SUBCLASSES OF CLOSE-TO-CONVEX AND QUASI-CONVEX FUNCTIONS WITH RESPECT TO OTHER POINTS

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Abstract. In this paper, we introduce new subclasses of close-to-convex and quasi-convex functions with respect to symmetric and conjugate points. The coefficient estimates for functions belonging to these classes are obtained.

1. Introduction

Let $U$ be the class of functions which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$ given by

$$\omega(z) = \sum_{k=1}^{\infty} c_k z^k$$

and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| \leq 1$, $z \in E$.

Let $S$ denote the class of functions $f$ which are analytic and univalent in $E$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in E. \quad (1.2)$$

Let $S^*_s$ be the subclass of functions $f(z) \in S$ and satisfying the condition

$$\Re \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \ z \in E.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [11].

Also, let $S^*_c$ be the subclass of functions $f(z) \in S$ and satisfying the condition

$$\Re \left( \frac{zf'(z)}{f(z) + f(z)} \right) > 0, \ z \in E.$$

These functions are called starlike with respect to conjugate points and were introduced by El-Ashwah and Thomas [3]. Further results on starlike functions with respect to symmetric points or conjugate points can be found in [13-15].

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Then, Das and Singh [2] introduced another class $C_s$, namely convex functions with respect to symmetric points and satisfying the condition
\[
\Re \left( \frac{(zf'(z))'}{(f(z) - f(-z))'} \right) > 0, \quad z \in E.
\]

Suppose that $f$ and $g$ are two analytic functions in $E$. Then, we say that the function $g$ is subordinate to the function $f$, and we write $g(z) \prec f(z)$, $z \in E$, if there exists a Schwarz function $\varpi(z)$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ such that $g(z) = f(\varpi(z))$, $z \in E$.

In view of subordination definition, Goel and Mehrok [4] introduced a subclass of $S_\alpha^* (A, B)$ denoted by $S_\alpha^* (A, B)$.

Let $S_\alpha^* (A, B)$ be the class of functions of the form (1.2) and satisfying the condition
\[
\frac{2zf'(z)}{f(z) - f(-z)} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E.
\]

Following them, many authors introduced the analogue definitions by extension as follows (see [1, 7]).

**Definition 1.1.** (i) Let $S_\alpha^* (A, B)$ be the subclass of $S$ consisting of functions given by (1.2) satisfying the condition
\[
\frac{2zf'(z)}{f(z) + f(\bar{z})} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E.
\]

(ii) Let $C_s (A, B)$ be the subclass of $S$ consisting of functions given by (1.2) satisfying the condition
\[
\frac{2zf'(z)}{f(z) - f(-z)} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E.
\]

(iii) Let $C_c (A, B)$ be the subclass of $S$ consisting of functions given by (1.2) satisfying the condition
\[
\frac{2zf'(z)}{f(z) + f(\bar{z})} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E.
\]

Motivated by the previous classes, Tang and Deng [5] recently introduced the following classes of functions with respect to symmetric and conjugate points.

**Definition 1.2.** (i) Let $M_s (\alpha, \mu, A, B)$ be the subclass of $S$ consisting of functions given by (1.2) satisfying the condition
\[
\frac{2\alpha \mu z^3 f'''(z) + 2(2\alpha \mu + \alpha - \mu)z^2 f''(z) + 2zf'(z)}{\alpha \mu z^2 (f(z) - f(-z))'' + (\alpha - \mu)z(f(z) - f(-z))' + (1 - \alpha + \mu)(f(z) - f(-z))} < \frac{1 + Az}{1 + Bz},
\]
where $-1 \leq B < A \leq 1$, $0 \leq \mu \leq \alpha \leq 1$, and $z \in E$.

(ii) Let $M_c (\alpha, \mu, A, B)$ be the subclass of $S$ consisting of functions given by (1.2) satisfying the condition
\[
\frac{2\alpha \mu z^3 f'''(z) + 2(2\alpha \mu + \alpha - \mu)z^2 f''(z) + 2zf'(z)}{\alpha \mu z^2 (f(z) + f(\bar{z}))'' + (\alpha - \mu)z(f(z) + f(\bar{z}))' + (1 - \alpha + \mu)(f(z) + f(\bar{z}))} < \frac{1 + Az}{1 + Bz},
\]
where $-1 \leq B < A \leq 1$, $0 \leq \mu \leq \alpha \leq 1$, and $z \in E$.

