ ($z \in U$). Replacing $z$ by $\xi_k^j$ $z$ ($j = 0, 1, ..., k-1$) in (3.2), we observe that (3.2) becomes

$$\frac{\xi_k^j z (I_p^m(\lambda, \ell)f(z))'}{p f_{p,k}^m(\lambda, \ell; \xi_k^j z)} = \varphi(w(\xi_k^j z)) \ (z \in U).$$

(3.3)

We note that

$$f_{p,k}^m(\lambda, \ell; \xi_k^j z) = \xi_k^j f_{p,k}^m(\lambda, \ell; z) \ (z \in U).$$

Thus, by letting $j = 0, 1, ..., k-1$ in (3.3), successively, and summing the resulting equations, we have

$$\frac{z (f_{p,k}^m(\lambda, \ell; z))'}{p f_{p,k}^m(\lambda, \ell; z)} = \frac{1}{k} \sum_{j=0}^{k-1} \varphi(w(\xi_k^j z)) \ (z \in U).$$

(3.4)

From (3.4), we get

$$\left( \frac{f_{p,k}^m(\lambda, \ell; z)}{f_{p,k}^m(\lambda, \ell; z)} \right) - \frac{p}{z} \frac{1}{k} \sum_{j=0}^{k-1} \left[ \frac{\varphi(w(\xi_k^j z)) - 1}{z} \right] \ (z \in U),$$

(3.5)

which, upon integration, yields

$$\log \left( \frac{f_{p,k}^m(\lambda, \ell; z)}{z^p} \right) = \frac{p}{k} \sum_{j=0}^{k-1} \int_0^z \varphi(w(\xi_k^j \xi)) - 1 \frac{d\xi}{\xi}.$$ 

(3.6)

Then, the assertion (3.1) of Theorem 4 can now easily obtained from (3.6).

**Theorem 5.** Let $f \in S_{p,k}^m(\lambda, \ell; \varphi)$. Then

$$I_p^m(\lambda, \ell)f(z) = \frac{z}{p} \int_0^z \zeta^{p-1} \varphi(w(\zeta)). \exp \left( \frac{p}{k} \sum_{j=0}^{k-1} \int_0^\zeta \varphi(w(\xi_k^j \xi)) - 1 \frac{d\xi}{\xi} \right) d\zeta,$$

(3.7)

where $w(z)$ is analytic in $U$ and satisfy $w(0) = 1$ and $|w(z)| < 1$ ($z \in U$).
Suppose that $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. Then, from (3.1) and (3.2), we have

\[
(I_p^m (\lambda, \ell) f(z))' = \frac{pf_{p,k}^m(\lambda, \ell; z)}{z} \varphi(w(z)) \\
= pz^{p-1} \varphi(w(z)), \exp \left( \frac{p}{k} \sum_{j=0}^{k-1} \frac{\varphi(w(z_j^k))}{z} - 1 \right) d\xi,
\]

which, upon integration, leads us easily to the assertion (3.7) of Theorem 5.

**Remark 5.** Putting $p = \lambda = 1$ and $\ell = m = 0$ in Theorem 5, we obtain the result obtained by Wang et al. [20, Theorem 6].

Moreover, in view of Lemma 3 and Theorem 1, we can get integral representation for the function class $S^m_{p,k}(\lambda, \ell; \varphi)$.

**Theorem 6.** Let $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. Then

\[
I_p^m (\lambda, \ell) f(z) = p_0 \zeta^{p-1} \varphi(w_2(\zeta)) \exp \left( \frac{p}{0} \varphi(w_1(\xi)) - 1 \right) d\xi,
\]

where $w_j(z)(j = 1, 2)$ are analytic in $U$ with $w_j(0) = 0$ and $|w_j(z)| < 1(z \in U; j = 1, 2)$.

