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A DISCUSSION ON EULER METHOD: A REVIEW

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Abstract. Notwithstanding the efforts of earlier workers some fundamental aspects of an introductory course on numerical methods have been overlooked. This paper dwells on this aspect. It is sometimes felt that the step size is root cause for such misconceptions. But it is hardly felt that clubbing of the forward Euler algorithm with the backward Euler algorithm for convenience can lead to serious misconceptions and of course it depends on the nature of the problems. With this in mind we consider few such problems that falls under the category of Ordinary Differential Equation (ODE), a differential equation containing one or more derivatives of the dependent variable. However nothing is discussed on the error due to nesting of forward Euler method in backward Euler method. It is shown in this paper that there is a maximum value of step size up to which the numerical algorithms are stable. This tutorial paper attempt to illustrate these issues by taking some practical example that has exact analytical solutions, so that pitfalls of the numerical algorithm can be vividly illustrated. Moreover, the discussion of linear systems offers considerable insight into the solution of non-linear equations.

1. Introduction

In real world problems, like, how to construct a bridge over a river, to know the motion of a pendulum, etc. we always try to get the "exact answer" of the problems. For that matter we need to construct an exact mathematical model of the problem. But it is not possible to incorporate every aspect of the problem in constructing the model. The simple reason can be elaborated easily by considering the problem of construction of the bridge. The various forces that act on the bridge are: the gravity of the earth, the density of the traffic crossing the bridge, the number of people crossing the bridge at any instant, gravity of the moon, etc. For exact mathematical modelling it is necessary to incorporate all such effects. Some of the forces have major effect and the other produces minor effects. It is therefore important to strip off some of the unimportant features in the design and to reduce the uncertainties in the initial phase in order to develop the relevant mathematical model to an acceptable form. These types of complications are also observed in many types of problems in physics, chemistry, biology, economics, engineering, etc. Thus one begins with an acceptable and mathematical tractable model and

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adding few terms here and there as perturbations (Figure 1). The most important mathematical tool used in modelling such and many others in physical sciences is differential equation. The differential equation or ordinary differential equation (ODE) in many an occasion is too complicated to be solved analytically. Even if an analytical solution is available it is so complicated that it is of little use. The numerical solution is thus the only way to obtain information about the system. It is worthwhile to note that numerical methods are employed to calculate the values by the dependent variable over a range of independent variable.

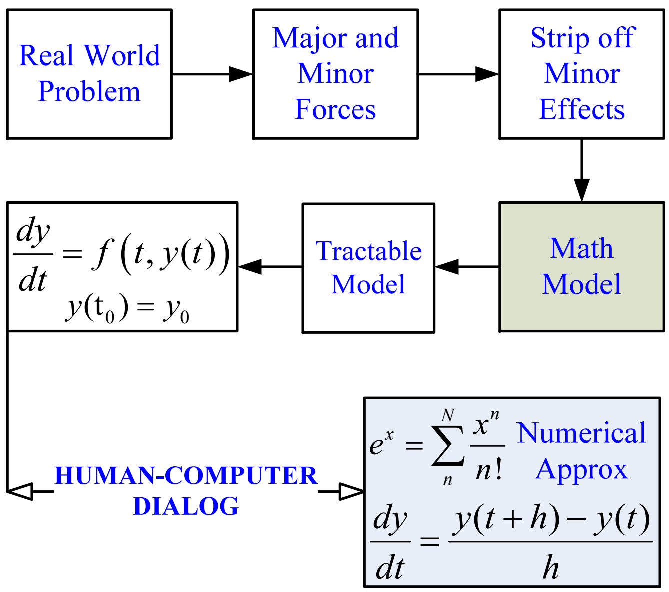

Figure 1. Relation between real world and computer world problem formulation

At this point it is worthwhile to remember that certain subsidiary conditions are necessary for the solution of the differential equation. When the subsidiary conditions are specified at single value of the independent variable then the problem is said to an initial value problem (IVP). When the subsidiary conditions are given at more than one value of the independent value it is called a boundary value problem (BVP).

2. Use of Elementary Difference

The use of elementary difference methods to obtain approximate solution of differential equations or initial value problems was first reported in 1768 by Leonhard Euler, Beethoven of Mathematics ([1],[2]). As such it is indispensable to discretizing the independent variable. Thus the method is ideally suited for using computers to obtain numerical solution as it discretizes the independent variable since computers understand discrete variables only. Euler gave the idea of programming long before the introduction of concept of programming in 1842 by Ada Lovelace, the daughter of Lord Byron. Numerical methods are no-longer laborious since computers will do the work. Effectively Euler's idea transforms a differential equation of an ODE to an algebraic equation. Thus the solution of an initial-value problem is simply a procedure that produces approximate solutions at particular points using only the operations of addition, subtraction, multiplication, division, and functional evaluations ([3]-[5]). Euler's method abolishes mismatch between the two worlds - the analogue world in which we humans live and the discrete / digital world in which computers survive.

3. Numerical Approach

Numerical methods are of course approximate, but with good care the approximations can be made to be so good that the values calculated by an analytical method and those by a numerical method are virtually identical. The advanced numerical methods are themselves complicated but do yield results which cannot be obtained by other simple methods ([6],[7]). But the principle of all the methods can be illustrated using simple methods ([8]-[10]). As such, a typical introductory course in differential equations and modeling with differential equations exposes the students to forward Euler and backward Euler methods, the oldest and simplest algorithm, in solving boundary value and initial value problems. It is felt that in such an attempt misconceptions, if creep in, should be dispelled. Notwithstanding the efforts of earlier workers ([11]-[16]) some fundamental aspects of an introductory course on numerical methods have been overlooked. This paper dwells on this aspect. It is sometimes felt that the step size is root cause for such misconceptions. But it is hardly felt that clubbing of the forward Euler algorithm with the backward Euler algorithm for convenience can lead to serious misconceptions and of course it depends on the nature of the problems ([17]-[19]). With this in mind we consider few such problems that falls under the category of Ordinary Differential Equation (ODE), a differential equation containing one or more derivatives of the dependent variable. It may also contain functions of both independent and dependent variables.

