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ON (C,2)(E,1) PRODUCT MEANS OF FOURIER SERIES AND ITS CONJUGATE SERIES

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Abstract. This paper introduces the concept of (C,2)(E,1) product operators and establishes two new theorems on (C,2)(E,1) product summability of Fourier series and its conjugate series. The results obtained in the paper further extend several known results on linear operators.

1. Introduction

A study of Nörlund (N, p_n) summability of Fourier series and its conjugate series was made by Jadia [12], Pandey [22] and Singh [23]. Khare [13] studied generalized Nörlund (N, p, q) summability of Fourier Series and its conjugate series. (N, p, q) method includes (N, p_n) method of summability as a special case. Singh & Singh [24] studied Fourier series and its conjugate series using almost (N, p, q) summability method and Mittal & Kumar [16] used the method of matrix summability to study Fourier Series and its conjugate series. Matrix method includes (N, p_n) and (N, p, q) methods of summability as special cases. Thereafter, Chandra & Dixit [9] studied $|B|$ and $|E, q|$ and summability of Fourier series and its allied series.

Studies on trigonometric approximation of functions in L_p-norm using different linear operators such as Hölder, Nörlund, Riesz, Euler, Borel etc. were made by several researchers like Mohapatra & Sahney [18], Mohapatra & Chandra ([19], [20]), Holland, Mohapatra & Sahney [11], Chandra ([1], [2], [3], [4], [5], [6], [7], [8]) and Mohapatra & Russell [17].

Studies on degree of approximation of a function belonging to different class of functions by product summability methods were made by Lal & Singh [15] and Nigam [21].

The aim of the present paper is to study Fourier series and its conjugate series by product operators. The advantage of considering product operators over linear operators can be understood with the observation that the infinite series, which

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is neither summable by the left linear operators nor by the right linear operators individually, is summable to some number by the product operators obtained from the same linear operators placed in the same sequential order. Thus, the method of product operators is more powerful than the methods of individual linear operators. Moreover, in studies of error estimates $E_n(f)$ through Trigonometric Fourier Approximation (TFA), product operators give better approximation than individual linear operators.

Therefore, in this paper, (C,2)(E,1) product summability method is introduced and two theorems on (C,2)(E,1) summability of Fourier series and its conjugate series are established under a very general condition.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with s_n for its n^{th} partial sum.

Let $\{t_n^{(E,1)}\}$ denote the sequence of $(E,1)$ mean of the sequence $\{s_n\}$. If the $(E,1)$ transform of s_n is defined as

$$t_n^{(E,1)}(f;x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k(f;x) \to s \text{ as } n \to \infty \tag{1}$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by the $(E,1)$ method (Hardy [10]).

Let $\{t_n^{(C,2)}\}$ denote the sequence of $(C,2)$ mean of the sequence $\{s_n\}$. If the $(C,2)$ transform of s_n is defined as

$$t_n^{(C,2)}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)s_k(f;x) \to s \text{ as } n \to \infty \tag{2}$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by $(C,2)$ method (Cesàro method).

Thus if

$$t_n^{(C,2)(E,1)}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{n}{\nu} s_\nu(f;x) \to s \text{ as } n \to \infty, \tag{3}$$

where $\{t_n^{(C,2)(E,1)}\}$ denote the sequence of $(C,2)(E,1)$ product mean of the sequence s_n, the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by $(C,2)(E,1)$ method.
We observe that $(C,2)(E,1)$ method is regular.

Let f be a 2π-periodic and Lebesgue integrable function. The Fourier series associated with f at a point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{4}$$

with partial sums $s_n\,(f;x)$.

The conjugate series of Fourier series (4) of f is given by

$$\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)\equiv\sum_{n=1}^{\infty}B_n(x) \tag{5}$$

with partial sums $\tilde{s}_n\,(f;x)$.
Throughout this paper, we will call (5) as conjugate Fourier series of function f.

