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ON THE APPLICATION OF MEAN SQUARE CALCULUS FOR SOLVING RANDOM DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the random Finite Difference Methods are used in solving random differential initial value problems of first order. The random Finite Difference method is presented and the conditions for the mean square convergence are established. Numerical examples show that random Finite Difference method gives good results. The some statistical properties of the numerical solutions are computed through numerical case studies.

1. Introduction

Random ordinary differential equations are defined as equations which contain contain random input variables [3]. Most scientific, engineering, physical, chemical and biological problems, which are very important for scientific and technological progress, have been traditionally formulated through mathematical models based on ordinary or partial differential equations, where the data (initial conditions, source term and/or coefficients) are expressed by means of numerical values or deterministic functions [5]. In recent years, the solution of a stochastic differential equation is gotten when we evaluate the probability density function of this solution. We can use several methods; see [1], [2], [4], [5], [6], [7], [8], [9], [10]. This paper deals with random differential initial value problems of the form

$$\begin{aligned} \dot{X}(t) &= F(X(t),t), \qquad t_0 < t < t_e, \\ X(t_0) &= X_0 \end{aligned} \tag{1}$$

where X_0 is a second order random variable and, the unknown $X(t)$ as well as the second member $F(X(t),t)$ are second order stochastic processes. In this paper the random Finite Difference methods are used to obtain an approximate solution for Equation 1. This paper is organized as follows. Section 2 deals with some preliminary definitions, results, notations and examples. Section 3 is addressed to the presentation and the proof of the convergence for the random Forward Finite Difference Scheme in mean square sense. In Section 4, the statistical properties for the exact and numerical solutions are studied. Last section 5 is devoted to conclusions.

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2. Preliminaries

2.1. **Mean Square Calculus.**

Definition 2.1. Consider the properties of a class of real r.v.'s $X_1, X_2, \ldots, X_n$ whose second moments,$E\left\{X_1^2\right\}, E\left\{X_2^2\right\}, \ldots E\left\{X_n^2\right\}$ are finite. In this case, they are called "*second order random variables*", (2.r.v's) [4].

Definition 2.2. The linear vector space of second order random variables with inner product, norm and distance, is called an L_2 -space. A s.p.$\{X(t),\ t \in T\}$ is called a "*second order stochastic process*" (2.s.p) if for $t_1, t_2, \ldots, t_n$, the r.v's $\{X(t_1), X(t_2), \ldots, X(t_n)\}$, are elements of L_2-space. A second order s.p. $\{X(t), t \in T\}$ [4] is characterized by

$$\|X(t)\|^2 = E\left\{X^2(t)\right\} < \infty, t \in T.$$

2.1.1. *Convergence in Mean square.* A sequence of r.v.'s $\{X_n\}$ converges in m.s. to a r.v. X as $n \to \infty$ if

$$\lim_{n \to \infty} \|X_n - X\| = 0$$

This type of convergence is often expressed by

$$X_n \xrightarrow{\text{m.s.}} X \qquad \text{or} \qquad l.i.m_{n\to\infty} X_n = X$$

The symbol l.i.m. denotes the limit in the mean square sense.

2.1.2. *Mean Square Continuity.* A 2-s.p. $\{X(t) : t \in T\}$ is said to be m.s. continuous at $t \in T$ if

$$l.i.m_{\tau\to\infty} X(t+\tau) = X(t), \quad \text{for} \quad t+\tau \in T, \text{or}$$
$$\lim_{\tau \to 0} \|X(t+\tau) - X(t)\|_2 = 0.$$

2.1.3. *Mean Square Differentiation.* The concept of m.s. differentiation follows naturally from that of m.s. continuity. A 2-s.p. $\{X(t) : t \in T\}$ has a m.s. derivative $\dot{X}(t)$ at $t \in T$ if

$$\lim_{\tau \to 0} \frac{X(t+\tau) - X(t)}{\tau} = \dot{X}(t), \quad \text{for} \quad t+\tau \in T, \text{or}$$
$$\lim_{\tau \to 0} \left\| \frac{X(t+\tau) - X(t)}{\tau} - \dot{X}(t) \right\|_2 = 0.$$

