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CUBIC SPLINE SOLUTION OF FRACTIONAL BAGLEY-TORVIK EQUATION

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ABSTRACT. Fractional calculus is a natural extension of the integer order calculus and recently, a large number of applied problems have been formulated on fractional differential equations. Analytical solution of many applications, where the fractional differential equations appear, cannot be established. Therefore, cubic polynomial spline function is considered to find approximate solution for fractional boundary value problems (FBPs). Convergence analysis of the method is considered. Some illustrative examples are presented and the obtained results reveal that the proposed technique is very effective, convenient and quite accurate to such considered problems.

1. INTRODUCTION

In the last few decades, many phenomena in science and engineering are described within the framework of the theory of fractional differential equations. Boundary value problems of fractional order occur in the description of many physical processes of stochastic transport, the investigation of liquid filtration in a strongly porous medium, cellular systems, diffusion wave, control theory, signal processing and oil industries, [9]. In particular, the 1/2-order derivative or 3/2-order derivative describe the frequency-dependent damping materials quite satisfactorily, and the Bagley-Torvik equation with 1/2-order derivative or 3/2-order derivative describes motion of real physical systems, an immersed plate in a Newtonian fluid and a gas in a fluid, respectively. For details we may refer to ([1]-[5],[15],[17]-[18],[23]-[26]). It is these successful applications of fractional-order derivatives that draw the researchers attention to fractional calculus such as in [33], the authors considered the numerical solution of the fractional boundary value problem (FBVP) $D^{-\alpha}y''(x)+p(x)y=g(x)$, $0\leq\alpha<1$, $x\in[a,b]$, with Dirichlet boundary conditions using quadratic polynomial spline, also in [34] the authors used cubic polynomial spline function based method in combined with shooting method to find approximate solution of second order FBVP with Dirichlet boundary conditions, (see [6]-[8],[12]-[13],[15]-[16],[19]-[22],[27]-[29]).

Key words and phrases. Cubic polynomial spline, Fractional boundary value problem, Bagley-Torvik equation, Dirichlet boundary conditions, Error bound.
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In this paper, we consider the numerical solution of the following Bagley-Torvik equation:

$$y^{''}(x) + (\eta D^{\alpha} + \mu)y = f(x), \quad m-1 \leq \alpha < m, \ x \in [a,b] \tag{1}$$

Subject to boundary conditions:

$$y(a) - A_1 = y(b) - A_2 = 0 \tag{2}$$

where $A_i (i = 1, 2), \mu, \eta$ are real constants and $m = 1$ or 2 . The function $f(x)$ is continuous on the interval $[a, b]$ and the operator D^{α} represents the Caputo fractional derivative. When $\alpha = 0$, Equation (1) is reduced to the classical second order boundary value problem.

The main objective of this work is to use cubic polynomial spline function to establish a new numerical method for the FBVP (1-2). This approach has its own advantage that it not only provides continuous approximations to $y(x)$, but also $y^{(j)}(x), j = 1, 2$ for at every point of the range of integration ([25],[30]-[32]).

This paper is organized as follows: In section 2, we introduce some definitions and theorem necessary to our work. Derivation of our method is established in section 3. Convergence analysis of the new method is presented in section 4. In section 5, we apply our method to singular boundary value problem of fractional order. In section 6, numerical results are included to show the applications and advantages of our method.

2. Preliminaries

In this section, definitions of fractional derivative and integral, used in our work, will be presented. There are different definitions for fractional derivatives, the most commonly used ones are the Riemann-Liouville and the Caputo derivatives. Let $f(x)$ be a function defined on (a, b), then

Definition 1 [16] The Riemann-Liouville fractional derivative

$$^{R}D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int\limits_0^x (x-t)^{m-\alpha-1} f(t) dt, \ \alpha > 0, m-1 < \alpha < m$$

where Γ is the gamma function.

