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THE UNIFIED SYSTEM BETWEEN LORENZ AND CHEN SYSTEMS: A DISCRETIZATION PROCESS

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Abstract. The unified chaotic system that contains the Lorenz and Chen systems was introduced. In this paper we are interested in discretizing that system. Fixed points and their asymptotic stability of the discrete system obtained are investigated. Some dynamical behaviors such as chaotic attractor, bifurcation and chaos are discussed.

1. Introduction

It was notable that both Lorenz and Chen system share some common properties such as they have the same symmetry, stability and dissipativity. In [8], a unified chaotic system was introduced that contains the Lorenz and Chen system as two dual systems as the two extremes to its spectrum. On the other hand, Some examples of dynamical systems generated by piecewise constant arguments have been studied in [2]-[3]. Here we propose a discretization process to obtain the discrete version of that unified system.

Consider the unified chaotic system

$$\begin{aligned} x^{\cdot} &= (25\beta+10)(y-x), \qquad\qquad t\in(0,T], \\ y^{\cdot} &= (28-35\beta)x - xz + (29\beta-1)y, \\ z^{\cdot} &= xy - \frac{\beta+8}{3}z, \end{aligned} \tag{1.1}$$

where $\beta \in [0,1]$.

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Meanwhile, consider the corresponding system with piecewise constant arguments given by

$$\begin{aligned} x(t) &= (25\beta+10)(y(r[\frac{t}{r}]) - x(r[\frac{t}{r}])), \\ y(t) &= (28-35\beta)x(r[\frac{t}{r}]) - x(r[\frac{t}{r}])z(r[\frac{t}{r}]) + (29\beta-1)y(r[\frac{t}{r}]), \\ z(t) &= x(r[\frac{t}{r}])y(r[\frac{t}{r}]) - \frac{\beta+8}{3}z(r[\frac{t}{r}]). \end{aligned} \tag{1.2}$$

Let $t \in [0, r)$, then $[\frac{t}{r}] = 0$, and the solution of (1.2) is given by

$$\begin{aligned} x_1(t) &= x_o + t((25\beta+10)(y_o - x_o)), \quad t \in [0, r) \\ y_1(t) &= y_o + t((28-35\beta)x_o - x_o z_o + (29\beta-1)y_o), \quad t \in [0, r) \\ z_1(t) &= z_o + t(x_o y_o - \frac{\beta+8}{3}z_o), \quad t \in [0, r). \end{aligned}$$

Let $t \in [r, 2r)$, then $[\frac{t}{r}] = 1$, and the solution of (1.2) is given by

$$\begin{aligned} x_2(t) &= x_1(r) + (t-r)((25\beta+10)(y_1 - x_1)), \quad t \in [0, r) \\ y_2(t) &= y_1(r) + (t-r)((28-35\beta)x_1 - x_1 z_1 + (29\beta-1)y_1), \quad t \in [0, r) \\ z_2(t) &= z_1(r) + (t-r)(x_1 y_1 - \frac{\beta+8}{3}z_1), \quad t \in [0, r). \end{aligned}$$

Repeating the process we get

$$\begin{aligned} x_{n+1}(t) &= x_n(nr) + (t-nr)((25\beta+10)(y_n(nr) - x_n(nr))), \quad t \in [r, 2r) \\ y_{n+1}(t) &= y_n(nr) + (t-nr)((28-35\beta)x_n(nr) - x_n(nr)z_n(nr) + (29\beta-1)y_n(nr)), \\ z_{n+1}(t) &= z_n(nr) + (t-nr)(x_n(nr)y_n(nr) - \frac{\beta+8}{3}z_n(nr)), \end{aligned}$$

as $t \to (n+1)r$,

$$\begin{aligned} x_{n+1}((n+1)r) &= x_n(nr) + r((25\beta+10)(y_n(nr) - x_n(nr))), \quad t \in [nr, (n+1)r) \\ y_{n+1}((n+1)r) &= y_n(nr) + r((28-35\beta)x_n(nr) - x_n(nr)z_n(nr) + (29\beta-1)y_n(nr)), \\ z_{n+1}((n+1)r) &= z_n(nr) + r(x_n(nr)y_n(nr) - \frac{\beta+8}{3}z_n(nr)). \end{aligned}$$

