Electronic Journal of Mathematical Analysis and Applications
Vol. 1(2) July 2013, pp. 273-284.
ISSN: 2090-792X (online)
http://ejmaa.6te.net/

SUBCLASSES OF CLOSE-TO-CONVEX AND QUASI-CONVEX FUNCTIONS WITH RESPECT TO OTHER POINTS

HUO TANG, GUAN-TIE DENG

ABSTRACT. In this paper, we introduce new subclasses of close-to-convex and quasi-convex functions with respect to symmetric and conjugate points. The coefficient estimates for functions belonging to these classes are obtained.

1. INTRODUCTION

Let U be the class of functions which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$ given by

$$\omega(z) = \sum_{k=1}^{\infty} c_k z^k \tag{1.1}$$

and satisfying the conditions $\omega(0) = 0,\ |\omega(z)| \leq 1,\ z \in E$.

Let S denote the class of functions f which are analytic and univalent in E of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in E. \tag{1.2}$$

Let S_s^* be the subclass of functions $f(z) \in S$ and satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0,\ z \in E.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [11].

Also, let S_c^* be the subclass of functions $f(z) \in S$ and satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0,\ z \in E.$$

These functions are called starlike with respect to conjugate points and were introduced by El-Ashwah and Thomas [3]. Further results on starlike functions with respect to symmetric points or conjugate points can be found in [13-15].

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions, close-to-convex, quasi-convex, symmetric, conjugate, coefficient estimate.

Submitted Mar. 3, 2013.

Then, Das and Singh [2] introduced another class C_s, namely convex functions with respect to symmetric points and satisfying the condition

$$Re\left(\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0,\ z \in E.$$

Suppose that f and g are two analytic functions in E. Then, we say that the function g is subordinate to the function f, and we write $g(z) \prec f(z),\ z \in E$, if there exists a Schwarz function $\varpi(z)$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ such that $g(z) = f(\varpi(z)),\ z \in E$.

In view of subordination definition, Goel and Mehrok [4] introduced a subclass of S_s^* denoted by $S_s^*(A, B)$.

Let $S_s^*(A, B)$ be the class of functions of the form (1.2) and satisfying the condition

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in E.$$

Following them, many authors introduced the analogue definitions by extension as follows (see [1,7]).

Definition 1.1. (i) Let $S_c^*(A, B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in E.$$

(ii) Let $C_s(A, B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in E.$$

(iii) Let $C_c(A, B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)+\overline{f(\overline{z})})'} \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in E.$$

Motivated by the pervious classes, Tang and Deng [5] recently introduced the following classes of functions with respect to symmetric and conjugate points.

Definition 1.2. (i) Let $M_s(\alpha, \mu, A, B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2\alpha\mu z^3 f'''(z) + 2(2\alpha\mu+\alpha-\mu)z^2 f''(z) + 2zf'(z)}{\alpha\mu z^2(f(z)-f(-z))'' + (\alpha-\mu)z(f(z)-f(-z))' + (1-\alpha+\mu)(f(z)-f(-z))} \prec \frac{1+Az}{1+Bz},$$

where $-1 \le B < A \le 1,\ 0 \le \mu \le \alpha \le 1$, and $z \in E$.

(ii) Let $M_c(\alpha, \mu, A, B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2\alpha\mu z^3 f'''(z) + 2(2\alpha\mu+\alpha-\mu)z^2 f''(z) + 2zf'(z)}{\alpha\mu z^2(f(z)+\overline{f(\overline{z})})'' + (\alpha-\mu)z(f(z)+\overline{f(\overline{z})})' + (1-\alpha+\mu)(f(z)+\overline{f(\overline{z})})} \prec \frac{1+Az}{1+Bz},$$

where $-1 \le B < A \le 1,\ 0 \le \mu \le \alpha \le 1$, and $z \in E$.