**Proof.** Suppose that $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. Then we find from (1.11) that

\[
z \left( \frac{f_{p,k}^m(\lambda, \ell; z)}{pf_{p,k}^m(\lambda, \ell; z)} \right) = \varphi(w_1(z)) (z \in U),
\]

where $w_1(z)$ is analytic in $U$ with $w_1(0) = 1$. Thus, by similarly applying the method of proof of Theorem 4, we find that

\[
f_{p,k}^m(\lambda, \ell; z) = z^p \exp \left( \frac{p}{0} \varphi(w_1(\xi)) - 1 \right) d\xi
\]

It now follows from (3.2) and (3.11) that

\[
(I_p^m (\lambda, \ell) f(z))' = \frac{pf_{p,k}^m(\lambda, \ell; z)}{z} \varphi(w_2(z)) \\
= pz^{p-1} \varphi(w_2(z)), \exp \left( \frac{p}{0} \varphi(w_1(\xi)) - 1 \right) d\xi,
\]

where $w_j(z)(j = 1, 2)$ are analytic in $U$ with $w_j(0) = 0$ and $|w_j(z)| < 1(z \in U; j = 1, 2)$. Integrating both sides of (3.12), we will obtain the assertion (3.9) of Theorem 6.

### 4. Convolution properties

In this section, we derive some convolution properties for the class $S^m_{p,k}(\lambda, \ell; \varphi)$.

**Theorem 7.** Let $f \in S^m_{p,k}(\lambda, \ell; \varphi)$. Then

\[
f(z) = \left[ p_0 \zeta^{p-1} \varphi(w(\zeta)) \exp \left( \frac{p}{k} \sum_{j=0}^{k-1} \frac{\varphi(w(\zeta_j^k))}{z} - 1 \right) d\xi \right] * \\
* \left( \sum_{n=0}^{\infty} \left( \frac{p+\ell}{p+\ell+\lambda n} \right)^m z^{n+p} \right),
\]

where $w(z)$ is analytic in $U$ with $w(0) = 1$ and $|w(z)| < 1 (z \in U)$. 
Proof. In view of (1.4) and (3.7), we know that
\[ p_0 \zeta^{p-1} \varphi(w(\zeta)) \exp \left( \frac{\zeta \sum_{j=0}^{k-1} \varphi(w(\epsilon_1^j \xi)) - 1}{\xi} \right) d\zeta \]
\[ = \left( z^p + \sum_{n=1}^{\infty} \left( \frac{p + \ell + \lambda_n}{p + \ell} \right)^m z^{n+p} \right) * f(z) = \phi_{p,\lambda,\ell}^m(z) * f(z). \tag{4.2} \]
Thus, from (4.2), we can easily get the assertion (4.1) of Theorem 7.

**Theorem 8.** Let \( f \in S_{p,k}^m(\lambda, \ell; \varphi) \). Then
\[ f(z) = \left[ p_0 \zeta^{p-1} \varphi(w_2(\zeta)) \exp \left( \frac{\zeta \varphi(w_1(\xi)) - 1}{\xi} \right) \right] d\zeta \]
\[ = \left( z^p + \sum_{n=0}^{\infty} \left( \frac{p + \ell + \lambda_n}{p + \ell} \right)^m z^{n+p} \right) * f(z) = \phi_{p,\lambda,\ell}^m(z) * f(z). \tag{4.3} \]
where \( w_j(z)(j = 1, 2) \) are analytic in \( U \) with \( w_j(0) = 0 \) and \(|w_j(z)| < 1(z \in U; j = 1, 2)\).

Proof. In view of (1.4) and (3.9), we know that
\[ p_0 \zeta^{p-1} \varphi(w_2(\zeta)) \exp \left( \frac{\zeta \varphi(w_1(\xi)) - 1}{\xi} \right) d\zeta \]
\[ = \left( z^p + \sum_{n=0}^{\infty} \left( \frac{p + \ell + \lambda_n}{p + \ell} \right)^m z^{n+p} \right) * f(z) = \phi_{p,\lambda,\ell}^m(z) * f(z). \tag{4.4} \]
Thus, from (4.4), we easily obtain (4.3).