That is

$$\begin{aligned} x_{n+1}((n+1)r) &= x_n + r((25\beta+10)(y_n - x_n)), \quad t \in [nr, (n+1)r) \\ y_{n+1}((n+1)r) &= y_n + r((28-35\beta)x_n - x_n z_n + (29\beta-1)y_n), \\ z_{n+1}((n+1)r) &= z_n + r(x_n y_n - \frac{\beta+8}{3}z_n). \end{aligned} \tag{1.3}$$

Moreover, if we consider the equations

$$\begin{aligned} x(t) &= (25\beta+10)(y(r[\frac{t-r}{r}]) - x(r[\frac{t-r}{r}])), \\ y(t) &= (28-35\beta)x(r[\frac{t-r}{r}]) - x(r[\frac{t-r}{r}])z(r[\frac{t-r}{r}]) + (29\beta-1)y(r[\frac{t-r}{r}]), \\ z(t) &= x(r[\frac{t-r}{r}])y(r[\frac{t-r}{r}]) - \frac{\beta+8}{3}z(r[\frac{t-r}{r}]). \end{aligned}$$

We can apply the same procedure to obtain the discretization of the second order difference equation

$$\begin{aligned} x_{n+1} &= x_n + r((25\beta+10)(y_{n-1}-x_{n-1})), \quad t \in [0, r) \\ y_{n+1} &= y_n + r((28-35\beta)x_{n-1} - x_{n-1}z_{n-1} + (29\beta-1)y_{n-1}), \\ z_{n+1} &= z_n + r(x_{n-1}y_{n-1} - \frac{\beta+8}{3}z_{n-1}). \end{aligned} \tag{1.4}$$

Remark 1. *It should be noticed that the system (1.3) can be obtained also by applying Euler's discretization method* [12]. *However, Euler's method fails in obtaining system (1.4).*

2. Fixed points and their asymptotic stability

Now we study the asymptotic stability of the fixed points of the system (1.3) which has three fixed points if $(\beta+8)(9-2\beta) > 0$:

- $fix_1 = (0, 0, 0)$
- $fix_2 = (\sqrt{(\beta+8)(9-2\beta)}, \sqrt{(\beta+8)(9-2\beta)}, 27-6\beta)$
- $fix_3 = (-\sqrt{(\beta+8)(9-2\beta)}, -\sqrt{(\beta+8)(9-2\beta)}, 27-6\beta)$

The last two fixed points are symmetrically placed with respect to $z-$axis.

By considering a Jacobian matrix for one of these fixed points and calculating their eigenvalues, we can investigate the stability of each fixed point based on the roots of the system characteristic equation [7].

Linearizing the system (1.3) about fix_1 yields the following characteristic equation

$$P(\lambda) \equiv \lambda^3 + (B+C-1)\lambda^2 + (BC-B+A)\lambda + (AC-A-D) = 0,$$

where:
$A = 1 + r(4\beta - 11) - r^2(25\beta+10)(29\beta-1)),$
$B = -2 - r(-54\beta - 11) + r^2(25\beta+10)(28-35\beta),$
$C = r\frac{\beta+8}{3},$
$D = r^3(25\beta+10)(28-35\beta)(\frac{\beta+8}{3}) - r^2(25\beta+10)(28-35\beta).$

Now let $a_1 = (B+C-1)$, $a_2 = (BC-B+A)$, $a_3 = (AC-A-D)$. From the Jury test, if $P(1) > 0$, $P(-1) < 0$, and $a_3 < 1$, $|b_3| > b_1$, $c_3 > |c_2|$, where $b_3 = 1-a_3^2$, $b_2 = a_1 - a_3a_2$, $b_1 = a_2 - a_3a_1$, $c_3 = b_3^2 - b_1^2$, and $c_2 = b_3b_2 - b_1b_2$, then the roots of $P(\lambda)$ satisfy $\lambda < 1$ and thus fix_1 is a asymptotically stable.