As a special case, when $\mu = 0$, we obtain

$$M_s(\alpha, 0, A, B) = M_s(\alpha, A, B)\ and\ M_c(\alpha, 0, A, B) = M_c(\alpha, A, B),$$

introduced and studied by Selvaraj and Vasanthi [12].
In this paper, we introduce the class $K_s^*(\alpha,\mu,A,B;C,D)$ consisting of analytic functions f of the form (1.2) and satisfying

$$\frac{2\alpha\mu z^3 f'''(z)+2(2\alpha\mu+\alpha-\mu)z^2f''(z)+2zf'(z)}{\alpha\mu z^2(g(z)-g(-z))''+(\alpha-\mu)z(g(z)-g(-z))'+(1-\alpha+\mu)(g(z)-g(-z))}\prec\frac{1+Cz}{1+Dz},$$

where $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in M_s(\alpha,\mu,A,B),\ -1\le D\le B<A\le C\le 1,\ 0\le\mu\le\alpha\le 1$, and $z\in E$.
Also, we introduce the class $K_c^*(\alpha,\mu,A,B;C,D)$ consisting of analytic functions f of the form (1.2) and satisfying

$$\frac{2\alpha\mu z^3 f'''(z)+2(2\alpha\mu+\alpha-\mu)z^2f''(z)+2zf'(z)}{\alpha\mu z^2(g(z)+\overline{g(\bar z)})''+(\alpha-\mu)z(g(z)+\overline{g(\bar z)})'+(1-\alpha+\mu)(g(z)+\overline{g(\bar z)})}\prec\frac{1+Cz}{1+Dz},$$

where $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in M_c(\alpha,\mu,A,B),\ -1\le D\le B<A\le C\le 1,\ 0\le\mu\le\alpha\le 1$, and $z\in E$.
We note that
(i) for $\mu=0$, $K_s^*(\alpha,0,A,B;C,D)=K_s^*(\alpha,A,B;C,D)$ and $K_c^*(\alpha,0,A,B;C,D)=K_c^*(\alpha,A,B;C,D)$ (see Tang and Deng [6])
(ii) for $\alpha=\mu=0$, $K_s^*(0,0,A,B;C,D)=K_s(A,B;C,D)$ (see Mehrok et al.[10]) and $K_c^*(0,0,A,B;C,D)=K_c(A,B;C,D)$
(iii) for $\alpha=\mu=0,\ C=1$ and $D=-1$, $K_s^*(0,0,A,B;1,-1)=K_s(A,B)$ (see Janteng and Halim [8]) and $K_c^*(0,0,A,B;1,-1)=K_c(A,B)$
(iv) for $\alpha=\mu=0,\ A=C=1$ and $B=D=-1$, $K_s^*(0,0,1,-1;1,-1)\equiv K_s$ and $K_c^*(0,0,1,-1;1,-1)\equiv K_c$
(v) for $\alpha=1$ and $\mu=0$, $K_s^*(1,0,A,B;C,D)=K_s^*(A,B;C,D)$ and $K_c^*(1,0,A,B;C,D)=K_c^*(A,B;C,D)$
(vi) for $\alpha=1,\ \mu=0,\ C=1$ and $D=-1$, $K_s^*(1,0,A,B;1,-1)=K_s^*(A,B)$ (see Janteng and Halim [9]) and $K_c^*(1,0,A,B;1,-1)=K_c^*(A,B)$
(vii) for $\alpha=1,\ \mu=0,\ A=C=1$ and $B=D=-1$, $K_s^*(1,0,1,-1;1,-1)\equiv K_s^*$ and $K_c^*(1,0,1,-1;1,-1)\equiv K_c^*$.
By the definition of subordination, it follows that $f\in K_s^*(\alpha,\mu,A,B;C,D)$ if and only if

$$\frac{2\alpha\mu z^3 f'''(z)+2(2\alpha\mu+\alpha-\mu)z^2f''(z)+2zf'(z)}{\alpha\mu z^2(g(z)-g(-z))''+(\alpha-\mu)z(g(z)-g(-z))'+(1-\alpha+\mu)(g(z)-g(-z))}=\frac{1+C\omega(z)}{1+D\omega(z)}=P(z),\ \omega(z)\in U, \tag{1.3}$$

and that $f\in K_c^*(\alpha,\mu,A,B;C,D)$ if and only if

$$\frac{2\alpha\mu z^3 f'''(z)+2(2\alpha\mu+\alpha-\mu)z^2f''(z)+2zf'(z)}{\alpha\mu z^2(g(z)+\overline{g(\bar z)})''+(\alpha-\mu)z(g(z)+\overline{g(\bar z)})'+(1-\alpha+\mu)(g(z)+\overline{g(\bar z)})}=\frac{1+C\omega(z)}{1+D\omega(z)}=P(z),\ \omega(z)\in U, \tag{1.4}$$

where

$$P(z)=1+\sum_{n=1}^{\infty}p_nz^n. \tag{1.5}$$

In the next section, we discuss the coefficient estimates for functions belonging to the classes $K_s^*(\alpha,\mu,A,B;C,D)$ and $K_c^*(\alpha,\mu,A,B;C,D)$.