3.1. **Euler' Idea for Solving an ODE: Marching Method.** Augustine Louis Cauchy proves convergence of the Euler's idea in 1824. In this proof, Cauchy uses the implicit Euler method. Incidentally, Taylor's theorem, published in 1714, enables one to approximate $y(t+h)$ in terms of $y(t)$ and its derivatives. However, the idea behind Euler's Method is to use the concept of local linearity to join multiple small line segments so that they make up an approximation of the actual curve of $y(t)$ versus t. This concept is illustrated for a well posed problem in the (Figure 2) and (Figure 3). Euler's method is based on the definition of derivative. Remember that dy/dt indicates slope of y versus t curve. Once the slope, $f(t, y(t))$ is known ordinate y_1 can be calculated by multiplying the slope with the equidistant step size . In this way one marches step by step till the time interval is covered. For this reason Euler's method is sometimes called Marching Method.

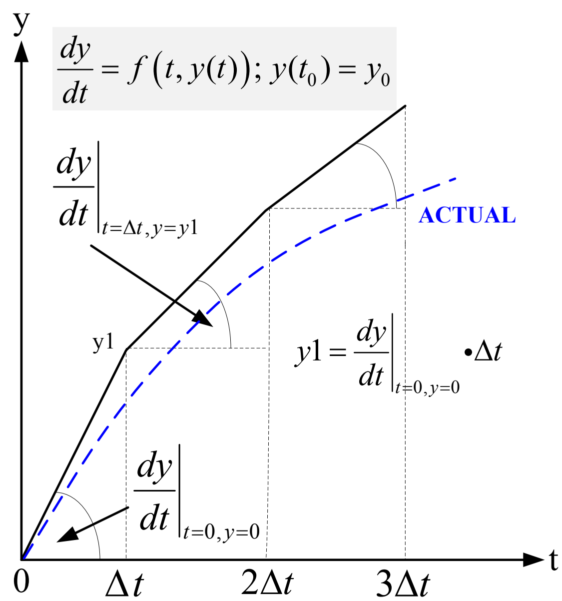

FIGURE 2. Illustration of Euler Algorithm

3.2. **Apparently Known.** At this point it is important to note a couple of points. Before starting the numerical exercise it is important to check whether the problem is a well posed one or not. For a function $f(t, y)$ to be well posed, it is necessary that it is continuous in the interval and satisfies Lipschitz's condition in the variable on the interval (Appendix A). First, the equidistant time step should be small enough so that successive steps can march along the actual curve. On doing so a very large number of N of time steps will be necessary to cover the specified time interval. Not only this approach is inefficient and expensive but truncation and round off errors will increase with N, thereby resulting in poor accuracy. In practice, it is therefore necessary to use as large a step as possible. In this connection backward Euler's method with larger step size can be used without affecting numerical stability. But it may take longer time for computation due to its implicit character. Secondly, one needs to consider the importance of the initial condition. To appreciate consider the following differential equation:

$$\frac{dy}{dt} = 4y - 5e^{-t}, \ i.e. \ \ y(t) = e^{-t} + Be^{+4t}$$

Here B is integration constant. Note that here the dominant part is e^{4t}.Thus the initial condition determines the nature of the solution to a large extent. Moreover, any error in the computation is likely to affect the solution to very adversely.

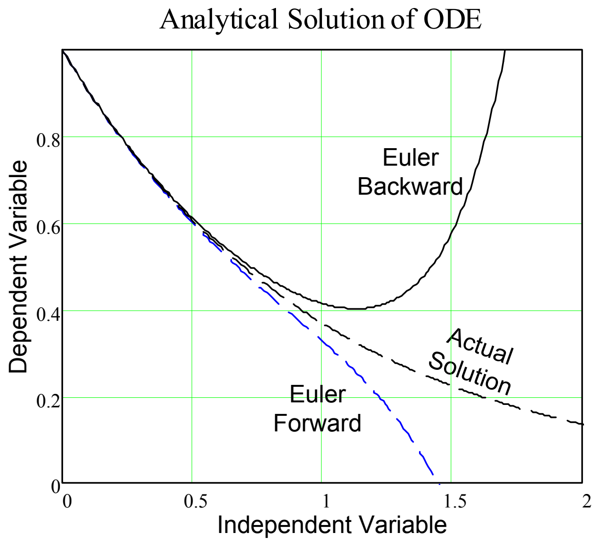

Figure 3. Analytical Solution of ODE

Numerical methods are of course approximate, but with good care, the approximations can be made to be so good that the values calculated by an analytical method and by numerical method are virtually identical. The advanced numerical methods are themselves complicated but do yield results which cannot be obtained by other simple methods. But the principle of all the methods can be illustrated using simple methods.

4. Charging of a Capacitor

In order to illustrate the objectives of the paper, we consider a simple practical problem of charging a capacitor through a resistance as shown in the (Figure 4).

It is required to plot the variation of voltage $V(t)$ across the capacitance with time. It can be done in two ways by hardware experiment or by numerical experiment, based on computer simulation program. The behavior of the system is governed by

$$E = RC\frac{dV}{dt} + V(t) \tag{1}$$

with the initial condition $V(0) = 0$.
Note that this is the governing equation when an ideal capacitor is used. But capacitors with dielectrics in between the plates have two types of losses. One is a conduction loss, representing the flow of actual charge through the dielectric. The other is a dielectric loss due to movement of the atoms or molecules in a varying