As a special case, when $\mu = 0$, we obtain
\[
M_s (\alpha, 0, A, B) = M_s (\alpha, A, B) \quad \text{and} \quad M_c (\alpha, 0, A, B) = M_c (\alpha, A, B),
\]
introduced and studied by Selvaraj and Vasanth [12].

In this paper, we introduce the class $K_s^*(\alpha, \mu, A, B; C, D)$ consisting of analytic functions $f$ of the form (1.2) and satisfying

$$\frac{2\alpha \mu z^2 f''''(z) + 2(2\alpha \mu + \alpha - \mu)z^2 f'''(z) + 2zf'(z)}{\alpha \mu z^2(g(z) - g(-z))'' + (\alpha - \mu)z(g(z) - g(-z))' + (1 - \alpha + \mu)(g(z) - g(-z))} \leq \frac{1 + Cz}{1 + Dz},$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_s(\alpha, \mu, A, B), -1 \leq D \leq B < A \leq C \leq 1, 0 \leq \mu \leq \alpha \leq 1$, and $z \in \mathbb{E}$.

Also, we introduce the class $K_c^*(\alpha, \mu, A, B; C, D)$ consisting of analytic functions $f$ of the form (1.2) and satisfying

$$\frac{2\alpha \mu z^2 f''''(z) + 2(2\alpha \mu + \alpha - \mu)z^2 f'''(z) + 2zf'(z)}{\alpha \mu z^2(g(z) + g(\overline{z}))'' + (\alpha - \mu)z(g(z) + g(\overline{z}))' + (1 - \alpha + \mu)(g(z) + g(\overline{z}))} \leq \frac{1 + Cz}{1 + Dz},$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_c(\alpha, \mu, A, B), -1 \leq D \leq B < A \leq C \leq 1, 0 \leq \mu \leq \alpha \leq 1$, and $z \in \mathbb{E}$.

We note that

(i) for $\mu = 0$, $K_s^*(\alpha, 0, A, B; C, D) = K_s^*(\alpha, A, B; C, D)$ and $K_s^*(\alpha, 0, A, B; C, D) = K_s^*(\alpha, A, B; C, D)$ (see Tang and Deng [6])

(ii) for $\alpha = \mu = 0$, $K_s^*(0, 0, A, B; C, D) = K_s(0, A, B; C, D)$ (see Mehrok et al.[10]) and $K_s^*(0, 0, A, B; C, D) = K_s(A, B; C, D)$

(iii) for $\alpha = \mu = 0$, $C = 1$ and $D = -1$, $K_s^*(0, 0, A, B; 1, -1) = K_s(A, B)$ (see Janteng and Halim [8]) and $K_s^*(0, 0, A, B; 1, -1) = K_s(A, B)$

(iv) for $\alpha = \mu = 0$, $A = C = 1$ and $B = D = -1$, $K_s^*(0, 0, 1, -1; 1, -1) = K_s$ and $K_s^*(0, 0, 1, -1; 1, -1) = K_s$.

(v) for $\alpha = 1$ and $\mu = 0$, $K_s^*(1, 0, A, B; C, D) = K_s(A, B; C, D)$ and $K_s^*(1, 0, A, B; C, D) = K_s(A, B; C, D)$

(vi) for $\alpha = 1$, $\mu = 0$, $C = 1$ and $D = -1$, $K_s^*(1, 0, A, B; 1, -1) = K_s(A, B)$ (see Janteng and Halim [9]) and $K_s^*(1, 0, A, B; 1, -1) = K_s(A, B)$

(vii) for $\alpha = 1$, $\mu = 0$, $A = C = 1$ and $B = D = -1$, $K_s^*(1, 0, 1, -1; 1, -1) = K_s$ and $K_s^*(1, 0, 1, -1; 1, -1) = K_s$.