While linearizing the system (1.3) about fix_2 or fix_3 yields the following characteristic equation

$$F(\lambda) \equiv \lambda^3 + a_{11}\lambda^2 + a_{22}\lambda + a_{33} = 0,$$

where:
$a_{11} = 3 - 11r + 4\beta r,$

$a_{22} = 25.3\beta r^3 + 31\beta^2 r^2 + 315\beta r^2 + 717.3\beta r + 29.6r - 59.3r^2 - 7.3r^3 - 3,$
$a_{33} = -r^2(25\beta + 10)(1 - 29\beta)(1 - r(\frac{\beta+8}{3}) - r^3(25\beta + 10)(\beta + 8)(9 - 2\beta),$

From the Jury test, if $F(1) > 0$, $F(-1) < 0$, and $a_{33} < 1$, $|b_{33}| > b_{11}$, $c_{33} > |c_{22}|$, where $b_{33} = 1 - a_{33}^2$, $b_{22} = a_{11} - a_{33}a_{22}$, $b_{11} = a_{22} - a_{33}a_{11}$, $c_{33} = b_{33}^2 - b_{11}^2$, and $c_{22} = b_{33}b_{22} - b_{11}b_{22}$, then the roots of $F(\lambda)$ satisfy $\lambda < 1$ and thus fix_2 or fix_3 is a asymptotically stable.

3. Attractors, bifurcation and chaos

In this section we show by numerical experiments that the dynamical behavior of the dynamica systems (1.3) and (1.4) is strongly affected by the change in β. In all simulations we take $r = 0.01$
Take $\beta = -4.2$ in (1.3) (Figure (1)).
Take $\beta = -2.5$ in (1.3) (Figure (2)).
Take $\beta = 0$ in (1.3) (Figure (3)).
Take $\beta = 0.3$ in (1.3) (Figure (4)).
Take $\beta = -1.5$ in (1.4) (Figure (5)).
Take $\beta = -1$ in (1.4) (Figure (6)).
Take $\beta = -0.9$ in (1.4) (Figure (7)).
Take $\beta = -0.7$ in (1.4) (Figure (8)).