2.1.4. *Mean Square Analyticity.* A 2-s.p. $\{X(t) : t \in T\}$ is said to be m.s. analytic on T if it can be expanded in the m.s. convergent series

$$X(t) = \sum_{n=0}^{\infty} \frac{X^{(n)}(t_0)}{n!}(t-t_0)^n, \qquad t, t_0 \in T.$$

2.1.5. *Mean Square Integration.* Let $\{X(t), t \in T\}$ be a 2-s.p. defined on$[a,b] \subset T$. Let $f(t,u)$ be an ordinary function defined on the same interval for t and Riemann integrable for every$u \in U$. We form the random variable

$$Y_n(u) = \sum_{k=1}^{n} f(t_k^{'}) X(t_k^{'})(t_k - t_{k-1})$$

Since L_2-space is linear, $Y_n(u)$ is an element of the L_2-space. It is a r.v. defined for each partition P_n and for $u \in U$.

Definition 2.3. If, for $u \in U$,

$$\text{l.i.m}_{n \underset{\triangle_{n\to 0}}{\longrightarrow} \infty} Y_n\left(u\right) = Y\left(u\right)$$

exists for some sequence of subdivisions P_n, the s.p. $Y(u)$, $u \in U$, is called the "*mean square Riemann integral*" or "*m.s. Riemann integral*" of $f(t,u)X(t)$ over the interval $[a,b]$ and is denoted by

$$Y(u) = \int_a^b f(t,u)X(t)dt$$

it is independent of the sequence of subdivisions as well as the positions of $t_k^{'} \in [t_k, t_{k-1})$.

Theorem 2.4 (Fundamental theorem of the m.s. calculus)**.** *Let $\{X(t), t \in T\}$ be a 2-s.p. m.s. differentiable on $T = [t_0, t]$, such that $X(t)$ is m.s. Riemann integrable on T, then one gets*

$$\int_{t_0}^{t} \dot{X}(u)du = X(t) - X(t_0).$$

Example 2.5. *Let Y be a 2-r.v. and let us consider the 2-s.p. $Y(t) = Y.t$ for t lying in the interval T and applying the formula of integration by parts for $h(t,u) \equiv 1$ one gets*

$$\int_{t_0}^{t} Y du = (t - t_0).$$

Proposition 2.6. *If $\{X(t), t \in T\}$ is a 2-s.p. m.s. continuous on $T = [t_0, t]$, then*

$$\left\| \int_{t_0}^{t} X(u)du \right\| \leq \int_{t_0}^{t} \left\| X\left(u\right) \right\| du \leq M_X\left(t - t_0\right),$$

$$M_X = \max_{t_0 \leq u \leq t} \left\| X\left(u\right) \right\|.$$

3. Random Initial Value Problem (RIVP)

3.1. Existence and Uniqueness.

Theorem 3.1. *Consider the random initial value problem* (1)*. If $F : S \times T \to L_2$ is continuous, satisfies the m.s. Lipschitz condition*

$$\left\| F(X,t) - F(Y,t) \right\|_2 \leq k(t) \left\| X - Y \right\|_2 \tag{2}$$

$$\textit{where} \int_{t_0}^{t_e} k(t)dt < \infty.$$

Then there exists a unique m.s. solution for any initial condition $X_0 \in L_2$ [3]*.*

3.2. The Convergence of Finite Difference Methods for Random Differential Equations in (m.s.) Sense.

Where X_0 is a second order random variable and, the unknown $X(t)$ as well as the second member $F(X(t),t)$ are second order stochastic processes defined on some probability space are solved using the random finite difference method.

Definition 3.2. Let $g : T \longrightarrow L_2$ is an m.s.bounded function and let $h > 0$ then the "m.*s.modulus of continuity of g*" is the function

$$w(g,h) = \sup_{|t-t^*|\leq h} \|g(t) - g(t^*)\|, t, t^* \in T.$$

The function g is said to be m.s uniformly continuous in T if $\mathrm{l.i.m}_{h\to 0} w(g,h) = 0$ [5].