We use the following notations:

$$\phi\,(t)=\phi\,(x,t)=f\,(x+t)+f\,(x-t)-2f\,(x)$$
$$\psi\,(t)=\psi\,(x,t)=f\,(x+t)-f\,(x-t)$$
$$K_n\,(t)=\frac{1}{\pi\,(n+1)\,(n+2)}\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\left\{\left(\begin{array}{c}k\\ \nu\end{array}\right)\ \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]$$
$$\bar{K}_n\,(t)=\frac{1}{\pi\,(n+1)\,(n+2)}\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)\ \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\,(t/2)}\right]$$

2. Main Theorems

We prove the following theorems:

2.1. **Theorem 1.** Let $\{c_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that

$$C_n=\sum_{\nu}^{n}c_\nu\to\infty\ as\ n\to\infty.$$

If

$$\Phi\,(t)=\int_0^t|\phi\,(u)\ |\,du=o\left[\frac{t}{\alpha\left(\frac{1}{t}\right).C_\tau}\right]\ as\ t\to+0, \tag{6}$$

where $\alpha\,(t)$ is a positive, monotonic and non-increasing function of t and

$$\log\,(n+1)=O\left[\{\alpha\,(n+1)\}\ \ C_{n+1}\right],\ \text{as}\ n\to\infty \tag{7}$$

then the Fourier series (4) is summable $(C,2)\,(E,1)$ to $f\,(x)$.

2.2. **Theorem 2.** Let $\{c_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that

$$C_n=\sum_{\nu}^{n}c_\nu\to\infty\ as\ n\to\infty.$$

If

$$\Psi\,(t)=\int_0^t|\psi\,(u)\ |\,du=o\left[\frac{t}{\alpha\left(\frac{1}{t}\right)\ C_\tau}\right]\ as\ t\to+0, \tag{8}$$

where $\alpha\,(t)$ is a positive, monotonic and non-increasing function of t,

$$2^\tau\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)=O(n+1)(n+2) \tag{9}$$

and condition (7) holds then the conjugate Fourier series (5) is summable $(C,2)\,(E,1)$ to

$$\bar{f}\,(x) = -\frac{1}{2\pi}\int_0^{2\pi} \psi\,(t)\ \ cot\left(\frac{t}{2}\right)\ dt$$

at every point where this integral exists.

3. Lemmas

For the proof of our theorems, following lemmas are required:

3.1. **Lemma 1.** For $0 \leq t \leq \frac{1}{n+1},\ |K_n\,(t)| = O(n+1)$.

Proof. For $0 \leq t \leq \frac{1}{n+1}$, $\sin nt \leq n \sin t$

$$
\begin{aligned}
|K_n\,(t)| &\leq \frac{1}{\pi\,(n+1)\,(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\sum_{\nu=0}^{k}\left(\begin{array}{c} k \\ \nu \end{array}\right)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right| \\
|K_n\,(t)| &\leq \frac{1}{\pi\,(n+1)\,(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\sum_{\nu=0}^{k}\left(\begin{array}{c} k \\ \nu \end{array}\right)\frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right]\right| \\
&\leq \frac{1}{\pi\,(n+1)\,(n+2)}\left|\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\,(2k+1)\sum_{\nu=0}^{k}\left(\begin{array}{c} k \\ \nu \end{array}\right)\right]\right| \\
&= \frac{1}{\pi\,(n+1)\,(n+2)}\sum_{k=0}^{n}[(n-k+1)\,(2k+1)] \\
&= \frac{n+1}{\pi\,(n+1)\,(n+2)}\sum_{k=0}^{n}(2k+1) - \frac{1}{\pi\,(n+1)\,(n+2)}\sum_{k=0}^{n}[k\,(2k+1)] \\
&= \frac{1}{\pi(n+2)}\sum_{k=0}^{n}(2k+1) - \frac{1}{\pi\,(n+1)\,(n+2)}\left[2\sum_{k=0}^{n}k^2 + \sum_{k=0}^{n}k\right] \\
&= \frac{(n+1)^2}{\pi(n+2)} - \frac{1}{\pi\,(n+1)\,(n+2)}\left[\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}\right] \\
&= \frac{(n+1)^2}{\pi(n+2)} - \frac{n(2n+1)}{3\pi(n+2)} - \frac{n}{2\pi(n+2)} \\
&= \frac{2n^2+7n+6}{6\pi(n+2)} \\
&= O\,(n+1)
\end{aligned}
$$

3.2. **Lemma 2.** For $\frac{1}{n+1} \leq t \leq \pi,\ |K_n\,(t)| = O\left(\frac{1}{t}\right)$.