By the definition of subordination, it follows that $f \in K_s^*(\alpha, \mu, A, B; C, D)$ if and only if

$$\frac{2\alpha \mu z^2 f''''(z) + 2(2\alpha \mu + \alpha - \mu)z^2 f'''(z) + 2zf'(z)}{\alpha \mu z^2(g(z) - g(-z))'' + (\alpha - \mu)z(g(z) - g(-z))' + (1 - \alpha + \mu)(g(z) - g(-z))} \leq \frac{1 + C\omega(z)}{1 + D\omega(z)} = P(z), \omega(z) \in U,$$  \hspace{1cm} (1.3)

and that $f \in K_c^*(\alpha, \mu, A, B; C, D)$ if and only if

$$\frac{2\alpha \mu z^2 f''''(z) + 2(2\alpha \mu + \alpha - \mu)z^2 f'''(z) + 2zf'(z)}{\alpha \mu z^2(g(z) + g(\overline{z}))'' + (\alpha - \mu)z(g(z) + g(\overline{z}))' + (1 - \alpha + \mu)(g(z) + g(\overline{z}))} \leq \frac{1 + C\omega(z)}{1 + D\omega(z)} = P(z), \omega(z) \in U,$$  \hspace{1cm} (1.4)

where

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$  \hspace{1cm} (1.5)

In the next section, we discuss the coefficient estimates for functions belonging to the classes $K_s^*(\alpha, \mu, A, B; C, D)$ and $K_c^*(\alpha, \mu, A, B; C, D)$.
2. SOME PRELIMINARY LEMMAS

We shall require the following lemmas for proving our main results.

Lemma 2.1 (see [4]). If $P(z)$ is given by (1.3), (1.4) and (1.5), then for $-1 \leq D < C \leq 1$, 

$$|p_n| \leq (C - D), \quad n = 1, 2, \ldots .$$

Lemma 2.2 (see [5]). Let $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_s(\alpha, \mu, A, B)$. Then for $n \geq 1, 0 \leq \mu \leq \alpha \leq 1$,

$$|b_{2n}| \leq \frac{(A - B)}{2^n \cdot n! [1 + (2n - 1)(\alpha - \mu + 2n\alpha\mu)]} \prod_{j=1}^{n-1} (A - B + 2j),$$

$$|b_{2n+1}| \leq \frac{(A - B)}{2^n \cdot n! [1 + 2n(\alpha - \mu + (2n + 1)\alpha\mu)]} \prod_{j=1}^{n-1} (A - B + 2j).$$

Lemma 2.3 (see [5]). Let $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_s(\alpha, \mu, A, B)$. Then for $n \geq 1, 0 \leq \mu \leq \alpha \leq 1$,

$$|b_{2n}| \leq \frac{(A - B)}{(2n - 1)! [1 + (2n - 1)(\alpha - \mu + 2n\alpha\mu)]} \prod_{j=1}^{2n-2} (A - B + j),$$

$$|b_{2n+1}| \leq \frac{(A - B)}{(2n)! [1 + 2n(\alpha - \mu + (2n + 1)\alpha\mu)]} \prod_{j=1}^{2n-1} (A - B + j).$$

3. MAIN RESULTS

Unless otherwise mentioned, we shall assume in the reminder of this paper that 

$-1 \leq D \leq B < A \leq C \leq 1$, $0 \leq \mu \leq \alpha \leq 1$, and $z \in E$.

Theorem 3.1. Let $f \in K_s^*(\alpha, \mu, A, B; C, D)$, then for $n \geq 1$,

$$|a_{2n}| \leq \frac{(C - D)}{2^n \cdot n! [1 + (2n - 1)(\alpha - \mu + 2n\alpha\mu)]} \prod_{j=1}^{n-1} (A - B + 2j), \quad (3.1)$$

$$|a_{2n+1}| \leq \frac{1}{(2n + 1)! [1 + 2n(\alpha - \mu + (2n + 1)\alpha\mu)]} \times \left\{ \frac{1}{(C - D) + \frac{(A - B)}{2n}} \left[ \frac{1}{2^n \cdot (n - 1)!} \prod_{j=1}^{n-1} (A - B + 2j) \right] \right\} . \quad (3.2)$$