Lemma 3.3. *Let $Y(t)$ be a 2-s.p., m.s.continuous on the interval $T = [t_0, t_e]$. Then, there exists $\xi \in [t_0, t_e]$ such that*

$$\int_{t_0}^{t_e} Y(s)ds = Y(\xi)(t_e - t_0), \qquad t_0 < \xi < t_1. \tag{3}$$

Theorem 3.4. *Let $X(s)$ be a m.s.differentiable 2-s.p. in $]t_0, t_1[$ and m.s.continuous in $T = [t_0, t_1]$. Then, there exists $\xi \in [t_0, t_e]$ such that*

$$X(t) - X(t_0) = \dot{X}(\xi)(t - t_0)$$

3.2.1. *The Convergence of Random Finite Difference Scheme.* Let us consider the random initial value problem (1) under the following hypotheses on $F : S \times T \to L_2$ with $S \subseteq L_2$

- C1: $F(X,t)$ is m.s. randomly bounded time uniformly continuous.
- C2: $F(X,t)$ satisfies the m.s. Lipschitz condition (2)

Note that condition C2 guarantees the m.s. continuity of $F(X,t)$ with respect to the first variable while C1 guarantees the continuity of $F(X,t)$ with respect to the second variable. Hence and from the inequality

$$\left\| F(X,t) - F(Y,t') \right\| \leq \|F(X,t) - F(Y,t)\| + \left\| F(Y,t) - F(Y,t') \right\|,$$

one gets the m.s. continuity of $F(X,t)$ with respect to both variable.

Let us introduce the random Forward Finite Difference Method for problem (1) defined by

$$\begin{aligned} X_{n+1} &= x_n + hF(X_n, t_n), \qquad n \geq 0, \\ X_0 &= X(t_0) \end{aligned} \tag{4}$$

where X_n, $F(X_n, t_n)$ are 2-r.v.'s, $h = t_n - t_{n-1}$, with $t_n = t_0 + nh$,for $n = 0, 1, 2, \ldots$ we wish to prove that under hypotheses C1 and C2, the Forward Finite Difference method (4) is m.s. convergent in the station sense, i.e., fixed $t \in [t_0, t_e]$ and taking n so that $t_n = t_0 + nh$, the m.s. error

$$e_n = X_n - X(t) = X_n - X(t_n), \tag{5}$$

tends to zero in L_2, as$h \to 0,\ n \to \infty$ with $t - t_0 = nh$.

Note that under hypotheses C1 and C2, Theorem 5.1.2 of [3] guarantees the existence and uniqueness of a m.s. solution $X(t) \in [t_n, t_{n+1}] \subset [t_0, t_e]$, and by using the m.s. fundamental theorem of calculus, i.e., Theorem 2.4 it follows that

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \dot{X}(u)du, \qquad n \geq 0. \tag{6}$$

From (4)-(6) it follows that

$$\begin{aligned} e_{n+1} - e_n &= (X_{n+1} - X_n) - (X(t_{n+1}) - X(t_n)) \\ &= hF(X_n, t_n) - \int_{t_n}^{t_{n+1}} \dot{X}(u)du. \end{aligned} \tag{7}$$

Note also that $F(X_n, t_n) \in L_2$, the first term appearing in the right-hand side of (7) can be written as follows, see example 2.5,

$$hF(X_n, t_n) = F(X_n, t_n)(t_{n+1} - t_n) = \int_{t_n}^{t_{n+1}} F(X_n, t_n)du. \tag{8}$$

by (7), (8) and using that $\dot{X}(u) = F(X(u), u)$, one gets

$$\begin{aligned} e_{n+1} &= e_n + \int_{t_n}^{t_{n+1}} [F(X_n, t_n) - \dot{X}(u)]du \\ &= e_n + \int_{t_n}^{t_{n+1}} [F(X_n, t_n) - F(X(u), u)]du. \end{aligned} \tag{9}$$