Proof. For $\frac{1}{n+1} \leq t \leq \pi$, applying Jordan's lemma, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $\sin nt \leq 1$.

$$
\begin{aligned}
|K_n(t)| &\leq \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right] \right| \\
&\leq \frac{(n+1)}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{1}{\left(\frac{t}{\pi}\right)} \right] \right| \\
&- \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{k}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{1}{\left(\frac{t}{\pi}\right)} \right] \right| \\
&= \frac{1}{t(n+2)} \left| \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \right] \right| \\
&- \frac{1}{t(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{k}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \right] \right| \\
&= \frac{1}{t(n+2)} \sum_{k=0}^{n} 1 - \frac{1}{t(n+1)(n+2)} \sum_{k=0}^{n} k \\
&= \frac{(n+1)}{t(n+2)} - \frac{n(n+1)}{2t(n+1)(n+2)} \\
&\leq \frac{n+1}{t(n+2)} - \frac{n}{2t(n+2)} \\
&= O\left(\frac{1}{t}\right)
\end{aligned}
$$

3.3. **Lemma 3.** For $0 \leq t \leq \frac{1}{n+1}$, $\bar{K}_n(t) = O\left(\frac{1}{t}\right)$.

Proof. For $0 \leq t \leq \frac{1}{n+1}$, $\sin(t/2) \geq (t/\pi)$ and $|\cos nt| \leq 1$

$$
\begin{aligned}
\left|\bar{K}_n(t)\right| &= \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{n-k+1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)} \right] \right| \\
&\leq \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} \left[\frac{n-k+1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{|\sin(t/2)|} \right] \\
&\leq \frac{1}{t(n+1)(n+2)} \sum_{k=0}^{n} \left[\frac{n-k+1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \right] \\
&= \frac{1}{t(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) \\
&= O\left(\frac{1}{t}\right)
\end{aligned}
$$

3.4. **Lemma 4.** For $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and any n,

$$\left|\bar{K}_n(t)\right| = O\left(\frac{1}{t}\right)$$

Proof. For $0 \leq \frac{1}{n+1} \leq t \leq \pi,\ \sin(t/2) \geq (t/\pi)$

$$\begin{aligned}
\left|\bar{K}_n(t)\right| &= \frac{1}{\pi(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right| \\
&\leq \frac{1}{t(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right| \\
&\leq \frac{1}{t(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right|\left|e^{\frac{it}{2}}\right| \\
&\leq \frac{1}{t(n+1)(n+2)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right| \\
&\leq \frac{1}{t(n+1)(n+2)}\left|\sum_{k=0}^{\tau-1}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right| \\
&+\ \frac{1}{t(n+1)(n+2)}\left|\sum_{k=\tau}^{n}\left[\frac{n-k+1}{2^k}\ \mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right|, \qquad (10)
\end{aligned}$$

where τ denoted the integral part of $\frac{1}{t}$.

Now considering first term of (10),

$$\begin{aligned}
&\frac{1}{t(n+1)(n+2)}\left|\sum_{k=0}^{\tau-1}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\ e^{i\nu t}\right\}\right]\right| \\
&\leq \frac{1}{t(n+1)(n+2)}\left|\sum_{k=0}^{\tau-1}\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ \right|\left|e^{i\nu t}\right| \\
&\leq \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{\tau-1}\left[\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ \right] \\
&\leq \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{\tau-1}(n-k+1) \\
&= \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{\tau-1}(n+1) - \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{\tau-1}k \\
&= \frac{1}{t(n+2)}\sum_{k=0}^{\tau-1}1 - \frac{1}{t(n+1)(n+2)}\sum_{k=0}^{\tau-1}k \\
&= \frac{\tau-1}{t(n+2)} - \frac{\tau(\tau-1)}{t(n+1)(n+2)} \\
&\leq k\left(\frac{1}{t}\right) \qquad (11)
\end{aligned}$$