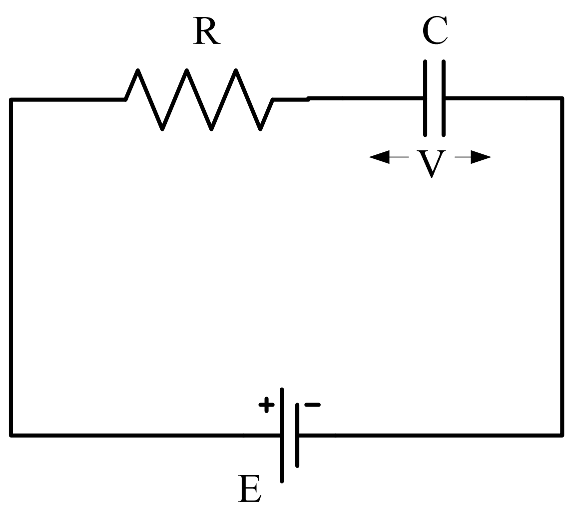

FIGURE 4. Charging of Capacitor

electric field. To account for these two effects a capacitance C is represented as

$$C = C_1 - jC_2$$

Moreover a capacitor with dielectric material varies with temperature. Thus C is to be represented with $C(T)$. This may make a capacitance time dependent, i.e. $C(t_1, T)$. Now the form of the simplified differential equation that governs the law of charging a condenser that can be expressed as

$$\frac{dy}{dt} = k\left[1 - y(t)\right]; y(0) = 0 \tag{2}$$

where$y(t) = {}^{V(t)}/_{E}$ and $k = {}^{1}/_{RC}$. In this case it assumes that capacitor is not only an ideal one but also it is not time dependent.
It has an exact solution

$$y(t) = 1 - \exp(-kt) \tag{3}$$

5. Geometrical Interpretation

To illustrate this we consider the simple problem mentioned in (1), having exact analytical solution so that the pitfalls of the numerical algorithm can be visibly illuminated (FIGURE 5).

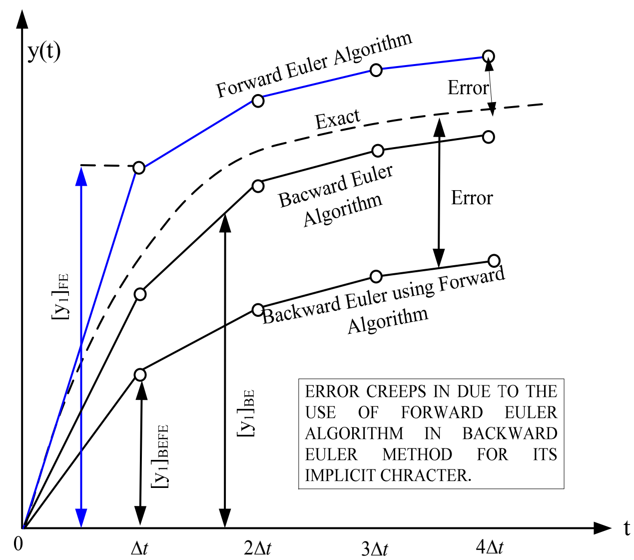

Figure 5. Geometrical Interpretation of Euler Algorithm

5.1. **The Euler's methods.** A typical form a first order ODE modelling a system is

$$\frac{dy}{dt} = f\left[t, y(t)\right], y(t_0) = y_0$$

That is, discretizing Euler writes

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} \cong f\left[t, y(t)\right] \tag{4}$$

$$y(t+\Delta t) \cong y(t) + \Delta t . f\left[t, y(t)\right] \tag{5}$$

Equation (4) merely reflects Euler's idea and provides a tangent line approximation to the first derivative. That is the unknown at each step is written explicitly in terms of previously computed values

$$and, \frac{y(t+\Delta t) - y(t)}{\Delta t} \cong f\left[t+\Delta t, y(t+\Delta t)\right] \tag{6}$$

$$i.e., y(t+\Delta t) = y(t) + \Delta t f\left[t+\Delta t, y(t+\Delta t)\right] \tag{7}$$

The second one, namely (6) or (7) is known as Implicit Finite Difference Euler Method in the sense that both the left hand side and right hand side of (6) and (7) include the unknown variable $y(t+\Delta t)$. Thus to find $y(t+\Delta t)$, it is necessary to import some other technique, like Newton-Rapson Method. The former one (4) is known as forward Euler's method or Explicit finite difference method in the sense that the left hand side of (5), namely $y(t+\Delta t)$, can be found by knowing the variable

in the earlier instant, namely, $y(t+\Delta t)$. Referring FIGURE 4, it is appreciated that $[y_1]_{FC}$ is obtained by multiplying the slope of the curve obtained from (2) with the tune step Δt, then we can repeat the process (as shown in FIGURE 5). Thus to obtained the corresponding for Backward Euler Method, we substitute $[y_1]_{FC}$ in the ODE to obtain the slope for Backward Euler Method. Thus knowing this slop, we get $[y_1]_{BE}$ by multiplying with Δt. Following the similar procedure and using (7) one can obtained $[y_1]_{BEFC}$.
It appears or it is obvious that in order to obtain reasonable accuracy, the step size must be made very small. On doing so a very large number of N of time steps will be necessary to cover the specified time interval. Not only this approach is inefficient and expensive but truncation and round off errors will increase with N, thereby resulting in poor accuracy. In practice, it is therefore necessary to use as large a step as possible. Thus in order to simplify the procedure of evaluation, Euler's Forward Method is imported in the use of Euler's Explicit Method, and (7) is written as

$$y(t+\Delta t) \cong y(t) + \Delta t f\left[t+\Delta t, y(t) + \Delta t f(t, y(t)\right] \tag{8}$$

If we add (5) and (7) then one gets

$$y(t+\Delta t) = y(t) + \frac{\Delta t}{2}\left[f\left(y(t)\right) + f\left(y(t+\Delta t)\right]\right. \tag{9}$$

This can be approximated as

$$y(t+\Delta t) = y(t) + \frac{\Delta t}{2}\left[f\left(y(t)\right) + f\left(y(t) + \Delta t f\left[t+\Delta t, y(t+\Delta t)\right]\right)\right] \tag{10}$$

Now the (10) gives the method of modified Euler method. Thus, to avoid difficulty in implementing Euler's Backward / Implicit Method, we approximate it to an explicit one. This turns the merits of Euler's Backward Method into demerits. It is likely that the students are led to misconception that Backward Euler's method is inferior to Forward Euler method. Moreover, solving numerically one has to be careful not to be misled by the outcome. Because errors may enters in many ways, e.g., wild simplification of computation methods, round-off errors, use of large step size to reduce computational cost, etc. Our purpose here is to illuminate these potholes by choosing simple problems and supplementing with geometrical / graphical method.