2. Some preliminary lemmas

We shall require the following lemmas for proving our main results.

Lemma 2.1 (see [4]). If $P(z)$ is given by (1.3), (1.4) and (1.5), then for $-1 \leq D < C \leq 1$,

$$|p_n| \leq (C-D),\ n = 1, 2, \cdots.$$

Lemma 2.2 (see [5]). Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_s(\alpha, \mu, A, B)$. Then for $n \geq 1$, $0 \leq \mu \leq \alpha \leq 1$,

$$|b_{2n}| \leq \frac{(A-B)}{2^n \cdot n![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{n-1}(A-B+2j),$$

$$|b_{2n+1}| \leq \frac{(A-B)}{2^n \cdot n![1+2n(\alpha-\mu+(2n+1)\alpha\mu)]} \prod_{j=1}^{n-1}(A-B+2j).$$

Lemma 2.3 (see [5]). Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_c(\alpha, \mu, A, B)$. Then for $n \geq 1$, $0 \leq \mu \leq \alpha \leq 1$,

$$|b_{2n}| \leq \frac{(A-B)}{(2n-1)![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{2n-2}(A-B+j),$$

$$|b_{2n+1}| \leq \frac{(A-B)}{(2n)![1+2n(\alpha-\mu+(2n+1)\alpha\mu)]} \prod_{j=1}^{2n-1}(A-B+j).$$

3. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $-1 \leq D \leq B < A \leq C \leq 1$, $0 \leq \mu \leq \alpha \leq 1$, and $z \in E$.

Theorem 3.1. Let $f \in K_s^*(\alpha, \mu, A, B; C, D)$, then for $n \geq 1$,

$$|a_{2n}| \leq \frac{(C-D)}{2^n \cdot n![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{n-1}(A-B+2j), \tag{3.1}$$

$$|a_{2n+1}| \leq \frac{1}{(2n+1)[1+2n(\alpha-\mu+(2n+1)\alpha\mu)]}$$
$$\times \left\{ \left[(C-D) + \frac{(A-B)}{2n}\right] \left[\frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1}(A-B+2j)\right] \right\}. \tag{3.2}$$

Proof. Since $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_s(\alpha, \mu, A, B)$, it follows that

$$2\alpha\mu z^3 g'''(z) + 2(2\alpha\mu + \alpha - \mu)z^2 g''(z) + 2zg'(z)$$
$$= [\alpha\mu z^2(g(z)-g(-z))'' + (\alpha-\mu)z(g(z)-g(-z))' + (1-\alpha+\mu)(g(z)-g(-z))]K(z) \tag{3.3}$$

for $z \in E$, with $Re(K(z)) > 0$, where $K(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$.
On equating the coefficients of like powers of z in (3.3), we get

$$2[1+(\alpha-\mu)+2\alpha\mu]b_2 = d_1,\ 2[1+2(\alpha-\mu)+6\alpha\mu]b_3 = d_2,$$

$$4[1+3(\alpha-\mu)+12\alpha\mu]b_4 = d_3 + [1+2(\alpha-\mu)+6\alpha\mu]b_3 d_1, \tag{3.4}$$

$$4[1+4(\alpha-\mu)+20\alpha\mu]b_5 = d_4 + [1+2(\alpha-\mu)+6\alpha\mu]b_3 d_2, \tag{3.5}$$

and continuing in this way, we obtain

$$2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]b_{2n} = d_{2n-1}+[1+2(\alpha-\mu)+6\alpha\mu]b_3d_{2n-3} + \cdots + [1+2(n-1)(\alpha-\mu)+2(2n-1)\alpha\mu]b_{2n-1}d_1, \tag{3.6}$$

$$2n[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]b_{2n+1} = d_{2n}+[1+2(\alpha-\mu)+6\alpha\mu]b_3d_{2n-2} + \cdots + [1+2(n-1)(\alpha-\mu)+2(2n-1)\alpha\mu]b_{2n-1}d_2. \tag{3.7}$$