Proof. Since $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_s(\alpha, \mu, A, B)$, it follows that

$$2\alpha\mu z^3 g'''(z) + 2(2\alpha\mu + \alpha - \mu) z^2 g''(z) + 2z g'(z)$$

$$= [\alpha\mu z^2 (g(z) - g(-z))'' + (\alpha - \mu) z (g(z) - g(-z))' + (1 - \alpha + \mu)(g(z) - g(-z))] K(z) \quad (3.3)$$

for $z \in E$, with $Re(K(z)) > 0$, where $K(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$. On equating the coefficients of like powers of $z$ in (3.3), we get

$$2[1 + (\alpha - \mu) + 2\alpha\mu] b_2 = d_1, \quad 2[1 + 2(\alpha - \mu) + 6\alpha\mu] b_3 = d_2,$$

$$4[1 + 3(\alpha - \mu) + 12\alpha\mu] b_4 = d_3 + [1 + 2(\alpha - \mu) + 6\alpha\mu] b_3 d_1,$$

$$4[1 + 4(\alpha - \mu) + 20\alpha\mu] b_5 = d_4 + [1 + 2(\alpha - \mu) + 6\alpha\mu] b_3 d_2. \quad (3.4)$$
and continuing in this way, we obtain
\[2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]|b_{2n} = d_{2n-1} + [1 + 2(\alpha - \mu) + 6\alpha\mu]|b_3d_{2n-3} + \ldots + [1 + 2(n-1)(\alpha - \mu) + 2(2n-1)\alpha\mu]|b_{2n-1}d_1,\]
(3.6)
\[2n[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha\mu]|b_{2n+1} = d_{2n} + [1 + 2(\alpha - \mu) + 6\alpha\mu]|b_3d_{2n-2} + \ldots + [1 + 2(n-1)(\alpha - \mu) + 2(2n-1)\alpha\mu]|b_{2n-1}d_2.\]
(3.7)

From (1.3) and (1.5), we have
\[
[z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + \ldots + 2na_{2n}z^{2n} + \ldots] + (2\alpha\mu + a_1 - \alpha)|2a_2z^2 + 6a_3z^3 + 12a_4z^4 + 20a_5z^5 + \ldots + (2n-1)2na_{2n}z^{2n} + \ldots| + \alpha\mu|6a_3z^3 + 24a_4z^4 + 60a_5z^5 + \ldots + (2n-1)2na_{2n}z^{2n} + \ldots|
\]

\[+ (2n-1)2n(a_{2n+1}z^{2n+1} + \ldots) = \left[(1+\alpha-\mu)[z+b_3z^3+b_5z^5+\ldots+b_{2n-1}z^{2n-1}+b_{2n+1}z^{2n+1} + \ldots] + (\alpha-\mu)[z+3b_3z^3+5b_5z^5+\ldots+(2n-1)b_{2n-1}z^{2n-1}+(2n+1)b_{2n+1}z^{2n+1} + \ldots] + \alpha\mu(6b_3z^3+20b_5z^5+\ldots+2n(2n+1)b_{2n+1}z^{2n+1} + \ldots)\right] \times [1+p_1z+p_2z^2+p_3z^3+p_4z^4+p_5z^5 + \ldots + p_{2n-1}z^{2n-1} + p_{2n}z^{2n} + \ldots].
\]

On equating the coefficients, we obtain
\[2[1+(\alpha-\mu)+2\alpha\mu]|a_2| = p_1, 3[1+2(\alpha-\mu)+6\alpha\mu]|a_3| = p_2 + [1+2(\alpha-\mu)+6\alpha\mu]|b_3,\]
(3.8)
\[4[1+3(\alpha-\mu)+12\alpha\mu]|a_4| = p_3 + [1+2(\alpha-\mu)+6\alpha\mu]|b_3p_1,\]
(3.9)
\[5[1+4(\alpha-\mu)+20\alpha\mu]|a_5| = p_4 + [1+2(\alpha-\mu)+6\alpha\mu]|b_3p_2 + [1+4(\alpha-\mu)+20\alpha\mu]|b_5,\]
(3.10)
and so
\[2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]|a_{2n} = p_{2n-1} + [1 + 2(\alpha - \mu) + 6\alpha\mu]|b_1p_{2n-3} + \ldots + [1 + 2(n-1)(\alpha - \mu) + 2(2n-1)\alpha\mu]|b_{2n-1}p_1,\]
(3.11)
\[(2n+1)[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha\mu]|a_{2n+1} = p_{2n} + [1 + 2(\alpha - \mu) + 6\alpha\mu]|b_3p_{2n-2} + \ldots + [1 + 2(n-1)(\alpha - \mu) + 2(2n-1)\alpha\mu]|b_{2n-1}p_2 + [1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha\mu]|b_{2n+1}.\]
(3.12)