**Theorem 9.** Let \( f \in A(p) \) and \( \varphi \in P \). Then \( f \in S_{p,k}^m(\lambda, \ell; \varphi) \) if and only if
\[ \frac{1}{z} \left[ \int f * \left[ \left( p z^p + \sum_{n=1}^{\infty} \left( \frac{p + \ell + \lambda_n}{p + \ell} \right)^m (n+p) z^{n+p} \right) \right. \right. \]
\[ \left. \left. \left. - p \varphi(e^{i\theta}) \left( z^p + \sum_{n=1}^{\infty} \left( \frac{p + \ell + \lambda_n}{p + \ell} \right)^m z^{n+p} \right) * \left( \frac{1}{k} \sum_{n=0}^{k-1} \frac{z^n}{1 - e^{i\theta} z} \right) \right] \right] \right] \neq 0 \]
\[ (z \in U; \ 0 \leq \theta < 2\pi). \tag{4.5} \]

Proof. Suppose that \( f \in S_{p,k}^m(\lambda, \ell; \varphi) \). Since
\[ \frac{z(I_p^m(\lambda, \ell) f(z))'}{p f_{p,k}^m(\lambda, \ell; z)} \prec \varphi(z), \]
is equivalent to
\[ \frac{z(I_p^m(\lambda, \ell) f(z))'}{p f_{p,k}^m(\lambda, \ell; z)} \neq \varphi(e^{i\theta}) \quad (z \in U; \ 0 \leq \theta < 2\pi), \tag{4.6} \]
it is easy to see that the condition (4.6) can be written as follows:
\[ \frac{1}{z} \left[ \left( I_p^m(\lambda, \ell) f(z) - p f_{p,k}^m(\lambda, \ell; z) \varphi(e^{i\theta}) \right) \left( z \in U; \ 0 \leq \theta < 2\pi \right) \right] \neq 0. \tag{4.7} \]
On the other hand, we know from (1.4) that
\[
z \left( I^m_p(\lambda, \ell) f \right)'(z) = \left( pz^p + \sum_{n=1}^{\infty} \left( \frac{p + \ell + \lambda n}{p + \ell} \right)^m (n + p)z^{n+p} \right) * f(z). \tag{4.8}
\]
Also, from the definition of \( f_{p,k}^m(\lambda, \ell; z) \), we have
\[
f_{p,k}^m(\lambda, \ell; z) = I^m_p(\lambda, \ell) f(z) * \left( \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{z^p}{1-\epsilon^\nu z} \right)
= \left( z^p + \sum_{n=1}^{\infty} \left( \frac{p + \ell + \lambda n}{p + \ell} \right)^m z^{n+p} \right) * \left( \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{z^p}{1-\epsilon^\nu z} \right) * f(z). \tag{4.9}
\]
Upon substituting from (4.8) and (4.9) in (4.7), we can easily obtain the convolution property (4.5) asserted by Theorem 9.

5. INTEGRAL-PRESERVING PROPERTIES

In this section, we prove some integral-preserving properties for the class \( S_{p,k}^m(\lambda, \ell; \varphi) \).

**Theorem 10.** Let \( \varphi \in P \) and
\[
Re \{ p\varphi(z) + \mu \} > 0 \quad (z \in U).
\]
If \( f \in S_{p,k}^m(\lambda, \ell; \varphi) \), then the function \( F(z) \in A(p) \) defined by
\[
F(z) = \frac{\mu + p}{z^{\mu}} \int_0^t f(t) \, dt \quad (\mu > -p; \ z \in U) \tag{5.1}
\]
belong to the class \( S_{p,k}^m(\lambda, \ell; \varphi) \).