Under hypothesis C1 and C2, $F(X, t)$ is a m.s. continuous with respect to both variables, the 2-s.p. defined by

$$G(u) = F(X_n, t_n) - F(X(u), u), \tag{10}$$

is m.s. continuous for $u \in [t_n, t_{n+1}]$. Taking norms in 9 and using proposition 2.6 it follows that

$$\|e_{n+1}\| \leq \|e_n\| + \int_{t_n}^{t_{n+1}} \|F(X_n, t_n) - F(X(u), u)\| \, du. \tag{11}$$

$$\begin{aligned} \|F(X_n, t_n) - F(X(u), u)\| &\leq \|F(X_n, t_n) - F(X(t_n), t_n)\| \\ &+ \|F(X(t_n), t_n) - F(X(u), t_n)\| \\ &+ \|F(X(u), t_n) - F(X(u), u)\| . \end{aligned} \tag{12}$$

For the two first terms, using hypothesis C2, one gets the following bounds

$$\begin{aligned} \|F(X_n, t_n) - F(X(t_n), t_n)\| &\leq k(t_n) \|X_n - X(t_n)\| \\ &= k(t_n) \|e_n\| , \end{aligned} \tag{13}$$

$$\|F(X(t_n), t_n) - F(X(u), t_n)\| \leq k(t_nt) \|X(t_n) - X(u)\| , \quad u \in [t_n, t_{n+1}]. \tag{14}$$

Note that applying 6 in $[t_n, u] \subset [t_n, t_{n+1}]$ and using again proposition 2.6, it follows that

$$\begin{aligned} \|X(u) - X(t_n)\| &= \left\| \int_{t_n}^{u} \dot{X}(v)dv \right\| \leq \int_{t_n}^{u} \left\| \dot{X}(v) \right\| dv \\ &\leq M_{\dot{X}}(u - t_n) \end{aligned} \tag{15}$$

where $M_{\dot{X}} = sup \left\{ \left\| \dot{X}(v) \right\| ; t_0 \leq v \leq t_e \right\}$.

From 14 and 15 one gets

$$\|F(X(t_n), t_n) - F(X(u), t_n)\| \leq k(t_n)hM_{\dot{X}}. \tag{16}$$

Let S_X be the bounded set in L_2 defined by the exact theoretical solution of problem (1),

$$S_X = \{X(t), \quad t_0 \leq t \leq t_e\}. \tag{17}$$

Then by hypothesis C1 and definition 3.2, we have

$$\|F(X(ut), t_n) - F(X(u), u)\| \leq \omega(S_X, h), \tag{18}$$

and by 13, 16 and 18, it follows that 12 can be written in the form

$$\|F(X_n, t_n) - F(X(u), u)\| \leq k(t_n) \|e_n\| + k(t_n)hM_{\dot{X}} + \omega(S_X, h).$$

Then

$$\int_{t_n}^{t_{n+1}} \|F(X_n,t_n) - F(X(u),u)\|\, du \le h\{k(t_n)\,\|e_n\| + k(t_n)hM_{\dot{X}} + \omega(S_X,h)\}$$

and hence, 11 takes the form

$$\|e_{n+1}\| \le \|e_n\|\,[1 + hk(t_n)] + h[\omega(S_X,h) + k(t_n)hM_{\dot{X}}]. \tag{19}$$

Then by substituting in 12 as shown in [4, 5] we have

$$\|e_{n+1}\| \le (1 + k(t_n)h)^{n+1}\,\|e_0\| + h[\omega(S_X,h)$$
$$+k(t_n)hM_{\dot{X}}][1 + (1 + k(t_n)h) + (1 + k(t_n)h)^2 + \cdots + (1 + k(t_n)h)^n].$$