5.2. **Geometrical approach to analytical solution.** Refer to (FIGURE 6) which plots y vs. t using the relation (8). From the figure one easily writes:

$$y_1 = k\Delta t$$
$$y_2 = k\Delta t + (1-k\Delta t).k\Delta t$$
$$y_3 = k\Delta t + (1-k\Delta t).k\Delta t + (1-k\Delta t)^2.k\Delta t$$

Thus

$$y_n = k\Delta t + (1-k\Delta t)k\Delta t + (1-k\Delta t)^2.k\Delta t + \dots\dots\dots\dots + (1-k\Delta t)^{n-1}.k\Delta t \tag{11}$$

$$(1-k\Delta t)y_n = k\Delta t.(1-k\Delta t) + (1-k\Delta t)^2 k\Delta t \dots\dots\dots\dots\dots + (1-k\Delta t)^n.k\Delta t$$

Therefore

$$y_n = 1-(1-k\Delta t)^n \tag{12}$$

i.e.,

$$y(n.\Delta t) = 1 - \left[(1-k\Delta t)^{\frac{1}{k\Delta t}}\right]^{kn.\Delta t}$$

When $\Delta t \to 0$, and $n \to \infty$, then using Sterling approximation it is easily shown that

$$y(t) = 1 - \exp(-kt) \tag{13}$$

Incidentally, a student can appreciate the significance of Sterling approximation,

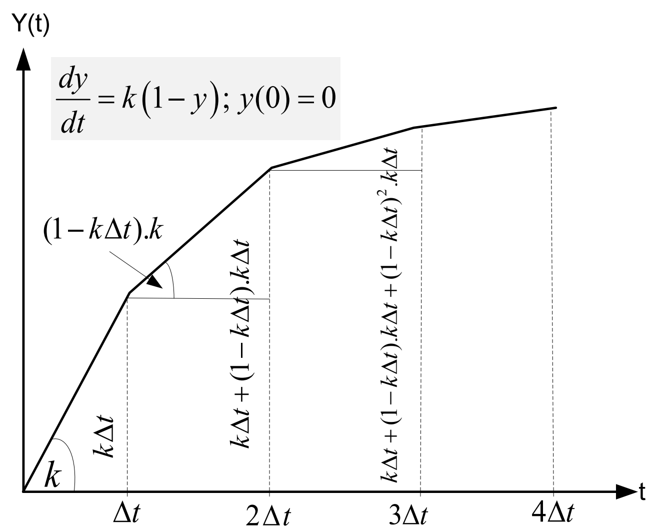

FIGURE 6. Graphical Solution of an ODE

viz, $\lim_{x\to 0}(1-x)^{\frac{1}{x}} = e^{-1}$ by plotting $(1-x)^{\frac{1}{x}}$ against x for small values (Appendix A, Fig A1). Equation (13) is exactly the same as given by (3). At this it is important to note that it may not be possible always to use the geometrical mean to arrive at the result.

6. Limitation in Euler Backward and Trapezoidal Algorithm

It is to be noted that the FIGURE 7 and FIGURE 8 does not truly reflect the advantages or disadvantages of the backward Euler method because in order to make these methods we have to approximate the "y_{n+1}" inside the function term $f(y_{n+1})$ by using forward Euler method. The effect is illustrated in FIGURE 7.

This can be illustrated as follows. Consider the first order ODE with following initial conditions, viz.., at $t = 0$, $y = y_0$.

$$\frac{dy}{dt} = f(y) \tag{14}$$

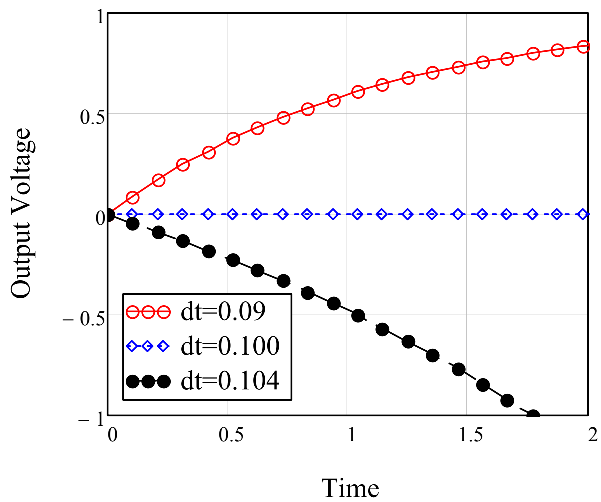

FIGURE 7. Backward Euler using Forward Euler

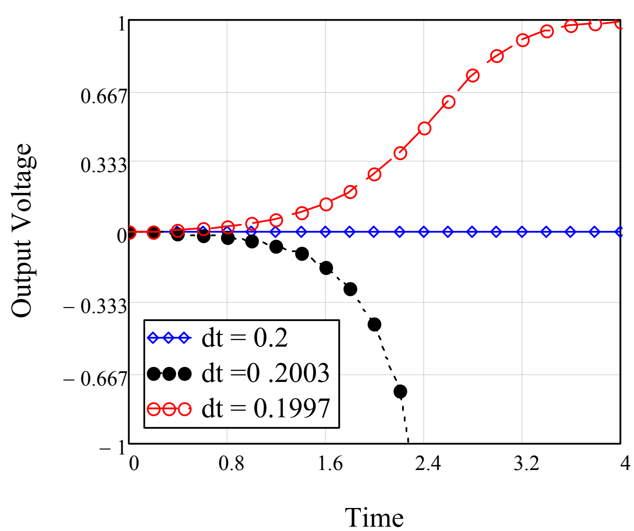

FIGURE 8. Trapezoidal Euler method using forward and backward algorithm

In the case of backward Euler algorithm we can write

$$y_1 = y_0 + f\left[y_0 + f(y_0)\Delta t\right]\Delta t$$

For the method to follow through

$$y_0 = y_0 + f\left[y_0 + f(y_0)\Delta t\right]\Delta t$$

That is

$$f\left[y_0 + f(y_0)\Delta t\right] = 0 \tag{15}$$

From (15) we can easily find the maximum value Δt for the algorithm will fall through. For example, in our case, $f(y) = 10(1 - y)$. Therefore,