Definition 2 [16]The Riemann-Liouville fractional integral

$$D_a^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0$$

Definition 3 [10] The Caputo fractional derivative

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int\limits_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) ds, \ \alpha > 0, m-1 < \alpha < m$$

The relation between the Riemann–Liouville operator and Caputo operator is given by

$$D^{\alpha} f(x) =^{R} D^{\alpha}[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-a)^k f^{(k)}(a)], \ \alpha > 0, m-1 < \alpha < m$$

Definition 4 [16] The Grünwald definition for fractional derivative is:

$$^{G}D^{\alpha}y(x)=\lim_{N\to\infty}\frac{1}{h^{\alpha}}\sum_{k=0}^{N}g_{\alpha,k}y(x-kh) \tag{3}$$

where the Grünwald weights are:

$$g_{\alpha,k}=\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \tag{4}$$

3. Consistency relations

In order to develop the spline approximation for the fractional differential equation (1-2) we introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into equal $n-$ parts.

$$x_i\ =a+ih,\quad x_0=a,\ x_n=b,\ h=\frac{b-a}{n},\quad i=0,1,2,...,n \tag{5}$$

Let $y(x)$ be the exact solution of (1) and S_i be an approximation to $y_i=y(x_i)$ obtained by the spline function $Q_i(x)$ passing through the points $(x_i,S_i\ )$ and $(x_{i+1},S_{i+1}\ )$ then in each subinterval the cubic polynomial spline segment $Q_i(x)$ has the form

$$Q_i(x)=a_i\ (x-x_i)^3+b_i\ (x-x_i)^2+c_i\ (x-x_i)+d_i,\quad i=0,1,2,...,n-1 \tag{6}$$

where a_i,b_i,c_i and d_i are constants to be determined. Following ([30]-[33]), we get the following recurrence relations

$$S_{i+1}\ -2S_i+S_{i-1}=\frac{h^2}{6}(M_{i+1}\ +4M_i+M_{i-1}),\qquad i=0,1,2,...,n-1 \tag{7}$$

where

$$M_i=\ f_i-\mu S_i-\eta D^{\alpha}S(x)|_{x=x_i}\ ,\quad i=0,1,2,...,n \tag{8}$$

where $f_i=f(x_i)$

Lemma 1 Let $y\in C^6[a,b]$ then the local truncation errors $t_i, i=1,2,...,n-1$ associated with the scheme (8) is:

$$t_i=\frac{-1}{12}h^4y_i^{(4)}+O(h^6),\quad i=0,1,2,...,n-1 \tag{9}$$

Proof: see [33].

In order to obtain a complete numerical solution for Eq.(1-2), we use the Grünwald definition (3-4) of the fractional derivative for discretizing the fractional term $D^{\alpha}S(x)|_{x=x_i}$ $i=0,1,2,...,n$ mentioned in Eq.(8).

Note that the normalized weights (4) depend only on the fractional order α and the index k . We have that:

$$g_{\alpha,0}=1,\ g_{\alpha,1}=-\alpha\ \text{ and } g_{\alpha,k}=\frac{(-\alpha)(-\alpha+1)...(-\alpha+k-1)}{k!},\forall k\geq 2. \tag{10}$$

It is well known that

$$(1+z)^p=\sum_{k=0}^{\infty}\binom{p}{k}z^k,\qquad \forall\,|z|\leq 1,\quad p>0 \tag{11}$$

where

$$\binom{p}{k}=\frac{(-1)^k\Gamma(k-p)}{\Gamma(-p)\Gamma(k+1)} \tag{12}$$

Then for $z=-1$, we have

$$\sum_{k=0}^{\infty}\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}=0 \tag{13}$$

Then from the above we can approximate the fractional term $D^{\alpha}S(x)|_{x=x_i}$ $i=0,1,2,...,n$ by:

$$D^{\alpha}S(x)|_{x=x_i}\cong\frac{1}{h^{\alpha}}\sum_{k=0}^{i}g_{\alpha,k}\ S(x_i-kh),\quad i=0,1,2,...,n \tag{14}$$