Since $[1 + (1 + k(t_n)h) + (1 + k(t_n)h)^2 + \cdots + (1 + k(t_n)h)^n]$ is geometrical series, then

$$[1 + (1 + k(t_n)h) + (1 + k(t_n)h)^2 + \cdots + (1 + k(t_n)h)^n] = \frac{(1 + k(t_n)h)^n - 1}{k(t_n)h}$$

Then

$$\|e_{n+1}\| \le (1 + k(t_n)h)^{n+1}\,\|e_0\|$$
$$+[\omega(S_X,h) + k(t_nt)hM_{\dot{X}}]\frac{[(1 + k(t_n)h)^n - 1]}{k(t_n)}$$

we have

$$\|e_n\| \le e^{nhk(t)}\,\|e_0\| + \frac{e^{nhk(t_n)} - 1}{k(t_n)}[k(t_n)hM_{\dot{X}} + \omega(S_X,h)].$$

At $h \longrightarrow 0, \|e_0\| = 0$ where $e_0 = X_0 - X(t_0) = 0$, the last inequality can be written in the form

$$\|e_n\| \le \frac{e^{nhk(t)} - 1}{k(t)}[k(t)hM_{\dot{X}} + \omega(S_X,h)]. \tag{20}$$

From 20, it follows that $\{e_n\}$ is m.s. convergent to zero and summarizing the following results has been established:

Theorem 3.5. *With the previous notation, under the hypothesis C1 and C2, the random Forward Finite Difference method (4) is m.s. convergent and discretization error e_n defined by (5) satisfies the inequality (20) for $t = t_0 + nh, \quad h > 0, \quad t_0 \le t \le t_e$.*

4. Numerical examples

In this section, some illustrative examples are presented.

4.1. **Case study: linear random initial value problem.** In this subsection, some linear initial value problems are considered

Example 4.1. *Consider the linear random differential equation*

$$\dot{X}(t) = Kt^2, \qquad X(t_0) = D, \qquad K \sim N(4,2), \qquad t \in [t_0, t_f]$$

where $\{X(t), t \in T\}$ is 2-order stochastic process, D is arbitrary constant, and K is a random variable has the Normal distribution

Solution **The exact solution**

$$X(t) = \frac{Kt^3}{3} + c$$

At $\quad t = t_0 \qquad \Longrightarrow \qquad X = D + \frac{K(t^3 - t_0^3)}{3}$ **The numerical solution**
Using the Random Finite Difference Method:

$$X_n = X_{n-1} + hf(X_{n-1}, t_{n-1}), \qquad X(t_0) = D$$

At $n = 1$
$X_1 = X_0 + hf(X_0, t_0) = D + hKt_0^2$
At $n = 2$
$X_2 = X_1 + hf(X_1, t_1) = D + hKt_0^2 + hKt_1^2 = D + hKt_0^2 + hK(t_0 + h)^2$
where $t_{i+1} = t_0 + ih$
At $n = 3$
$X_3 = X_2 + hf(X_2, t_2) = D + hKt_0^2 + hK(t_0 + h)^2 + hK(t_0 + 2h)^2$
At $n = 4$
$X_4 = X_3 + hf(X_3, t_3) = D + hKt_0^2 + hK(t_0 + h)^2 + hK(t_0 + 2h)^2 + hK(t_0 + 3h)^2$
and so on ....
Then the general numerical solution is

$$\begin{aligned} X_n &= X_{n-1} + hf(X_{n-1}, t_{n-1}) \\ &= D + hKt_0^2 + hK(t_0 + h)^2 + hK(t_0 + 2h)^2 + hK(t_0 + 3h)^2 \\ &+ \cdots + hK(t_0 + (n-1)h)^2 \\ X_n &= D + hK \sum_{i=0}^{n-1} (t_0 + ih)^2 \end{aligned}$$

Also we can prove that

(1) $l.i.m_{n \to \infty} X_n = X.$

Proof. Since

$$l.i.m_{n \to \infty} X_n = X \qquad \Longleftrightarrow \qquad \lim_{n \to \infty} E|X_n - X|^2 = 0$$