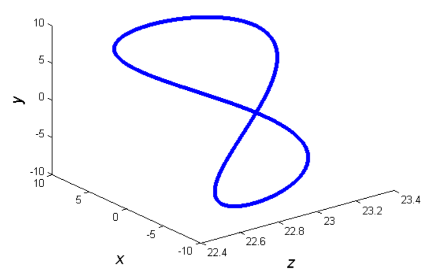

Figure 1. Regular attractor of (1.3) with $\beta = -4.2$

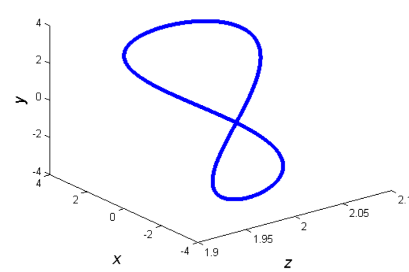

Figure 2. Regular attractor of (1.3) with $\beta = -2.5$

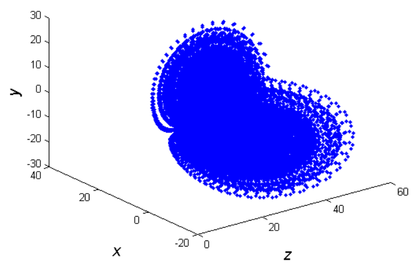

FIGURE 3. Chaotic attractor of (1.3) with $\beta = 0$

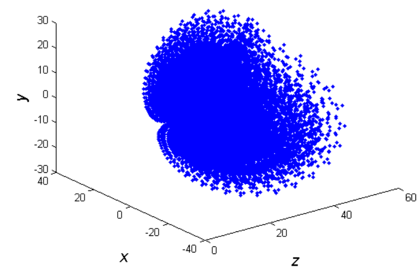

FIGURE 4. Chaotic attractor of (1.3) with $\beta = 0.3$

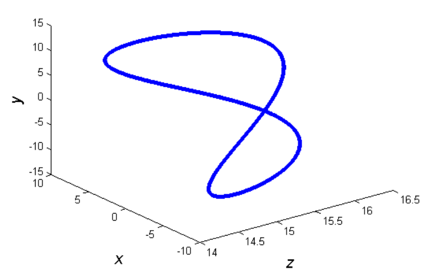

FIGURE 5. Regular attractor of (1.4) with $\beta = -1.5$

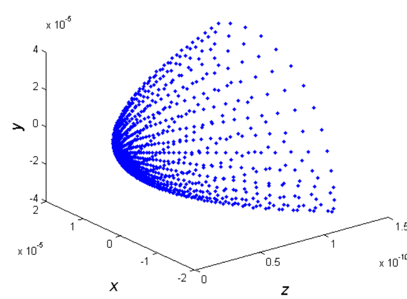

FIGURE 6. Chaotic attractor of (1.4) with $\beta = -1$

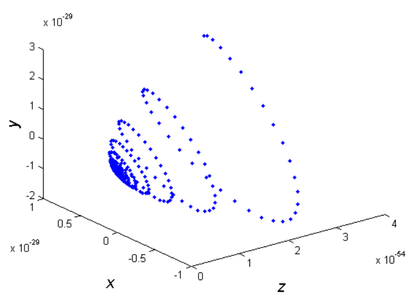

FIGURE 7. Chaotic attractor of (1.4) with $\beta = -0.9$

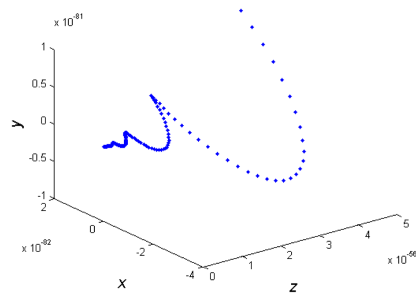

FIGURE 8. Chaotic attractor of (1.4) with $\beta = -0.7$

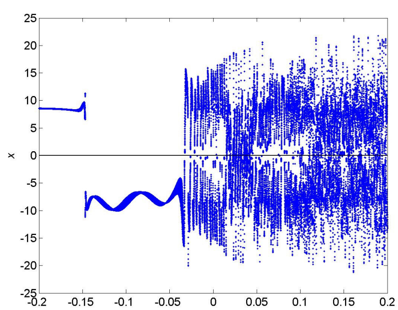

FIGURE 9. Bifurcation diagram of (1.3) as a function of x

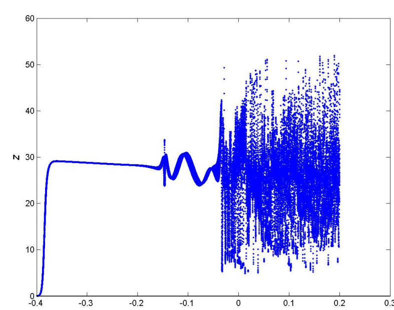

FIGURE 10. Bifurcation diagram of (1.3) as a function of z

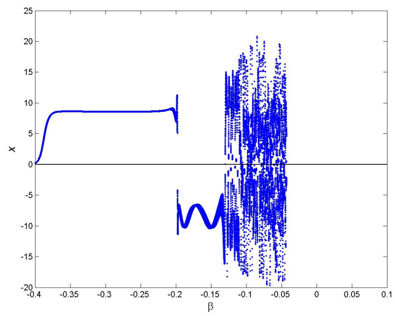

FIGURE 11. Bifurcation diagram of (1.4) as a function of x

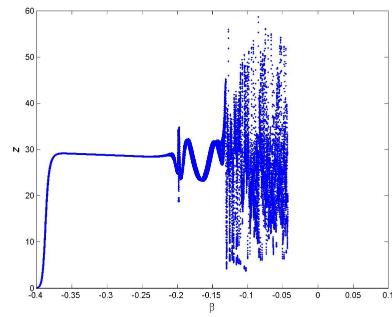

FIGURE 12. Bifurcation diagram of (1.4) as a function of z

4. Conclusion

A discretization process is applied to descretize the unified dynamical system between the Lorenz and Chen systems with piecewise constant arguments. We obtained first and second difference equations of the unified system. Euler's discretzation method was able to obtain first order difference equation while we were able here to obtain a second order difference equation. The range of the parameter β of the original system has been changed from $[0,1]$ to $[-0.9,0.3]$.

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