$$\begin{array}{l} f(y_0) = 10(1-0) \\ (1 - 0 - 10\Delta t) = 0 \end{array}$$

Therefore

$$\Delta t = 0.1 \tag{16}$$

Now in the case of trapezoidal method we can write

$$y_1 = y_0 + \left[f(y_0) + f\left[y_0 + f(y_0)\Delta t\right]\right]{}^{\Delta t}\!/\!_2$$

For the method to follow through

$$y_0 = y_0 + \left[f(y_0) + f\left[y_0 + f(y_0)\Delta t\right]\right]{}^{\Delta t}\!/\!_2$$

So that

$$f(y_0) + f\left[y_0 + f(y_0)\Delta t\right] = 0 \tag{17}$$

Considering the same example we get the maximum value of $\Delta t = 0.2$ after which the trapezoidal algorithm (nothing but an average of forward and backward Euler methods) will fall though (Figure 8). For the particular value of Δt (0.1 for backward and 0.2 for trapezoidal) the effect of round-off or truncation error is nil as because for this particular value of Δt the system becomes unstable. Simulation results confirm the theoretical results. It is clear from the Figure 8 that if the time step is increased beyond a certain value depending on the nature of the $f(y)$ the computation procedure falls through. This is because Euler backward method is used with the Euler forward method. Similar situation is observed also in case of a non-linear ODE (Appendix B). If we use Newton Rapshon or any other method to calculate then this effect does not appear. Figure 9, (Figure 10 justifies this. It is further found that deliberate introduction of round off and truncation errors do not affect much to the nature of Figure 7 and Figure 8 . Figure 11 shows three different solutions using three different Euler algorithms along with the exact solution in case of capacitor charging problem. From figure (Figure 10) it is clear that Backward Euler algorithm gives the better result. From the numerical solutions (Figure 9 , (Figure 10 and Figure 11) it is seen that the backward Euler method embedded with forward Euler method is inferior to forward Euler algorithm if the time steps are not large, so for the transient response is concerned. Even the stability zone is becomes small (Figure 14).

7. Stability Zone

All dielectrics (except vacuum) have two types of losses. One is a conduction loss, representing the flow of actual charge through the dielectric. The other is a dielectric loss due to movement or rotation of the atoms or molecules in an alternating electric field. Dielectric losses in water are the reason for food and drink getting hot in a microwave oven. One way of describing dielectric losses is to consider the permittivity as a complex number. Considering the most general case

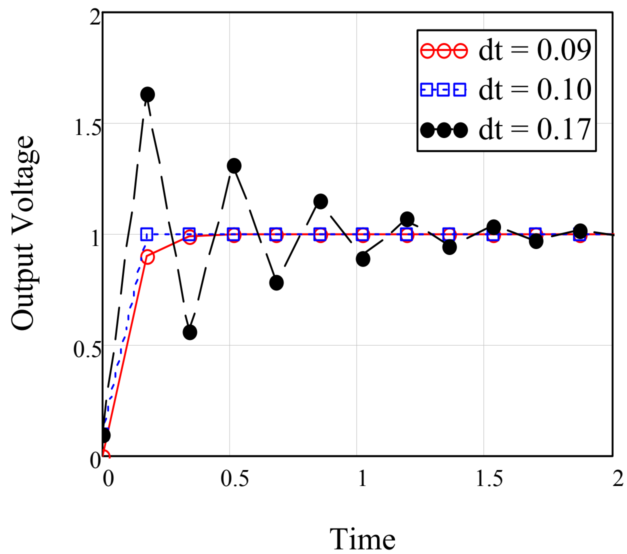

FIGURE 9. Numerical Solution of ODE for capacitor charging using Forward Euler algorithm.

when the capacitance is placed by a complex one due to the complex nature of the dielectric constant: we can write

$$k = \frac{1}{RC} = s = \sigma + j\omega \tag{18}$$

Referring to (2), (4) and (5) we can write

$$V(n\Delta t) = E\left[1 - \left(1 - \frac{\Delta t}{RC}\right)^n\right] \tag{19}$$

From the physical condition of the problem, it is at once seen that

$$\lim_{n\Delta t\to\infty} V(n\Delta t) = E \tag{20}$$

Therefore, using (19) and (20) it is found for numerical stability

$$\left|1 - \frac{\Delta t}{RC}\right| < 1 \tag{21}$$

we find for numerical stability

$$\begin{gathered} |(1 - s\Delta t)| < 1 \\ (1 + \sigma\Delta t)^2 + (\omega\Delta t)^2 < 1 \end{gathered} \tag{22}$$

The stability boundary is shown in FIGURE 12. This is for the forward Euler Algorithm. Referring to (2) and (7) we can write