$X_n - X = nhKt_0^2 + n(n-1)Kh^2 t_0 + \frac{n(n+1)(2n+1)}{6} Kh^3 - \frac{K}{(} t^3 - {t_0}^3)3$
where
$h = \frac{t_n - t_0}{n}$
$X_n - X = Kt_0^2(t_n - t_0) + \frac{n-1}{n} K(t_n - t_0)^2 t_0 + \frac{(n+1)(2n+1)}{6n^2} K(t_n - t_0)^3 - \frac{K(t^3 - t_0^3)}{3}$
$|X_n - X|^2 = \left\{Kt_0^2(t_n - t_0) + \frac{n-1}{n} K(t_n - t_0)^2 t_0\right\}^2$
$+2\left\{\left[Kt_0^2(t_n - t_0) + \frac{n-1}{n} K(t_n - t_0)^2 t_0\right] \cdot \left[\frac{(n+1)(2n+1)}{6n}^2 K(t_n - t_0)^3 - \frac{K(t^3 - t_0^3)}{3}\right]\right\} +$
$\left[\frac{(n+1)(2n+1)}{6n^2} K(t_n - t_0)^3 - \frac{K(t^3 - t_0^3)}{3}\right]^2$
$E\,|X_n - X|^2 = t_0^4(t_n - t_0)^2 E\{K^2\} + \frac{2(n-1)}{n} t_0^3 (t_n - t_0)^3 E\{K^2\} + \left(\frac{(n-1)}{n}\right)^2 t_0^2 (t_n -$
$t_0)^4 E\{K^2\} + \frac{(n+1)(2n+1)}{3n^2} t_0^2 (t_n - t_0)^4 E\{K^2\} - \frac{2}{3} t_0^2 (t^3 - t_0^3)(t_n - t_0) E\{K^2\} +$
$\frac{(n-1)(n+1)(2n+1)}{3n^3} E\{K^2\} t_0 (t_n - t_0)^5 + \frac{2(n-1)}{3n} t_0 (t^3 - t_0^3)(t_n - t_0)^2 E\{K^2\} +$

$\left(\frac{(n+1)(2n+1)}{6n}\right)^2 \cdot (t_n - t_0)^6 E\{K^2\} - \frac{(n+1)(2n+1)}{n^2} E\{K^2\}(t^3 - t_0^3)(t_n - t_0)^3 + \frac{1}{9}(t^3 - t_0^3)^2 E\{K^2\}$

since $K \sim N(4,2)$ then $E\{K^2\} = 20$

$E|X_n - X|^2 = 20t_0^4(t_n - t_0)^2 + \frac{40(n-1)}{n} t_0^3(t_n - t_0)^3 + 20\left(\frac{(n-1)}{n}\right)^2 t_0^2(t_n - t_0)^4 + \frac{20(n+1)(2n+1)}{3n^2} t_0^2(t_n - t_0)^4 - \frac{40}{3} t_0^2(t^3 - t_0^3)(t_n - t_0) + \frac{20(n-1)(n+1)(2n+1)}{3n^3} t_0(t_n - t_0)^5 + \frac{40(n-1)}{3n} t_0(t^3 - t_0^3)(t_n - t_0)^2 + 20\left(\frac{(n+1)(2n+1)}{6n}\right)^2 (t_n - t_0)^6 - \frac{20(n+1t)(2n+1t)}{9n^2}(t^3 - t_0^3)(t_n - t_0)^3 + \frac{20}{9}(t^3 - t_0^3)^2$

$\lim_{n\to\infty} E|X_n - X|^2 = 20t_0^4(t_n - t_0)^2 + 40t_0^3(t_n - t_0)^3 + 20t_0^2(t_n - t_0)^4 + \frac{40}{3} t_0^2(t_n - t_0)^4 - \frac{40}{3} t_0^2(t^3 - t_0^3)(t_n - t_0) + \frac{40}{3} t_0(t_n - t_0)^5 - \frac{40}{3} t_0(t^3 - t_0^3)(t_n - t_0)^2 + \frac{20}{9}(t_n - t_0)^6 - \frac{40}{9}(t^3 - t_0^3)(t_n - t_0)^3 + \frac{20}{9}(t^3 - t_0^3)^2.$