By using Lemma 2.1 and (3.8), we have
\[|a_2| \leq \frac{(C - D)}{2 \cdot 1 + (\alpha - \mu) + 2\alpha\mu}, \quad |a_4| \leq \frac{(A - B) + 2(C - D)}{3 \cdot 2 \cdot [1 + 2(\alpha - \mu) + 6\alpha\mu]}.
\]
Again, by applying Lemma 2.1 and using (3.4) and (3.5), we obtain from (3.9) and (3.10)
\[|a_4| \leq \frac{(C - D)(A - B + 2)}{4 \cdot 2 \cdot [1 + 3(\alpha - \mu) + 12\alpha\mu]}, \quad |a_5| \leq \frac{(A - B + 2)|A - B + 4(C - D)|}{5 \cdot 8 \cdot [1 + 4(\alpha - \mu) + 20\alpha\mu]}
\]

It follows that (3.1) and (3.2) hold for \(n = 1, 2\). We now prove (3.1) and (3.2) by induction.

Equations (3.11) and (3.12), together with Lemma 2.1, yield
\[|a_{2n}| \leq \frac{(C - D)}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]} \left[1 + \sum_{k=1}^{n-1} [1 + 2k(\alpha - \mu) + 2(2k + 1)\alpha\mu]|b_{2k+1}| \right].
\]
(3.13)
\[|a_{2n+1}| \leq \frac{1}{(2n + 1)[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha\mu]} \times \left(\frac{C - D}{(C - D)}\right).
\]
\[ \times \left\{ 1 + \sum_{k=1}^{n-1} \left[ 1 + 2k(\alpha - \mu) + 2(2k + 1)\alpha|b_{2k+1}| \right] \right\} + [1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha|b_{2n+1}|]. \] (3.14)

Again, using Lemma 2.1 in (3.7), we have

\[ |b_{2n+1}| \leq \frac{(A - B)}{2n[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha]} \left[ 1 + \sum_{k=1}^{n-1} [1 + 2k(\alpha - \mu) + 2(2k + 1)\alpha|b_{2k+1}|] \right]. \] (3.15)

Using (3.15) in (3.14), we obtain

\[ \left| a_{2n+1} \right| \leq \frac{1}{(2n + 1)[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha]} \times \left\{ \left( C - D \right) + \frac{(A - B)}{2n} \right\} \left[ 1 + \sum_{k=1}^{n-1} [1 + 2k(\alpha - \mu) + 2(2k + 1)\alpha|b_{2k+1}|] \right]. \] (3.16)

We suppose that (3.1) and (3.2) hold for \( k = 3, 4, \cdots, (n - 1). \)

Using Lemma 2.2 in (3.13) and (3.16), we get

\[ \left| a_{2n} \right| \leq \frac{(C - D)}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha]} \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{2^{k} \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right], \] (3.17)

\[ \left| a_{2n} \right| \leq \frac{1}{(2n + 1)[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha]} \times \left\{ \left( C - D \right) + \frac{(A - B)}{2n} \right\} \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{2^{k} \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right]. \] (3.18)

In order to prove (3.1), it is sufficient to show that

\[ \frac{(C - D)}{2m[1 + (2m - 1)(\alpha - \mu) + 2m(2m - 1)\alpha]} \left[ 1 + \sum_{k=1}^{m-1} \frac{(A - B)}{2^{k} \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right] \]

\[ = \frac{(C - D)}{2^{m} \cdot m! [1 + (2m - 1)(\alpha - \mu) + 2m(2m - 1)\alpha]} \prod_{j=1}^{m-1} (A - B + 2j) \quad (m = 3, 4, \cdots, n). \] (3.19)

Thus, (3.19) is valid for \( m = 3. \)

Let us assume that (3.19) is true for all \( m, \ 3 < m \leq (n - 1). \) Then from (3.17), we have

\[ \frac{(C - D)}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha]} \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{2^{k} \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right] \]

\[ = \frac{(n - 1)}{n} \times \left\{ \frac{(C - D)}{2^{n-1} \cdot (n - 1)! [1 + (2(n - 1)(\alpha - \mu) + 2(n - 1)(2n - 1)\alpha]} \right\} \]

\[ \times \left[ 1 + \sum_{k=1}^{n-2} \frac{(A - B)}{2^{k} \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right]. \]
Thus, (3.19) holds for \( m = n \), and, hence (3.1) follows. Next, we prove (3.2).