4. Convergence analysis

Let $Y=(y_i)$, $S=(s_i)$, $C=(c_i)$, $T=(t_i)$ and $E=(e_i)=Y-S$ be $(n-1)$-dimensional column vectors. Then, we can write the system given by (10) as follows:

$$NS=h^2BM+C \tag{15}$$

where the matrices N,B and the vector C are given below

$$N=\begin{bmatrix} -3 & 1 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -3 \end{bmatrix}$$

$$B=\begin{bmatrix} 4 & 1 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 1 & 4 \end{bmatrix},\quad C=\begin{bmatrix} -A_1+\frac{h^2}{6}M_0 \\ \cdot \\ \cdot \\ \cdot \\ -A_2+\frac{h^2}{6}M_n \end{bmatrix}$$

From Eqs.(11) and (3.14), the vector M can be written as:

$$M=F-\mu S-\eta h^{-\alpha}(GS+G_0) \tag{16}$$

Where the vectors F,G_0 and the matrix G are given below respectively:

$$F=(\ f_1 \quad f_2 \quad . \quad . \quad . \quad f_{n-2} \quad f_{n-1}\ )^t \tag{17}$$

$$G_0=A_1(\ g_{\alpha,1} \quad g_{\alpha,2} \quad . \quad . \quad . \quad g_{\alpha,n-2} \quad g_{\alpha,n-1}\ )^t \tag{18}$$

and $G = \begin{bmatrix} g_{\alpha,0} & & & & & & \\ g_{\alpha,1} & g_{\alpha,0} & & & & & \\ g_{\alpha,2} & g_{\alpha,1} & g_{\alpha,0} & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ g_{\alpha,n-3} & g_{\alpha,n-4} & & & g_{\alpha,1} & g_{\alpha,0} & \\ g_{\alpha,n-2} & g_{\alpha,n-3} & & & g_{\alpha,2} & g_{\alpha,1} & g_{\alpha,0} \end{bmatrix}$

where $g_{\alpha,i},\ \ i = 0, 1, 2, ..., n$ are the Grünwald weights.
Substituting from Eq.(16) into Eq.(15) we get:

$$(N + \mu h^2 B + \eta h^{2-\alpha} BG)S = h^2 B(F - \eta h^{-\alpha} G_0) + C$$

and

$$(N + \mu h^2 B + \eta h^{2-\alpha} BG)Y = h^2 B(F - \eta h^{-\alpha} G_0) + C + T$$

Then the error equation can be written as

$$(N + \mu h^2 B + \eta h^{2-\alpha} BG)E = T \tag{19}$$

In the following we need the following lemma.
Lemma 2 [14] If M is square matrix of order n and $\|M\| < 1$, then $(1 + M)^{-1}$ exists and $\left\|(1 + M)^{-1}\right\| < 1/(1 - \|M\|)$.
Lemma 3 The matrix $(N + \mu h^2 B + \eta h^{2-\alpha} BG)$ given by (19) is nonsingular, provided that

$$\psi(\mu + 2\eta m h^{-\alpha}) < 1 \tag{20}$$

Proof: Rewrite the error equation (19) , we get

$$E = (I + \mu h^2 N^{-1} B + \eta h^{2-\alpha} N^{-1} BG)^{-1} N^{-1} T$$

Using Lemma 2, we have

$$\|E\| \leq \frac{\left\|N^{-1}\right\| \|T\|}{1 - \mu h^2 \left\|N^{-1}\right\| \|B\| - \eta h^{2-\alpha} \left\|N^{-1}\right\| \|B\| \|G\|} \tag{21}$$

Provided that

$$\mu h^2 \left\|N^{-1}\right\| \|B\| - \eta h^{2-\alpha} \left\|N^{-1}\right\| \|B\| \|G\| < 1 \tag{22}$$