From (1.3) and (1.5), we have

$$\left[z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots\right]+(2\alpha\mu+\alpha-\mu)[2a_2z^2+6a_3z^3+12a_4z^4 +20a_5z^5+\cdots+(2n-1)2na_{2n}z^{2n}+\cdots]+\alpha\mu[6a_3z^3+24a_4z^4+60a_5z^5+\cdots +(2n-1)2n(2n+1)a_{2n+1}z^{2n+1}+\cdots] = \Big[(1+\alpha-\mu)[z+b_3z^3+b_5z^5+\cdots+b_{2n-1}z^{2n-1}+b_{2n+1}z^{2n+1} +\cdots]+(\alpha-\mu)[z+3b_3z^3+5b_5z^5+\cdots+(2n-1)b_{2n-1}z^{2n-1}+(2n+1)b_{2n+1}z^{2n+1}+\cdots] +\alpha\mu[6b_3z^3+20b_5z^5+\cdots+2n(2n+1)b_{2n+1}z^{2n+1}+\cdots]\Big]\times[1+p_1z+p_2z^2+p_3z^3+p_4z^4+p_5z^5 +\cdots+p_{2n-1}z^{2n-1}+p_{2n}z^{2n}+\cdots].$$

On equating the coefficients, we obtain

$$2[1+(\alpha-\mu)+2\alpha\mu]a_2 = p_1,\ 3[1+2(\alpha-\mu)+6\alpha\mu]a_3 = p_2+[1+2(\alpha-\mu)+6\alpha\mu]b_3, \tag{3.8}$$

$$4[1+3(\alpha-\mu)+12\alpha\mu]a_4 = p_3+[1+2(\alpha-\mu)+6\alpha\mu]b_3p_1, \tag{3.9}$$

$$5[1+4(\alpha-\mu)+20\alpha\mu]a_5 = p_4+[1+2(\alpha-\mu)+6\alpha\mu]b_3p_2+[1+4(\alpha-\mu)+20\alpha\mu]b_5, \tag{3.10}$$

and so

$$2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]a_{2n} = p_{2n-1}+[1+2(\alpha-\mu)+6\alpha\mu]b_3p_{2n-3} + \cdots + [1+2(n-1)(\alpha-\mu)+2(2n-1)\alpha\mu]b_{2n-1}p_1, \tag{3.11}$$

$$(2n+1)[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]a_{2n+1} = p_{2n}+[1+2(\alpha-\mu)+6\alpha\mu]b_3p_{2n-2} +\cdots+[1+2(n-1)(\alpha-\mu)+2(2n-1)\alpha\mu]b_{2n-1}p_2+[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]b_{2n+1}. \tag{3.12}$$

By using Lemma 2.1 and (3.8), we have

$$|a_2| \le \frac{(C-D)}{2\cdot 1\cdot[1+(\alpha-\mu)+2\alpha\mu]},\ |a_3| \le \frac{(A-B)+2(C-D)}{3\cdot 2\cdot[1+2(\alpha-\mu)+6\alpha\mu]}.$$

Again, by applying Lemma 2.1 and using (3.4) and (3.5), we obtain from (3.9) and (3.10)

$$|a_4| \le \frac{(C-D)(A-B+2)}{4\cdot 2\cdot[1+3(\alpha-\mu)+12\alpha\mu]},\ |a_5| \le \frac{(A-B+2)[(A-B)+4(C-D)]}{5\cdot 8\cdot[1+4(\alpha-\mu)+20\alpha\mu]}.$$

It follows that (3.1) and (3.2) hold for $n=1,2$. We now prove (3.1) and (3.2) by induction.

Equations (3.11) and (3.12), together with Lemma 2.1, yield

$$|a_{2n}| \le \frac{(C-D)}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\left[1+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu)+2(2k+1)\alpha\mu]|b_{2k+1}|\right], \tag{3.13}$$

$$|a_{2n+1}| \le \frac{1}{(2n+1)[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]}\times\Big\{(C-D)$$

$$\times\left[1+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu)+2(2k+1)\alpha\mu]|b_{2k+1}|\right]+[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]|b_{2n+1}|\Bigg\}. \tag{3.14}$$

Again, using Lemma 2.1 in (3.7), we have

$$|b_{2n+1}|\leq\frac{(A-B)}{2n[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]}\left[1+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu)+2(2k+1)\alpha\mu]|b_{2k+1}|\right]. \tag{3.15}$$

Using (3.15) in (3.14), we obtain

$$|a_{2n+1}|\leq\frac{1}{(2n+1)[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]}$$

$$\times\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\times\left[1+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu)+2(2k+1)\alpha\mu]|b_{2k+1}|\right]\right\}. \tag{3.16}$$

We suppose that (3.1) and (3.2) hold for $k=3,4,\cdots,(n-1)$.