Now considering second term of (10) and using Abel's Lemma

$$\begin{aligned}
&\frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=\tau}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c} k \\ \nu \end{array}\right)\ e^{i\nu\,t}\right\}\right]\right| \\
&\leq \frac{1}{t\,(n+1)\,(n+2)}\sum_{k=\tau}^{n}\frac{n-k+1}{2^k}\ \max_{0\leq\ m\leq\ k}\ \left|\sum_{\nu=0}^{k}\left(\begin{array}{c} k \\ \nu \end{array}\right)\ e^{i\nu\,t}\right| \\
&\leq \frac{k}{t\,(n+1)\,(n+2)}2^{\tau}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right) \qquad (12)
\end{aligned}$$

Combining (10) to (12), we get

$$\bar{K}_n\,(t) \leq k\left(\frac{1}{t}\right)+k\left\{\left(\frac{1}{t\,(n+1)\,(n+2)}\right)2^{\tau}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)\right\} \qquad (13)$$

4. Proof of Main Theorems

4.1. **Proof of Theorem 1.** Following Titchmarsh [25] and using Riemann-Lebesgue theorem, $s_n(f;x)$ of the series (1.4) is given by

$$s_n\,(f;x)-f\,(x)=\frac{1}{2\pi}\int_0^{\pi}\phi\,(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Using (1), the $(E,1)$ transform of $s_n\,(f;x)$ is given by

$$t_n^{(E,1)}-f\,(x)=\frac{1}{\pi\ 2^{n+1}}\int_0^{\pi}\phi\,(t)\left\{\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\ \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

The $(C,2)\,(E,1)$ transform of $s_n\,(f;x)$ is given by

$$\begin{aligned}
t^{(C,2)(E,1)}-f\,(x)&=\frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi\,(t)}{\sin\frac{t}{2}}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c} k \\ \nu \end{array}\right)\sin\left(\nu+\frac{1}{2}\right)t\right\}dt\right] \\
&=\int_0^{\pi}\ \phi\,(t)\ \ K_n\,(t)\,dt
\end{aligned}$$

In order to prove the theorem, we have to show under our assumptions that

$$\int_0^{\pi}\ \phi\,(t)\ \ K_n\,(t)\,dt=o\,(1)\ \text{ as } n\to\infty$$

For $0<\delta<\pi$, we have

$$\begin{aligned}
\int_0^{\pi}\ \phi\,(t)\ \ K_n\,(t)\,dt&=\left[\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\delta}+\int_{\delta}^{\pi}\right]\phi\,(t)\ \ K_n\,(t)\,dt \\
&=I_{1.1}+I_{1.2}+I_{1.3}\ \text{(say)} \qquad (14)
\end{aligned}$$

We consider,

$$\begin{aligned}
|I_{1.1}| &\leq \int_0^{\frac{1}{n+1}} |\phi(t)|\,|K_n(t)|\,dt \\
&= O(n+1)\left[\int_0^{\frac{1}{n+1}} |\phi(t)|\ dt\right] \text{ using Lemma 1} \\
&= O(n+1)\left[o\left\{\frac{1}{(n+1)\ \ \alpha(n+1)\,C_{n+1}}\right\}\right] \text{ by (6)} \\
&= o\left\{\frac{1}{\alpha(n+1)\,C_{n+1}}\right\} \\
&= o\left\{\frac{1}{\log(n+1)}\right\} \text{ by (7)} \\
&= o(1),\ \text{ as } n \to \infty \qquad (15)
\end{aligned}$$