It was shown that [14]

$$\left\|N^{-1}\right\| = \frac{h^{-2}}{8}((b-a)^2 + h^2) = \psi h^{-2},\ \ \psi = \frac{((b-a)^2 + h^2)}{8}. \tag{23}$$

$$\|B\| = 1, \tag{24}$$

We have that $\|G\| = \sum_{i=0}^{n-2} |g_{\alpha,i}|$
From Eq.(21) we can conclude that

(1) When $0 < \alpha < 1$, we have $g_{\alpha,0} = 1$ and $g_{\alpha,i} < 0\ \forall i$ and $i \neq 0$. Then, we get that $\displaystyle\sum_{k=1}^{\infty} \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\,\Gamma(k+1)} = -1$ which leads to $\|G\| \leq 2$.
(2) When $1 < \alpha < 2$, we have $g_{\alpha,1} = -\alpha$ and $g_{\alpha,i} > 0\ \forall i$ and $i \neq 1$. Then, we get that $\displaystyle\sum_{\substack{k=0\\k\neq 1}}^{\infty} \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\,\Gamma(k+1)} = \alpha$ which leads to $\|G\| \leq 2\alpha$.

Then from the above we can finish to

$$\|G\| \leq 2m, \qquad \forall\ (m-1) < \alpha < m. \tag{25}$$

From equation (12)we have

$$\|T\| = \frac{1}{12} h^4 M_4 \tag{26}$$

where

$$M_4 = \max_{a \leq x \leq b} \left| y^{(4)}(x) \right| \tag{27}$$

Substituting from Eqns. (23-26) into Eq. (22) completes the proof of the lemma. As a consequence of Lemma 3, the discrete boundary value problem Eq.(15) has a unique solution if $\psi(\mu + 2\eta h^{-\alpha}) < 1$.
Then

$$\|E\| \leq \frac{\left\|N^{-1}\right\| \|T\|}{1 - \mu h^2 \left\|N^{-1}\right\| \|B\| - \eta h^{2-\alpha} \left\|N^{-1}\right\| \|B\| \|G\|} \cong O(h^2). \tag{28}$$

As a result of the above lemma we can write the following theorem
Theorem 3 Let $y(x)$be the exact solution of the continuous boundary value problem (1-2) and let $y(x_i), i = 1, 2, ..., n-1$, satisfy the discrete boundary value problem (15). Further, if$e_i = y(x_i) - S_i$, then $\|E\| \cong\ O\left(h^2\right)$ second order convergent method, which is given by Eq. (28), neglecting all errors due to round off.

5. Singular Boundary value problem of fractional order

The approach introduced in section 3 can also be used to find the solution of the following singular fractional differential equation

$$\varepsilon y''(x) + \eta D^{\alpha} y + \mu y = f(x),\ \ \varepsilon << 1, 0 \leq \alpha < 1,\ \ x \in [a, b], \tag{29}$$

with boundary conditions given by Eq.(2)
The consistency relations (10) hold but in this case we have

$$M_i = \frac{1}{\varepsilon}(f_i - \mu S_i - \eta D^{\alpha}\ S(x)|_{\ x = x_i})\ ,\ i = 0, 1, 2, ..., n, \tag{30}$$

6. Numerical examples

We will consider some numerical examples demonstrating the solution using cubic spline methods illustrated above. All calculations are implemented with MATLAB 7.

Example 6.1

Consider the fractional boundary value problem:

$$y''(x) + 0.5 D^{\alpha} y(x) + y(x) = 3 + x^2 \left(\frac{x^{-\alpha}}{\Gamma(3-\alpha)} + 1\right), \tag{31}$$

$$y(0) = 1\ \ , y(1) = 2. \tag{32}$$

The exact solution of Eq.(31) is

$$y(x) = x^2 + 1. \tag{33}$$

The numerical solutions using cubic spline are represented in Table 6.1 in case of $n = 8$ and $\alpha = 0.5$, while the error and the order of convergence for various values of $\alpha = 0$,$\alpha = 0.3$ and $\alpha = 0.5$ are represented in Table 6.2.