Using Lemma 2.2 in (3.13) and (3.16), we get

$$|a_{2n}|\leq\frac{(C-D)}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right], \tag{3.17}$$

$$|a_{2n+1}|\leq\frac{1}{(2n+1)[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]}$$

$$\times\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\right\}. \tag{3.18}$$

In order to prove (3.1), it is sufficient to show that

$$\frac{(C-D)}{2m[1+(2m-1)(\alpha-\mu)+2m(2m-1)\alpha\mu]}\left[1+\sum_{k=1}^{m-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$=\frac{(C-D)}{2^m\cdot m![1+(2m-1)(\alpha-\mu)+2m(2m-1)\alpha\mu]}\prod_{j=1}^{m-1}(A-B+2j)\ (m=3,4,\cdots,n). \tag{3.19}$$

Thus, (3.19) is valid for $m=3$.

Let us assume that (3.19) is true for all $m,\ 3<m\leq(n-1)$. Then from (3.17), we have

$$\frac{(C-D)}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$=\frac{(n-1)}{n}\times\left\{\frac{(C-D)}{2(n-1)[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\right.$$

$$\left.\times\left[1+\sum_{k=1}^{n-2}\frac{(A-B)}{2^k\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\right\}$$

$$+\frac{(C-D)}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\frac{(A-B)}{2^{n-1}\cdot(n-1)!}\prod_{j=1}^{n-2}(A-B+2j)$$

$$=\frac{(n-1)}{n}\times\frac{(C-D)}{2^{n-1}\cdot(n-1)![1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\prod_{j=1}^{n-2}(A-B+2j)$$

$$+\frac{(C-D)}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\frac{(A-B)}{2^{n-1}\cdot(n-1)!}\prod_{j=1}^{n-2}(A-B+2j)$$

$$=\frac{(C-D)}{2^{n}\cdot n![1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\prod_{j=1}^{n-2}(A-B+2j)(A-B+2(n-1))$$

$$=\frac{(C-D)}{2^{n}\cdot n![1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\prod_{j=1}^{n-1}(A-B+2j).$$

Thus, (3.19) holds for $m=n$, and, hence (3.1) follows. Next, we prove (3.2).

From (3.19), we have

$$1+\sum_{k=1}^{n-1}\frac{(A-B)}{2^{k}\cdot k!}\prod_{j=1}^{k-1}(A-B+2j)=\frac{1}{2^{n-1}\cdot(n-1)!}\prod_{j=1}^{n-1}(A-B+2j). \tag{3.20}$$

By using (3.20) in (3.18), we obtain

$$|a_{2n+1}|\leq\frac{1}{(2n+1)[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]}$$
$$\times\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\left[\frac{1}{2^{n-1}\cdot(n-1)!}\prod_{j=1}^{n-1}(A-B+2j)\right]\right\},$$

which proves (3.2).

Theorem 3.2. Let $f\in K_c^*(\alpha,\mu,A,B;C,D)$, then for $n\geq 1$,

$$|a_{2n}|\leq\frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}$$
$$\times\left\{\left[(C-D)+\frac{(A-B)}{2n-1}\right]\left[\frac{1}{(2n-2)!}\prod_{j=1}^{2n-2}(A-B+j)\right]\right\}, \tag{3.21}$$

$$|a_{2n+1}|\leq\frac{1}{(2n+1)[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]}$$
$$\times\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\left[\frac{1}{(2n-1)!}\prod_{j=1}^{2n-1}(A-B+j)\right]\right\}. \tag{3.22}$$

Proof. Since $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in M_c(\alpha,\mu,A,B)$, it follows that

$$2\alpha\mu z^3g'''(z)+2(2\alpha\mu+\alpha-\mu)z^2g''(z)+2zg'(z)$$
$$=[\alpha\mu z^2(g(z)+\overline{g(\bar{z})})''+(\alpha-\mu)z(g(z)+\overline{g(\bar{z})})'+(1-\alpha+\mu)(g(z)+\overline{g(\bar{z})})]K(z), \tag{3.23}$$

where $K(z)=1+d_1z+d_2z^2+d_3z^3+\cdots$.