From (3.19), we have

\[
1 + \sum_{k=1}^{n-1} \frac{(A - B)}{2^k \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) = \frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A - B + 2j).
\]

By using (3.20) in (3.18), we obtain

\[
|a_{2n+1}| \leq \frac{1}{(2n + 1)[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha\mu]} \times \left\{ \left( C - D \right) + \frac{(A - B)}{2n} \right\} \left[ \frac{1}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-1} (A - B + 2j) \right],
\]

which proves (3.2).

**Theorem 3.2.** Let \( f \in K^*_c(\alpha, \mu, A, B; C, D) \), then for \( n \geq 1 \),

\[
|a_{2n}| \leq \frac{1}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]} \times \left\{ \left( C - D \right) + \frac{(A - B)}{2n - 1} \right\} \left[ \frac{1}{(2n - 2)!} \prod_{j=1}^{2n-2} (A - B + j) \right],
\]

\[
|a_{2n+1}| \leq \frac{1}{(2n + 1)[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha\mu]} \times \left\{ \left( C - D \right) + \frac{(A - B)}{2n} \right\} \left[ \frac{1}{(2n - 1)!} \prod_{j=1}^{2n-1} (A - B + j) \right].
\]

**Proof.** Since \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in M_c(\alpha, \mu, A, B) \), it follows that

\[
2\alpha\mu z^3 g''(z) + 2(2\alpha\mu + \alpha - \mu)z^2 g'(z) + 2zg'(z) = [\alpha\mu z^2 (g(z) + g(\bar{z}))'' + (\alpha - \mu)z(g(z) + g(\bar{z}))' + (1 - \alpha + \mu)(g(z) + g(\bar{z}))]K(z),
\]

where \( K(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots \).

On equating the coefficients of like powers of \( z \) in (3.23), we get

\[
|1 + (\alpha - \mu) + 2\alpha\mu| b_2 = d_1,
\]

\[
2[1 + 2(\alpha - \mu) + 6\alpha\mu] b_3 = d_2 + [1 + (\alpha - \mu) + 2\alpha\mu] b_2 d_1,
\]

\[
+ 2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu] \times \frac{(A - B)}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-2} (A - B + 2j)
\]

\[
= \frac{(n-1)}{n} \times \frac{(C - D)}{2^{n-1} \cdot (n-1)!} \prod_{j=1}^{n-2} (A - B + 2j)
\]

\[
= \frac{(C - D)}{2^n \cdot n! [1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]} \prod_{j=1}^{n-1} (A - B + 2j)(A - B + 2(n-1))
\]