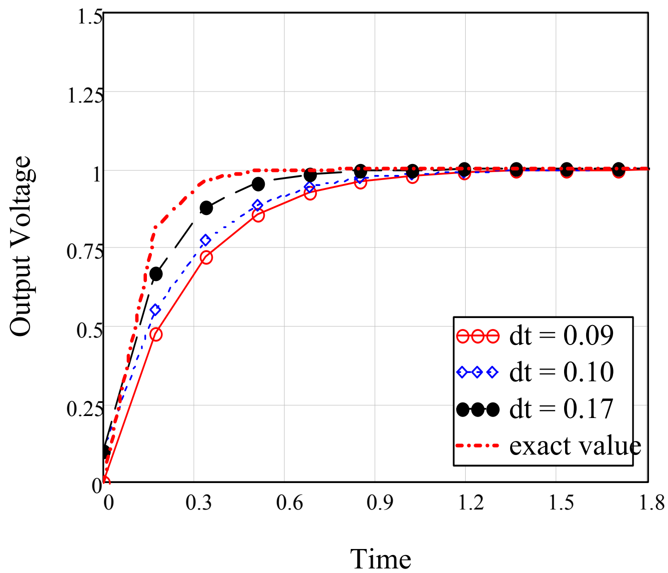

FIGURE 10. Numerical Solution of ODE for capacitor charging using Backward without Forward Euler.

$$V(n\Delta t) = E\left[1 - \frac{1}{\left(1+\frac{\Delta t}{RC}\right)^n}\right]$$

For stability of solution it is necessary

$$|1 + s\Delta t| > 1 \tag{23}$$

As the boundary condition of the problem suggests that the charge across the condenser must decay to zero in the steady state ($n \to \infty$) and so also

$$x(n\Delta t)$$

is also zero in the steady state. Noting that

$$s = \sigma \pm j\omega \tag{24}$$

$$(1 + \sigma\Delta t)^2 + (\omega\Delta t)^2 > 1 \tag{25}$$

This is shown in FIGURE 13 . This is for Backward Euler Algorithm. Backward Euler Algorithm is absolutely stable in the exterior of the unit circle coated

$$\sigma\Delta t = -1 + j0$$

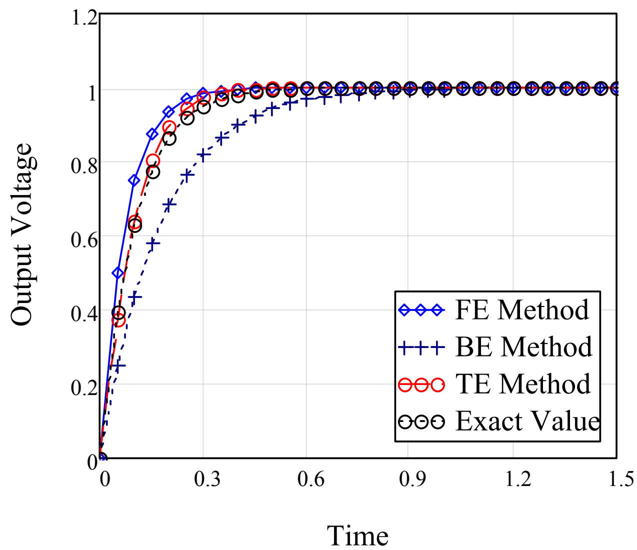

FIGURE 11. Numerical Solution of ODE for capacitor charging using three Euler algorithms.

7.1. **Nesting of forward Euler method in backward Euler method.** Using the forward Euler algorithm in backward Euler algorithm

$$V(n\Delta t) = E\left[1 - \left(1 - \frac{\Delta t}{RC} + \left(\frac{\Delta t}{RC}\right)^2\right)^n\right] \tag{26}$$

Incidentally, by adopting the limiting process of $n \to \infty$ and $\Delta t \to 0$, (26) can be written as

$$V(t) = \left(1 - e^{-\frac{t}{CR}}\right) \tag{27}$$

Actual significance of (27) for numerical approximation is easily appreciated by referring (Fig A1) [Appendix A]. For numerical stability it is required that

$$\left|1 - \frac{\Delta t}{RC} + \left(\frac{\Delta t}{RC}\right)^2\right| < 1$$

$$i.e., 0 < \frac{\Delta t}{RC} < 1$$

This is worse than that of forward Euler algorithm. For deterring the stability boundary, let us put

$$\frac{1}{RC} = \sigma + j\omega$$

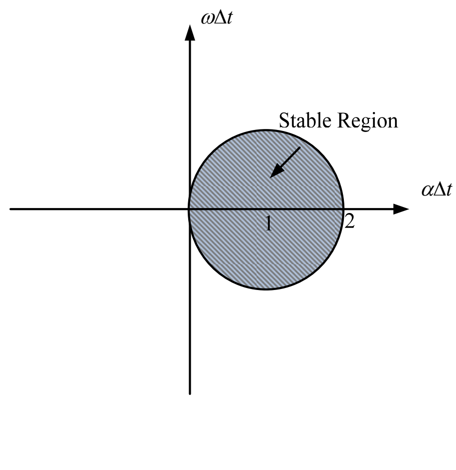

FIGURE 12. Stability region for Forward Euler algorithm.

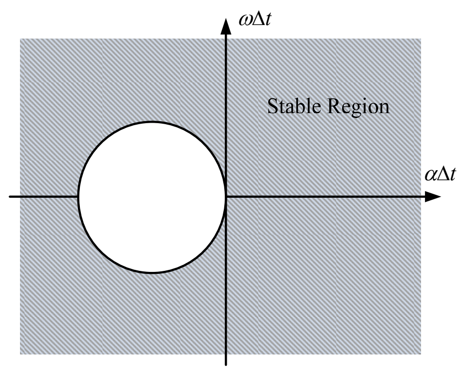

FIGURE 13. Stability region for Backward Euler algorithm

thus from (27)

$$\left|1-(\sigma+j\omega)\,\Delta t+((\sigma+j\omega)\,\Delta t)^2\right|<1$$

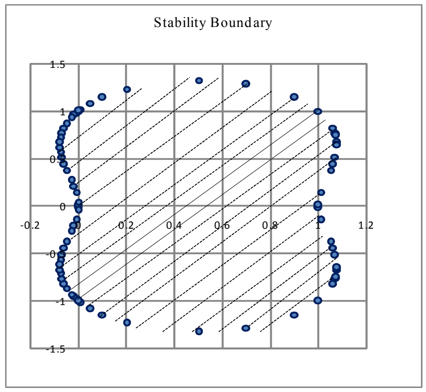

FIGURE 14. Stability region for backward Euler algorithm using forward algorithm.