On equating the coefficients of like powers of z in (3.23), we get

$$[1+(\alpha-\mu)+2\alpha\mu]b_2=d_1, \tag{3.24}$$

$$2[1+2(\alpha-\mu)+6\alpha\mu]b_3=d_2+[1+(\alpha-\mu)+2\alpha\mu]b_2d_1, \tag{3.25}$$

$$3[1+3(\alpha-\mu)+12\alpha\mu]b_4 = d_3+[1+(\alpha-\mu)+2\alpha\mu]b_2d_2+[1+2(\alpha-\mu)+6\alpha\mu]b_3d_1, \quad (3.26)$$

$$4[1+4(\alpha-\mu)+20\alpha\mu]b_5 = d_4+[1+(\alpha-\mu)+2\alpha\mu]b_2d_3+[1+2(\alpha-\mu)+6\alpha\mu]b_3d_2 +[1+3(\alpha-\mu)+12\alpha\mu]b_4d_1, \quad (3.27)$$

and continuing in this way, we obtain

$$(2n-1)[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]b_{2n} = d_{2n-1}+[1+(\alpha-\mu)+2\alpha\mu]b_2d_{2n-2} +\cdots+[1+(2n-2)(\alpha-\mu)+2n(2n-1)\alpha\mu]b_{2n-1}d_1, \quad (3.28)$$

$$2n[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]b_{2n+1} = d_{2n}+[1+(\alpha-\mu)+2\alpha\mu]b_2d_{2n-1} +\cdots+[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]b_{2n}d_1. \quad (3.29)$$

From (1.4) and (1.5), we have

$$[z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots]+(2\alpha\mu+\alpha-\mu)[2a_2z^2+6a_3z^3+12a_4z^4 +20a_5z^5+\cdots+(2n-1)2na_{2n}z^{2n}+\cdots]+\alpha\mu[6a_3z^3+24a_4z^4+60a_5z^5+\cdots +(2n-1)2n(2n+1)a_{2n+1}z^{2n+1}+\cdots] = \Big[(1+\alpha-\mu)[z+b_2z^2+b_3z^3+b_4z^4+b_5z^5+\cdots+b_{2n}z^{2n}+\cdots] +(\alpha-\mu)[z+2b_2z^2+3b_3z^3+4b_4z^4+5b_5z^5+\cdots+2nb_{2n}z^{2n}+\cdots] +\alpha\mu[2b_2z^2+6b_3z^3+12b_4z^4+20b_5z^5+\cdots+(2n-1)2nb_{2n}z^{2n}+\cdots]\Big] \times[1+p_1z+p_2z^2+p_3z^3+p_4z^4+p_5z^5+\cdots+p_{2n-1}z^{2n-1}+\cdots].$$

On equating the coefficients, we obtain

$$2[1+(\alpha-\mu)+2\alpha\mu]a_2 = p_1+[1+(\alpha-\mu)+2\alpha\mu]b_2, \quad (3.30)$$

$$3[1+2(\alpha-\mu)+6\alpha\mu]a_3 = p_2+[1+(\alpha-\mu)+2\alpha\mu]b_2p_1+[1+2(\alpha-\mu)+6\alpha\mu]b_3, \quad (3.31)$$

$$4[1+3(\alpha-\mu)+12\alpha\mu]a_4 = p_3+[1+(\alpha-\mu)+2\alpha\mu]b_2p_2+[1+2(\alpha-\mu)+6\alpha\mu]b_3p_1 +[1+3(\alpha-\mu)+12\alpha\mu]b_4, \quad (3.32)$$

$$5[1+4(\alpha-\mu)+20\alpha\mu]a_5 = p_4+[1+(\alpha-\mu)+2\alpha\mu]b_2p_3+[1+2(\alpha-\mu)+6\alpha\mu]b_3p_2 +[1+3(\alpha-\mu)+12\alpha\mu]b_4p_1+[1+4(\alpha-\mu)+20\alpha\mu]b_5, \quad (3.33)$$

and so

$$2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]a_{2n} = p_{2n-1}+[1+(\alpha-\mu)+2\alpha\mu]b_2p_{2n-2} +[1+2(\alpha-\mu)+6\alpha\mu]b_3p_{2n-3}+\cdots+[1+(2n-2)(\alpha-\mu)+2n(2n-1)\alpha\mu]b_{2n-1}p_1 +[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]b_{2n}, \quad (3.34)$$

$$(2n+1)[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]a_{2n+1} = p_{2n}+[1+(\alpha-\mu)+2\alpha\mu]b_2p_{2n-1} +[1+2(\alpha-\mu)+6\alpha\mu]b_3p_{2n-2}+\cdots+[1+(2n-1)(\alpha-\mu)+2n(2n+1)\alpha\mu]b_{2n}p_1 +[1+2n(\alpha-\mu)+2n(2n+1)\alpha\mu]b_{2n+1}. \quad (3.35)$$