\[
= \frac{(C - D)}{2^n \cdot n! [1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]} \prod_{j=1}^{n-1} (A - B + 2j).
\]
3[1 + 3(\alpha - \mu) + 12\alpha \mu] b_4 = d_3 + [1 + (\alpha - \mu) + 2\alpha \mu] b_2 d_2 + [1 + 2(\alpha - \mu) + 6\alpha \mu] b_3 d_1, \quad (3.26)
4[1 + 4(\alpha - \mu) + 20\alpha \mu] b_5 = d_4 + [1 + (\alpha - \mu) + 2\alpha \mu] b_2 d_3 + [1 + 2(\alpha - \mu) + 6\alpha \mu] b_3 d_2 \\
+ [1 + 3(\alpha - \mu) + 12\alpha \mu] b_4 d_1, \quad (3.27)
and continuing in this way, we obtain
\begin{align*}
(2n - 1)[1 + (2n - 1)(\alpha - \mu)] + 2n(2n - 1)\alpha \mu] b_{2n} &= d_{2n-1} + [1 + (\alpha - \mu) + 2\alpha \mu] b_{2n-2} \\
&+ \cdots + [1 + (2n - 2)(\alpha - \mu) + 2n(2n - 1)\alpha \mu] b_{2n-1}, \\
2n[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha \mu] b_{2n+1} &= d_{2n} + [1 + (\alpha - \mu) + 2\alpha \mu] b_{2n-1} \\
&+ \cdots + [1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha \mu] b_{2n}, \quad (3.29)
\end{align*}
From (1.4) and (1.5), we have
\begin{align*}
&[z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + \cdots + 2n a_{2n} z^{2n} + \cdots] + (2\alpha + \alpha - \mu) [2a_2 z^2 + 6a_3 z^3 + 12a_4 z^4 \\
&+ 20a_5 z^5 + \cdots + (2n - 1) 2na_{2n} z^{2n} + \cdots] + \alpha \mu [6a_3 z^3 + 24a_4 z^4 + 60a_5 z^5 + \cdots]
+(2n-1)2n(2n+1)a_{2n+1} z^{2n+1} + \cdots] + [1 + (\alpha - \mu) [z + 2b_2 z^2 + 3b_3 z^3 + 4b_4 z^4 + 5b_5 z^5 + \cdots + 2nb_{2n} z^{2n} + \cdots] \\
&+ \alpha \mu [2b_2 z^2 + 6b_3 z^3 + 12b_4 z^4 + 20b_5 z^5 + \cdots + (2n - 1) 2nb_{2n} z^{2n} + \cdots]]
\times [1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + p_5 z^5 + \cdots + p_{2n-1} z^{2n-1} + \cdots].
\end{align*}
On equating the coefficients, we obtain
\begin{align*}
2[1 + (\alpha - \mu) + 2\alpha \mu] a_2 &= p_1 + [1 + (\alpha - \mu) + 2\alpha \mu] b_2, \quad (3.30) \\
3[1 + 2(\alpha - \mu) + 6\alpha \mu] a_3 &= p_2 + [1 + (\alpha - \mu) + 2\alpha \mu] b_2 p_1 + [1 + 2(\alpha - \mu) + 6\alpha \mu] b_3, \quad (3.31) \\
4[1 + 3(\alpha - \mu) + 12\alpha \mu] a_4 &= p_3 + [1 + (\alpha - \mu) + 2\alpha \mu] b_2 p_2 + [1 + 2(\alpha - \mu) + 6\alpha \mu] b_3 p_1 \\
&+ [1 + 3(\alpha - \mu) + 12\alpha \mu] b_4, \quad (3.32) \\
5[1 + 4(\alpha - \mu) + 20\alpha \mu] a_5 &= p_4 + [1 + (\alpha - \mu) + 2\alpha \mu] b_2 p_3 + [1 + 2(\alpha - \mu) + 6\alpha \mu] b_3 p_2 \\
&+ [1 + 3(\alpha - \mu) + 12\alpha \mu] b_4 p_1 + [1 + 4(\alpha - \mu) + 20\alpha \mu] b_5, \quad (3.33)
\end{align*}
and so
\begin{align*}
2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha \mu] a_{2n} &= p_{2n-1} + [1 + (\alpha - \mu) + 2\alpha \mu] b_{2n-2} \\
&+ [1 + 2(\alpha - \mu) + 6\alpha \mu] b_3 p_{2n-3} + \cdots + [1 + (2n - 2)(\alpha - \mu) + 2n(2n - 1)\alpha \mu] b_{2n-1} \\
&+ [1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha \mu] b_{2n}, \quad (3.34) \\
(2n + 1)[1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha \mu] a_{2n+1} &= p_{2n} + [1 + (\alpha - \mu) + 2\alpha \mu] b_{2n-1} \\
&+ [1 + 2(\alpha - \mu) + 6\alpha \mu] b_3 p_{2n-2} + \cdots + [1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha \mu] b_{2n} p_1 \\
&+ [1 + 2n(\alpha - \mu) + 2n(2n + 1)\alpha \mu] b_{2n+1}, \quad (3.35)
\end{align*}
By using Lemma 2.1, (3.24), (3.25), (3.30), and (3.31), we have
\[|a_2| \leq \frac{(C - D) + (A - B)}{2 \cdot 1 \cdot 1 + (\alpha - \mu) + 2\alpha \mu}, \quad |a_3| \leq \frac{(A - B + 1)[(A - B) + 2(C - D)]}{3 \cdot 2 \cdot 1 + 2(\alpha - \mu) + 6\alpha \mu}, \]
Again, by applying Lemma 2.1 and using (3.24)-(3.27), we obtain from (3.32) and (3.33)
\[|a_4| \leq \frac{(A - B + 1)(A - B + 2)[(A - B) + 3(C - D)]}{4 \cdot 6 \cdot 1 + 3(\alpha - \mu) + 12\alpha \mu}, \]
\[|a_5| \leq \frac{(A - B + 1)(A - B + 2)(A - B + 3)[(A - B) + 4(C - D)]}{5 \cdot 24 \cdot 1 + 4(\alpha - \mu) + 20\alpha \mu}. \]
It follows that (3.21) and (3.22) hold for \( n = 1, 2 \). We now prove (3.21) by induction.
Equation (3.34), together with Lemma 2.1, yields
\[
|a_{2n}| \leq \frac{1}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]}
\]
\[
\times \left\{ (C - D) \left[ 1 + \sum_{k=1}^{n-1} [1 + (2k - 1)(\alpha - \mu) + 2k(2k - 1)\alpha\mu] |b_{2k}| \right.ight.
\]
\[
+ \sum_{k=1}^{n-1} [1 + 2k(\alpha - \mu) + 2k(2k + 1)\alpha\mu] |b_{2k+1}| \right] + [1 + (2n-1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu] b_{2n} \right\}. \tag{3.36}
\]
Again, by using Lemma 2.1 in (3.28), we have
\[
|b_{2n}| \leq \frac{(A - B)}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]}
\]
\[
\times \left[ 1 + \sum_{k=1}^{n-1} [1 + (2k - 1)(\alpha - \mu) + 2k(2k - 1)\alpha\mu] |b_{2k}| \right.
\]
\[
+ \sum_{k=1}^{n-1} [1 + 2k(\alpha - \mu) + 2k(2k + 1)\alpha\mu] |b_{2k+1}| \right]. \tag{3.37}
\]
Using (3.37) in (3.36), we obtain
\[
|a_{2n}| \leq \frac{1}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]}
\]
\[
\times \left\{ (C - D) + \frac{(A - B)}{2n - 1} \times \left[ 1 + \sum_{k=1}^{n-1} [1 + (2k - 1)(\alpha - \mu) + 2k(2k - 1)\alpha\mu] |b_{2k}| \right.ight.
\]
\[
+ \sum_{k=1}^{n-1} [1 + 2k(\alpha - \mu) + 2k(2k + 1)\alpha\mu] |b_{2k+1}| \right\}. \tag{3.38}
\]
We suppose that (3.21) holds for \( k = 3, 4, \ldots, (n - 1) \).