After simplify this equation we obtain $\lim_{n\to\infty} E|X_n - X|^2 = 0$ i.e,

$$l.i.m_{n\to\infty} X_n = X.$$

□

Example 4.2. *Consider the linear random differential equation*

$$\dot{X}(t) = At + B, \quad X(t_0) = D, \quad t \in [t_0, t_f]$$

where $\{X(t), \quad t \in T\}$ *is 2-order stochastic process,* B *is arbitrary constant,* A *and* D *are independent random variables with joint PDF*

$$f_{A,D}(a,d) = \frac{e^{-4} 2^{(A+D)}}{A!\ D!}, \qquad A, D = 0, 1, 2, \ldots.$$

<u>The exact solution</u>

since $\dot{X}(t) = At + B, \qquad X(t_0) = D$, then the exact solution is $X(t) = D + B(t - t_0) + \frac{A}{2}(t^2 - t_0^2)$.

<u>The numerical solution</u>

by using the forward finite difference method

$X_n = X_{n-1} + hf(X_{n-1}, t_{n-1})$ where $X(t_0) = D, \qquad f(X,t) = At + B$.

at $n = 1$

$X_1 = X_0 + hf(X_0, t_0) = D + h(At_0 + B) = D + hB + Aht_0$

at $n = 2$

$X_2 = X_1 + hf(X_1, t_1) = D + 2hB + Aht_0 + Aht_1$

Since $t_n = t_0 + (n-1)h \quad \Longrightarrow t_1 = t_0 + h$, then

$X_2 = D + 2hB + Aht_0 + Ah(t_0 + h)$

at $n = 3$

$X_3 = X_2 + hf(X_2, t_2) = D + 2hB + Aht_0 + Ah(t_0 + h) + h(At_2 + B) = D + 3hB + 3Aht_0 + 3Ah^2$

at $n = 4$

$X_4 = X_3 + hf(X_3, t_3) = D + 3hB + 3Aht_0 + 3Ah^2 + h(At_3 + B) = D + 4hB + 4Aht_0 + 6Ah^2$

and so on.... Then the general numerical solution is:

$$X_n = D + nhB + Ah\sum_{i=0}^{n-1}(t_0 + ih)$$

4.2. Case study: nonlinear random differential equations.

Example 4.3. *Consider the nonlinear random differential equation*

$$\dot{X}(t) = X^2 - A^2 X, \qquad X(0) = A^2, \qquad t \in [t_0, t_f]$$

where $\{X(t), \quad t \in T\}$ *is 2-order stochastic process, and* A *is a random variable has the exponential distribution* $(A \sim \exp(2))$.

The exact solution

We can write the equation in the form Bernoulli equation:

$$\dot{X}(t) + A^2 X = X^2, \qquad X(0) = A^2 \tag{21}$$

Then the exact solution of the equation 21 is $X(t) = A^2$.

The numerical solution

$$X_n = X_{n-1} + hf(X_{n-1}, t_{n-1}), \qquad X(0) = A^2$$

at $n = 1$
$X_1 = X_0 + hf(X_0, t_0) = A^2 + h(A^4 - A^2 \cdot A^2) = A^2$
at $n = 2$
$X_2 = X_1 + hf(X_1, t_1) = A^2 + h(X_1^2 - A^2 X_1) = A^2$
at $n = 3$
$X_3 = X_2 + hf(X_2, t_2) = A^2 + h(X_2^2 - A^2 X_2) = A^2$
And so on .... Then the general numerical solution is:

$$X_n = A^2.$$

5. Conclusions

In this paper, we have presented the proof of the mean square convergence of the random Finite Difference scheme. Moreover we take advantage of the random Finite Difference scheme for computing directly the main statistical properties like mean, variance and probability density function of the mean square approximations.

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