By using Lemma 2.1, (3.24), (3.25), (3.30), and (3.31), we have

$$|a_2| \leq \frac{(C-D)+(A-B)}{2\cdot 1\cdot[1+(\alpha-\mu)+2\alpha\mu]}, \quad |a_3| \leq \frac{(A-B+1)[(A-B)+2(C-D)]}{3\cdot 2\cdot[1+2(\alpha-\mu)+6\alpha\mu]}.$$

Again, by applying Lemma 2.1 and using (3.24)-(3.27), we obtain from (3.32) and (3.33)

$$|a_4| \leq \frac{(A-B+1)(A-B+2)[(A-B)+3(C-D)]}{4\cdot 6\cdot[1+3(\alpha-\mu)+12\alpha\mu]},$$

$$|a_5| \leq \frac{(A-B+1)(A-B+2)(A-B+3)[(A-B)+4(C-D)]}{5\cdot 24\cdot[1+4(\alpha-\mu)+20\alpha\mu]}.$$

It follows that (3.21) and (3.22) hold for $n=1,2$. We now prove (3.21) by induction.

Equation (3.34), together with Lemma 2.1, yields

$$|a_{2n}| \leq \frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}$$
$$\times\Bigg\{(C-D)\Bigg[1+\sum_{k=1}^{n-1}[1+(2k-1)(\alpha-\mu)+2k(2k-1)\alpha\mu]|b_{2k}|$$
$$+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu)+2k(2k+1)\alpha\mu]|b_{2k+1}|\Bigg]+[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]|b_{2n}|\Bigg\}. \tag{3.36}$$

Again, by using Lemma 2.1 in (3.28), we have

$$|b_{2n}| \leq \frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}$$
$$\times\Bigg[1+\sum_{k=1}^{n-1}[1+(2k-1)(\alpha-\mu)+2k(2k-1)\alpha\mu]|b_{2k}|$$
$$+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu)+2k(2k+1)\alpha\mu]|b_{2k+1}|\Bigg]. \tag{3.37}$$

Using (3.37) in (3.36), we obtain

$$|a_{2n}| \leq \frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}$$
$$\times\Bigg\{\Bigg[(C-D)+\frac{(A-B)}{2n-1}\Bigg]\times\Bigg[1+\sum_{k=1}^{n-1}[1+(2k-1)(\alpha-\mu)+2k(2k-1)\alpha\mu]|b_{2k}|$$
$$+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu)+2k(2k+1)\alpha\mu]|b_{2k+1}|\Bigg]\Bigg\}. \tag{3.38}$$

We suppose that (3.21) holds for $k=3,4,\cdots,(n-1)$.

Using Lemma 2.3 in (3.38), we get

$$|a_{2n}| \leq \frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\Bigg\{\Bigg[(C-D)+\frac{(A-B)}{2n-1}\Bigg]$$
$$\times\Bigg[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\Bigg]\Bigg\}. \tag{3.39}$$

In order to prove (3.21), it is sufficient to show that

$$\frac{1}{2m[1+(2m-1)(\alpha-\mu)+2m(2m-1)\alpha\mu]}\times\Bigg\{\Bigg[(C-D)+\frac{(A-B)}{2m-1}\Bigg]$$
$$\times\Bigg[1+\sum_{k=1}^{m-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{m-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\Bigg]\Bigg\}$$
$$=\frac{1}{2m[1+(2m-1)(\alpha-\mu)+2m(2m-1)\alpha\mu]}$$

$$\times\left\{\left[(C-D)+\frac{(A-B)}{2m-1}\right]\left[\frac{1}{(2m-2)!}\prod_{j=1}^{2m-2}(A-B+j)\right]\right\}\ (m=3,4,\cdots,n). \tag{3.40}$$

Thus, (3.40) is valid for $m=3$.