Table 6.1: exact, approximate and absolute error

x	*Exact Solution*	*Approximate Solution*	*Error*
0	1	1	0
0.125	1.015625	1.020078	$4.19E-3$
0.250	1.062500	1.065554	$2.52E-3$
0.375	1.140630	1.141389	$4.90E-5$
0.500	1.250000	1.247476	$3.59E-3$
0.625	1.390630	1.383746	$8.16E-3$
0.750	1.562500	1.550150	$1.37E-2$
0.875	1.765630	1.750225	$1.68E-2$
1	2	2	0

Table 6.2: maximum absolute error and order of convergence (O.C.)

h	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.5$	
	Error	*O.C.*	*Error*	*O.C.*	*Error*	*O.C.*
1/8	$7.39E-3$		$8.60E-3$		$1.68E-2$	
1/16	$2.09E-3$	1.82	$2.25E-3$	1.93	$4.35E-3$	1.95
1/32	$5.61E-4$	1.90	$6.27E-4$	1.84	$1.18E-3$	1.88
1/46	$1.47E-4$	1.93	$1.68E-4$	1.90	$3.27E-4$	1.85
1/128	$3.85E-5$	1.93	$4.42E-5$	1.92	$9.13E-5$	1.84

Example 6.2
Consider the boundary value problem:

$$y''(x) + \eta D^{\alpha} y(x) + \mu y(x) = f(x), \tag{34}$$

$$y(0) = y(1) = 0. \tag{35}$$

where $f(x) = 4x^2(5x-3) + \eta x^{4-\alpha}(\frac{120}{\Gamma(6-\alpha)}x - \frac{24}{\Gamma(5-\alpha)}) + \mu x^4(x-1)$.
The exact solution of Eq.(34) is $y(x) = x^4(x-1)$.
The numerical solution for$\eta = 0.5$, $\mu = 1$, n=8 and $\alpha = 0.3$ is represented in Table 6.3. Also, the error and the order of convergence for various values of $\alpha = 0$,$\alpha = 0.3$ and $\alpha = 0.5$ are represented in Table 6.4.

Table 6.3: exact, approximate and absolute error

x	*Exact Solution*	*Approximate Solution*	*Error*
0	0	0	0
0.125	-0.0002140	-0.00221	$2.00E-3$
0.250	-0.0029297	-0.00701	$4.08E-3$
0.375	-0.0123596	-0.01819	$5.83E-3$
0.500	-0.0312500	-0.03810	$6.85E-3$
0.625	-0.0572200	-0.06403	$6.81E-3$
0.750	-0.0791000	-0.08467	$5.57E-3$
0.875	-0.0732730	-0.07654	$3.26E-3$
1	0	0	0

Table 6.4:maximum absolute error and order of convergence

h	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.5$	
	$Error$	$O.C.$	$Error$	$O.C.$	$Error$	$O.C.$
$1/8$	$7.33E-3$		$6.85E-3$		$6.39E-3$	
$1/16$	$2.09E-3$	1.81	$1.93E-3$	1.83	$1.73E-3$	1.89
$1/32$	$5.34E-4$	1.97	$5.38E-4$	1.84	$4.95E-4$	1.81
$1/46$	$1.44E-4$	1.89	$1.52E-4$	1.82	$1.37E-4$	1.85
$1/128$	$4.02E-5$	1.84	$4.23E-5$	1.83	$3.69E-5$	1.89