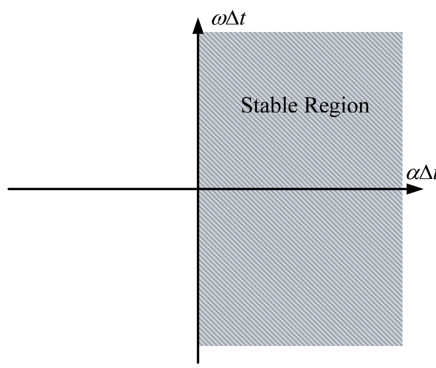

FIGURE 15. Stability region for Trapezoidal Euler algorithm.

The stability zone is shown in FIGURE 14 . shrinking the stability zone considerably. For a trapezoidal the corresponding equation for the stability is given by

$$Z = \frac{(1 - \frac{s}{2}.\Delta t)}{(1 + \frac{s}{2}\Delta t)} = \frac{\left[(1 - \frac{\sigma\Delta t}{2}) - \frac{j\omega\Delta t}{2}\right]}{\left[(1 + \frac{\sigma\Delta t}{2}) + j\frac{\omega\Delta t}{2}\right]}$$

$$|Z| = \sqrt{\frac{(1-\frac{\sigma\Delta t}{2})^2 + (\frac{\omega\Delta t}{2})^2}{(1+(\frac{\sigma\Delta t}{2})^2 + (\frac{\omega\Delta t}{2})^2}} \tag{28}$$

When $|Z| < 1$, $\sigma\Delta t\rangle 0$; When $|Z| = 1, \sigma\Delta t = 0$ and when $|Z|\rangle 1$, $\sigma\Delta t\langle 0$. Thus it is absolutely stable on the right half-plane in Figure 15.

8. Concluding Remarks

All the aspects of different Euler's methods have been looked into for initial value problems. It is found that when the range of integration is relatively short, relatively small step sizes can be used without excessive computing time. In such situation truncation and round-off-errors matter little. But when we consider another problem given by Hosking et al ([18]) the truncation and round-off-errors affects much the result (Appendix C). For such cases simple self-starting Euler's Methods can be used. But precautions should be taken about the initial condition of the problem depending on its nature. Moreover, before starting numerical exercise, it should be checked whether the problem is a well-posed one. When forward Euler algorithm is embedded in Euler's Backward Method, and relatively large step sizes are used, Backward Euler's methods and Trapezoidal Methods become unstable. These aspects need to be carefully looked into before applying such self-starting algorithms, even to non-linear problems. The effect of truncation and round-off error has been illustrated in Appendix C.

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Appendix - A

Well-posed Problem Let us consider the following first order initial value problem:

$$\frac{dy}{dt} = f\left[t, y(t)\right], y(t_0) = y_0$$

Before starting the numerical exercise it is important to check whether the problem is a (**??**) well posed one or not. For a function to be well posed it is necessary that it is continuous in the interval and satisfies Lipschitzs condition in the variable on the interval

$$D = \{(t, y)\ |t_1 < t < t_2 \ and\ y_1 < y < y_2\}$$

Incidentally, Jacques Hadamard defines the term well posed problem is one that obeys the following: 1. A solution of the differential exists. 2. The solution is unique. 3. The solution's behaviour hardly changes when there is a slight change in the initial condition. Example: This example is taken from ([17]). Show that the following initial value problem

$$\frac{dy}{dt} = f\left(t, y\right);\ 0 \le t \le 2, \quad y(0) = 0.5$$

Where,

$$f(t, y) = y - t^2 + 1$$

is well posed on

$$D = \{(t, y)\ |0 \le t \le 2 \ and\ -\infty \le y \le \infty\}$$

Solution: Because

$$\left|\frac{\partial\left(y - t^2 + 1\right)}{\partial y}\right| = |1| = 1$$

Thus, $f(t, y)$ satisfies a Lipschitz condition in y on D in the variable y with Lipschitz constant.

Significance of Stirlings Approximations

Significance of Stirlings approximations to forward Euler method and backward Euler method embedding forward Euler method is shown in Figure 16. Referring (19) and (26) we can write for forward Euler method and backward Euler method embedding forward Euler method respectively

$$f_1(x) = (1 - x)^{\frac{1}{x}}$$

and

$$f_2(x) = \left(1 - x + x^2\right)^{\frac{1}{x}}$$

where

$$x = \frac{\Delta t}{RC}$$

Appendix - B

Non-linear Differential Equation

Now we consider the example of synchronizing an oscillator using the phase-lock technique. Now the general form of synchronizing equation (16) is given as

$$\frac{d\phi}{dt} = \Omega - k\sin\phi$$

where, ϕ = phase difference between the local oscillator and the external signal,
Ω = initial frequency error between the two oscillations,
k = locking range. Let us apply Euler algorithm to realize the numerical solution of the non-linear ordinary differential equation for $\Omega < k$. Forward Euler algorithm:

$$\phi_{n+1} = \phi_n + \Delta t\,(\Omega - k\sin\phi_n)$$

Now we assume that $\{\phi_0 = 0$ Backward Euler algorithm:

$$\phi_{n+1} = \phi_n + \Delta t\,(\Omega - k\sin\phi_{n+1})$$

Nesting Backward Euler algorithm with Forward Euler algorithm we can write

$$\phi_{n+1} = \phi_n + \Delta t\,(\Omega - k\sin(\phi_n + \Delta t\,(\Omega - k\sin\phi_n)))$$

For the particular value of time step the effect of round-off or truncation error is nil as because for this particular value of Δt the system becomes unstable. Simulation results confirm the theoretical results. It is clear from the FIGURE 17 that if the time step is increased beyond a certain value depending on the nature of the $f(y)$ the computation procedure falls through.