Using Lemma 2.3 in (3.38), we get
\[
|a_{2n}| \leq \frac{1}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha\mu]} \times \left\{ (C - D) + \frac{(A - B)}{2n - 1} \times \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-2} (A - B + j) \right. \right.
\]
\[
+ \sum_{k=1}^{n-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-2} (A - B + j) \right] \right\}. \tag{3.39}
\]
In order to prove (3.21), it is sufficient to show that
\[
\frac{1}{2m[1 + (2m - 1)(\alpha - \mu) + 2m(2m - 1)\alpha\mu]} \times \left\{ (C - D) + \frac{(A - B)}{2m - 1} \times \left[ 1 + \sum_{k=1}^{m-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-2} (A - B + j) \right. \right.
\]
\[
+ \sum_{k=1}^{m-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-2} (A - B + j) \right] \right\} = \frac{1}{2m[1 + (2m - 1)(\alpha - \mu) + 2m(2m - 1)\alpha\mu]}
Thus, (3.40) is valid for \( m = 3 \).

Let us assume that (3.40) is true for all \( m, 3 < m \leq (n - 1) \). Then from (3.39), we have

\[
\begin{align*}
\times & \left\{ \left( (C - D) + \frac{(A - B)}{2m - 1} \right) \left[ \frac{1}{(2m - 2)!} \prod_{j=1}^{2m-2} (A - B + j) \right] \right\} \\
\frac{1}{2n[1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha \mu]} & \times \left\{ \left[ (C - D) + \frac{(A - B)}{2n - 1} \right] \\
\times \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-2} (A - B + j) + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right] \right\} \\
\frac{1}{[2(n-1)][1 + (2n - 1)(\alpha - \mu) + 2n(2n - 1)\alpha \mu]} & \times \left\{ (C - D) + \frac{(A - B)}{2n - 1} \right\} \\
\times \left[ 1 + \sum_{k=1}^{n-2} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-2} (A - B + j) + \sum_{k=1}^{n-2} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right] \\
\end{align*}
\]

Thus, (3.40) is valid for \( m = 3 \).
Thus, (3.40) holds for \( m = n \), and, hence (3.21) follows. Similarly, we can prove (3.22).

**Remark 3.1.** By taking \( \mu = 0 \) in Theorems 3.1 and 3.2, we obtain the results obtained by Tang and Deng [6, Theorems 6 and 11, respectively].

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