Example 6.3

Consider the singular fractional boundary value problem [26]

$$\varepsilon y''(x) + D^{\alpha} y = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}, \ \varepsilon << 1, \quad 1 \le \alpha < 2, \quad x \in [a,b], \tag{36}$$

$$y(0) = y(1) = 0 \tag{37}$$

In [26], the authors used the Laplace transform properties and they deduced that the exact solution of (36) is given by:

$$y(x) = C_1 x^{\alpha-1} E_{\alpha-1,\alpha}(-x^{\alpha-1}/\varepsilon) - \frac{x^{\alpha+1}}{\varepsilon} E_{\alpha-1,\alpha+2}(-x^{\alpha-1}/\varepsilon), \ C_1 = \frac{E_{\alpha-1,\alpha+2}(-1/\varepsilon)}{\varepsilon E_{\alpha-1,\alpha}(-1/\varepsilon)}$$

The numerical solution for $\alpha = 1.25, h = \frac{1}{16}$ and $\varepsilon = 10^{-1}$ is represented in Fig.6.1. This example had been solved also for $\alpha = 1.25$ and $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-5}$ and the results are represented in Fig. 6.2. Also, exact solution and approximate solution for various values of step size $h = \frac{1}{8}, \frac{1}{16}$ and $h = \frac{1}{32}$ are presented in Fig.6.3. and the error corresponding to each approximate solution in Fig.6.4.

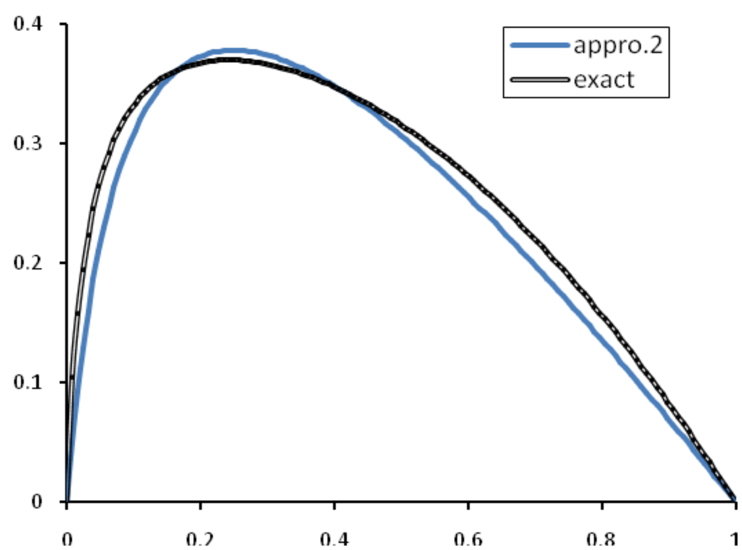

Fig. 6.1, the numerical solution of example 6.3 for $\alpha = 1.25$ and $\varepsilon = 10^{-1}$.

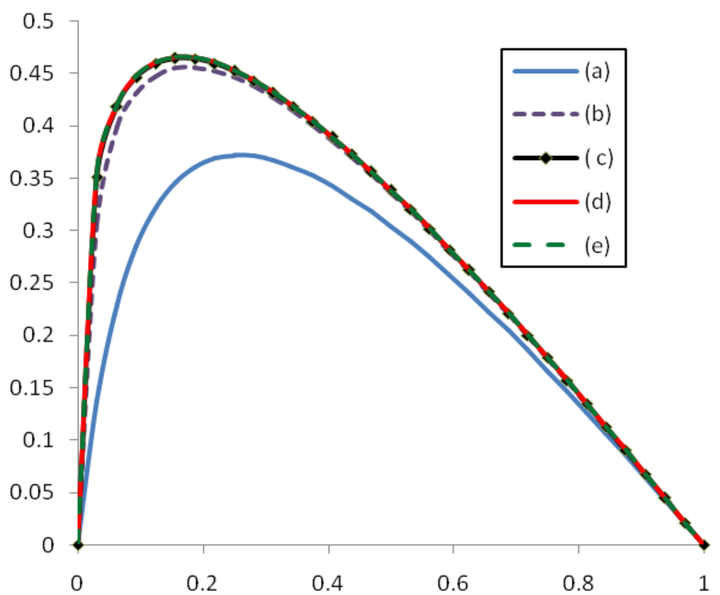

Fig. 6.2, the numerical solution of Example 6.3 for $\alpha = 1.25$ and
(a) $\varepsilon = 10^{-1}$, (b) $\varepsilon = 10^{-2}$, (c) $\varepsilon = 10^{-3}$,(d) $\varepsilon = 10^{-4}$,(e) $\varepsilon = 10^{-5}$