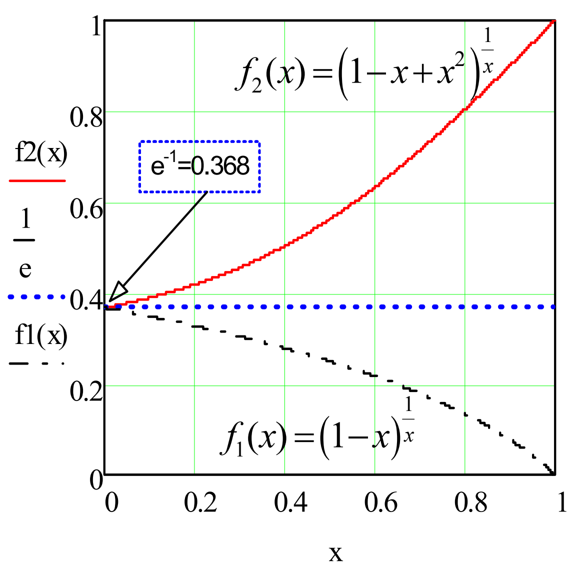

FIGURE 16. Significance of Stirlings Approximations to Forward Euler Method and Backward Euler Method embedding Forward .

Appendix - C

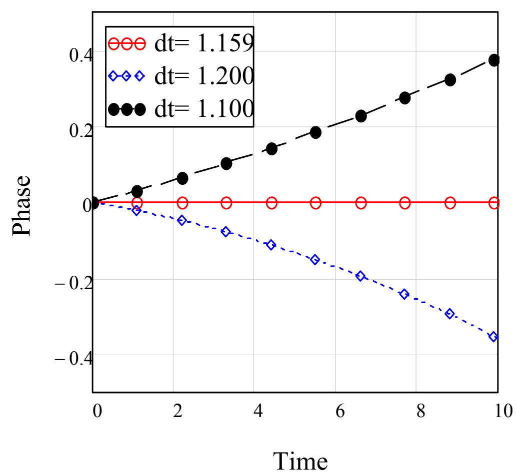

FIGURE 17. Nesting Backward Euler algorithm with Forward Euler algorithm.

Let us consider the ordinary differential equation of the form ([18]-[19])

$$\frac{dy}{dt} = \frac{t+y}{t-y}$$

, with initial condition (1) = 0 . This is an excellent example for the beginners in the sense that (i) It has two values of the dependent variable for a particular value of the independent variable within the solution range. As a result, the entire solution of this type of problem cannot be obtained with the marching methods. (ii) The solution of this equation shows vividly the effect of truncation and round-off errors (FIGURE 18 to FIGURE 21). (iii) It demonstrates that small step size does not always give better result. (iv) Outside the solution boundary the marching methods become widely unstable.

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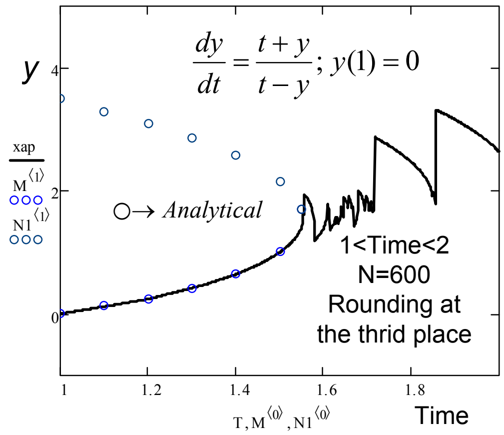

FIGURE 18. Effect of round-off error for time steps 600.

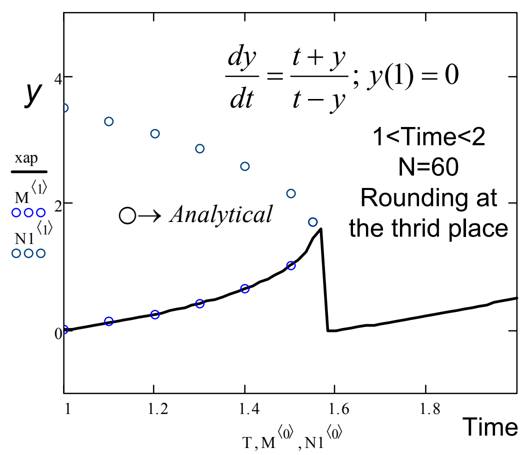

FIGURE 19. Effect of round-off error for time steps 60.

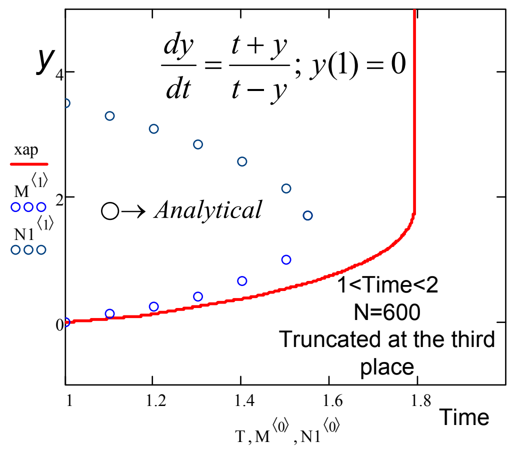

FIGURE 20. Effect of truncation for time steps 600.

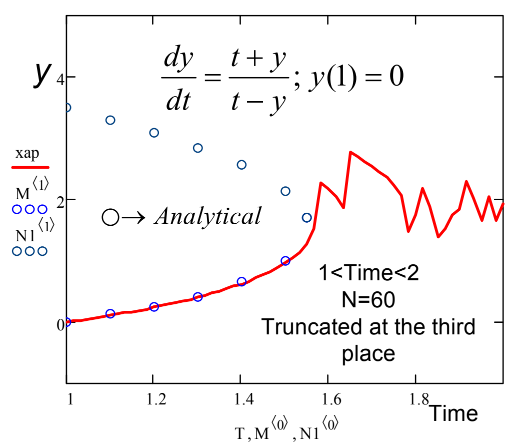

FIGURE 21. Effect of truncation for time steps 60.

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