Let us assume that (3.40) is true for all $m,\ 3<m\leq(n-1)$. Then from (3.39), we have

$$\frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\left\{\left[(C-D)+\frac{(A-B)}{2n-1}\right]\right.$$
$$\left.\times\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\right\}$$
$$=\left(\frac{n-1}{n}\right)\times\left\{\frac{1}{2(n-1)[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\left[(C-D)+\frac{(A-B)}{2n-1}\right]\right.$$
$$\left.\times\left[1+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\right\}$$
$$+\frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]$$
$$\times\left[\frac{(A-B)}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]$$
$$=\left(\frac{n-1}{n}\right)\times\frac{1}{2(n-1)[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]$$
$$\times\left[\frac{1}{(2(n-1)-2)!}\prod_{j=1}^{2n-4}(A-B+j)\right]+\frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}$$
$$\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]\left[\frac{(A-B)}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]$$
$$=\frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]$$
$$\times\left[\frac{1}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)(A-B+(2n-3))+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]$$
$$=\frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]$$
$$\times\left[\frac{1}{(2(n-1)-1)!}\prod_{j=1}^{2n-3}(A-B+j)+\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\right]$$
$$=\frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]}\times\left[(C-D)+\frac{(A-B)}{2n-1}\right]$$
$$\times\left[\frac{1}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)(A-B+(2n-2))\right]$$

$$= \frac{1}{2n[1+(2n-1)(\alpha-\mu)+2n(2n-1)\alpha\mu]} \times \left[(C-D)+\frac{(A-B)}{2n-1}\right]$$
$$\times \left[\frac{1}{(2n-2)!}\prod_{j=1}^{2n-2}(A-B+j)\right].$$

Thus, (3.40) holds for $m = n$, and, hence (3.21) follows. Similarly, we can prove (3.22).

Remark 3.1. By taking $\mu = 0$ in Theorems 3.1 and 3.2, we obtain the results obtained by Tang and Deng [6, Theorems 6 and 11, respectively].

Acknowledgements

The present investigation is partly supported by the Natural Science Foundation of China under Grant 11271045, the Higher School Doctoral Foundation of China under Grant 20100003110004 and the Natural Science Foundation of Inner Mongolia of China under Grant 2010MS0117.

References

[1] S. A. F. M. Dahhar and A. Janteng, A subclass of starlike functions with respect to conjugate points, Int. Math. Forum, 28 (4) (2009), 1373-1377.
[2] R. N. Das and P. Singh, On subclasses of schlicht mapping, Indian J. Pure Appl. Math., 8 (1977), 864-872.
[3] R. M. El-Ashwah and D. K. Thomas, Some subclasses of close-to-convex functions, J. Ramanujan Math. Soc., 2 (1987), 86-100.
[4] R. M. Goel and B. C. Mehrok, A subclass of starlike functions with respect to symmetric points, Tamkang J. Math., 13 (1) (1982), 11-24.
[5] Huo Tang and Guan-Tie Deng, Coefficient estimates for new subclasses of analytic functions with respect to other points, Tamkang J. Math., to appear.
[6] Huo Tang and Guan-Tie Deng, New subclasses of analytic functions with respect to symmetric and conjugate Points, J. Complex. Anal., Vol. 2013, Art. ID 578036, 9 pages.
[7] A. Janteng and S. A. Halim, A subclass of convex functions with respect to symmetric points, Proceedings of The 16th National Symposium on Science Mathematical, 2008.
[8] A. Janteng and S. A. Halim, Coefficient estimate for a subclass of close-to-convex functions with respect to symmetric points, Int. J. Math. Analysis, 3 (2009), 309-313.
[9] A. Janteng and S. A. Halim, A subclass of quasi-convex functions with respect to symmetric points, Appl. Math. Sci., 12 (3) (2009), 551-556.
[10] B. S. Mehrok, G. Singh, and D. Gupta, Coefficient estimate for a subclass of close-to-convex functions with respect to symmetric points, Tamkang J. Math., 42 (2) (2011), 217-222.
[11] K. Sakaguchi, On certain univalent mappings, J. Math. Soc. Japan., 11 (1959), 72-75.
[12] C. Selvaraj and N. Vasanthi, Subclasses of analytic functions with respect to symmetric and conjugate points, Tamkang J. Math., 42 (1) (2011), 87-94.
[13] J. Sokol, Some remarks on the class of functions starlike with respect to symmetric points, Folia Sci. Univ. Tech. Resov., 73 (1990), 79-91.
[14] J. Sokol, Function starlike with respect to conjugate points, Folia Sci. Univ. Tech. Resov., 85 (1991), 53-65.
[15] J. Sokol, A. Szpila, and M. Szpila, On some subclass of starlike functions with respect to symmetric points, Folia Sci. Univ. Tech. Resov., 85 (1991), 65-73.

Huo Tang
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China, and School of Mathematics and Statistics, Chifeng University, Chifeng 024000, Inner Mongolia, China
E-mail address: thth2009@163.com, thth2009@tom.com

Guan-Tie Deng
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
E-mail address: denggt@bnu.edu.cn