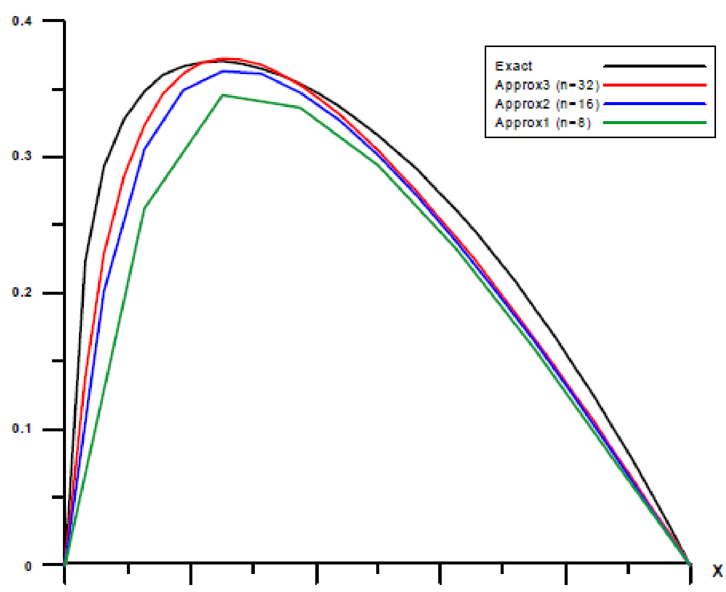

Fig. 6.3, Exact and approximate solutions of Example 6.3 with variable step size.

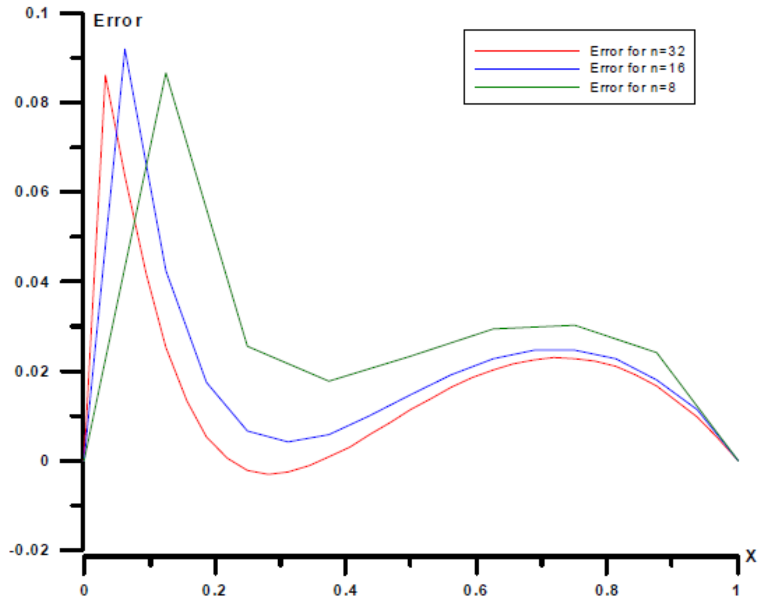

Fig. 6.4, The error corresponding to each approximate solution. .

6. Conclusion

In this paper, we used cubic polynomial spline based method to present an approximate solution for a class of boundary value problem of fractional order subject to Dirichlet boundary conditions. Our approach depends on approximating the fractional term using the Grünwald definition of the fractional derivative. This approach was used also to find the solution of singular fractional differential equation. Convergence analysis of the method was presented. Some numerical examples were included to illustrate the practical usefulness of the proposed methods.

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