**Proof.** Let \( f \in S_{p,k}^m(\lambda, \ell; \varphi) \). Then, from (5.1), we find that
\[
z \left( I^m_p(\lambda, \ell) F(z) \right)' + \mu I^m_p(\lambda, \ell) F(z) = (\mu + p) I^m_p(\lambda, \ell) f(z). \tag{5.2}
\]
Thus, in view of (1.6) and (5.1), we have
\[
z \left( F_{p,k}^m(\lambda, \ell; z) \right)' + \mu F_{p,k}^m(\lambda, \ell; z) = (\mu + p) f_{p,k}^m(\lambda, \ell; z). \tag{5.3}
\]
We now put
\[
H(z) = \frac{z \left( F_{p,k}^m(\lambda, \ell; z) \right)'}{p F_{p,k}^m(\lambda, \ell; z)} \quad (z \in U). \tag{5.4}
\]
Then \( H(z) \) is analytic in \( U \) and \( H(0) = 1 \). It follows from (5.3) and (5.4) that
\[
\mu + p H(z) = (\mu + p) \frac{F_{p,k}^m(\lambda, \ell; z)}{F_{p,k}^m(\lambda, \ell; z)}. \tag{5.5}
\]
Differentiating both sides of (5.5) logarithmically with respect to \( z \) and using Lemma 3, we obtain
\[
\frac{H(z) + \frac{z H'(z)}{\mu + p H(z)}}{p H(z)} = \frac{z \left( F_{p,k}^m(\lambda, \ell; z) \right)'}{p F_{p,k}^m(\lambda, \ell; z)} \prec \varphi(z). \tag{5.6}
\]
Since \( Re \{ p\varphi(z) + \mu \} > 0 \ (z \in U) \), it follows from (5.6) and Lemma 1 that \( H(z) \prec \varphi(z) \ (z \in U) \). Furthermore, we suppose that
\[ G(z) = \frac{z(I_p(\lambda, \ell)F(z))'}{pI^m_{p,k}(\lambda, \ell; z)} \quad (z \in U). \]

The remainder of the proof of Theorem 10 is similar to that of Theorem 1. We, therefore, choose to omit the analogous details involved. We thus find that

\[ G(z) \prec \varphi(z), \]

which implies that \( F(z) \in S^m_{p,k}(\lambda, \ell; \varphi). \) This completes the proof of Theorem 10.

**Theorem 11.** Let \( \varphi \in P \) and

\[ \text{Re} \{p\beta \varphi(z) + \mu\} > 0 \quad (z \in U). \]

If \( f \in S^m_{p,k}(\lambda, \ell; \varphi) \), then the function \( R(z) \in A(p) \) defined by

\[ I^m_{p}(\lambda, \ell)R(z) = \left\{ \frac{\mu + p\beta z}{z^\mu} \int_0^1 t^\mu I^m_{p}(\lambda, \ell)f(t)^\beta \, dt \right\}^{\frac{1}{\beta}} (z \in U) \quad (5.7) \]

belongs to the class \( S^m_{p,k}(\lambda, \ell; \varphi) \).

**Proof.** Suppose that \( f \in S^m_{p,k}(\lambda, \ell; \varphi) \). Then, by Definition 1, we have

\[ \frac{z}{pI^m_{p}(\lambda, \ell)f(z)} \prec \varphi(z). \quad (5.8) \]

We now set

\[ D(z) = \frac{z(I^m_{p}(\lambda, \ell)R(z))'}{pI^m_{p}(\lambda, \ell)R(z)}. \quad (5.9) \]

From (5.7), (5.8) and (5.9), we have

\[ \mu + p\beta D(z) = (\mu + p\beta) \left( \frac{I^m_{p}(\lambda, \ell)f(z)}{pI^m_{p}(\lambda, \ell)R(z)} \right)^\beta. \quad (5.10) \]

Using (5.7), (5.8) and (5.9), we can get

\[ D(z) + \frac{zD'(z)}{\mu + p\beta D(z)} = \frac{z(I^m_{p}(\lambda, \ell)f(z))'}{pI^m_{p}(\lambda, \ell)f(z)} \prec \varphi(z). \quad (5.11) \]

Since

\[ \text{Re} \{p\beta \varphi(z) + \mu\} > 0 \quad (z \in U), \]

it follows from (5.11) and Lemma 1 that

\[ D(z) \prec \varphi(z), \]

that is, that \( R(z) \in S^m_{p,k}(\lambda, \ell; \varphi). \) This completes the proof of Theorem 11.

**Remark 6** (i) Putting \( \lambda = 1 \) and \( \ell = 0 \) in the above results, we obtain corresponding results for the operator \( D^m_{p}; \)

(ii) Putting \( \ell = 0 \) in the above results, we obtain corresponding results for the operator \( D^m_{p}; \)

(iii) Putting \( \lambda = 1 \) in the above results, we obtain corresponding results for the operator \( I_p(m, \ell). \)
References


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