Now we consider,

$$\begin{aligned}
|I_{1.2}| &\leq \int_{\frac{1}{n+1}}^{\delta} |\phi(t)|\,|K_n(t)|\,dt \\
&= O\left[\int_{\frac{1}{n+1}}^{\delta} |\phi(t)|\ \left(\frac{1}{t}\right) dt\right] \text{using Lemma 2} \\
&= O\left[\left\{\frac{1}{t}\ \Phi(t)\ \right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2}\ \Phi(t)\ dt\right] \\
&= O\left[o\left\{\frac{1}{\alpha(1/t)\,C_\tau}\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} o\left(\frac{1}{t\ \alpha\left(\frac{1}{t}\right)C_\tau}\right) dt\right] \text{ by (6)}
\end{aligned}$$

Putting $\frac{1}{t} = u$ in second term,

$$\begin{aligned}
I_{1.2} &= O\left[o\left\{\frac{1}{\alpha(n+1)\,C_{n+1}}\right\} + \int_{\frac{1}{\delta}}^{n+1} o\left(\frac{1}{u\ \alpha(u)\,C_u}\right) du\right] \\
&= o\left\{\frac{1}{\alpha(n+1)\,C_{n+1}}\right\} + o\left\{\frac{1}{(n+1\ )\,\alpha(n+1)\,C_{n+1}}\right\}\int_{\frac{1}{\delta}}^{n+1} 1.du \\
&= o\left\{\frac{1}{\log(n+1)}\right\} + o\left\{\frac{1}{\log(n+1)}\right\} \text{ by (7)} \\
&= o(1) + o(1), \quad \text{as } n \to \infty \\
&= o(1), \quad \text{ as } n \to \infty. \qquad (16)
\end{aligned}$$

By Riemann- Lebesgue lemma and by regularity condition of the method of summability,

$$\begin{aligned}
|I_{1.3}| &\leq \int_{\delta}^{\pi} |\phi(t)|\ \ |K_n(t)|\ \ dt \\
&= o(1), \text{as } n \to \infty \qquad (17)
\end{aligned}$$

Combining (14) to (17),

$$t^{(C,2)(E,1)} - f(x) = o(1), \text{ as } n \to \infty.$$

This completes the proof of theorem 1.

4.2. **Proof of Theorem 2.** Let $\bar{s}_n(f;x)$ denotes the partial sum of series (5) then following Lal [14] and using Riemann- Lebesgue theorem, $\bar{s}_n(f;x)$ of series (5) is given by

$$\bar{s}_n(f;x) - \bar{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\ \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\ dt$$

Using (5), the $(E,1)$ transform of $\bar{s}_n(f;x)$ is given by

$$\bar{t}_n^{(E,1)} - \bar{f}(x) = \frac{1}{2^{n+1}\ \pi}\int_0^\pi \psi(t)\left\{\sum_{k=0}^{n}\begin{pmatrix} n \\ k\end{pmatrix}\ \frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

Now denoting $(C,2)(E,q)$ transform of $\bar{s}_n$ is given by

$$\begin{aligned}\bar{t}^{(C,2)(E,1)} - \bar{f}(x) &= \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^\pi \frac{\psi(t)}{\sin\frac{t}{2}}\left\{\sum_{\nu=0}^{k}\begin{pmatrix} k \\ \nu\end{pmatrix}\cos\left(\nu+\frac{1}{2}\right)t\right\}dt\right] \\ &= \int_0^\pi \psi(t)\ \bar{K}_n(t)\,dt\end{aligned}$$

In order to prove the theorem, we have to show under our assumptions that

$$\int_0^\pi \psi(t)\ \bar{K}_n(t)\ dt = o(1) \text{ as } \ n\to\infty$$

For $0<\delta<\pi$, we have

$$\begin{aligned}\int_0^\pi \psi(t)\ \bar{K}_n(t)\ dt &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\delta} + \int_\delta^\pi\right]\psi(t)\ \bar{K}_n(t)\ dt \\ &= I_{2.1} + I_{2.2} + I_{2.3} \text{ (say)}\end{aligned} \tag{18}$$

We consider,

$$\begin{aligned}|I_{2.1}| &\le \int_0^{\frac{1}{n+1}} |\psi(t)|\ \left|\bar{K}_n(t)\right|dt \\ &= O\left[\int_0^{\frac{1}{n+1}}\frac{1}{t}\,|\psi(t)|\ dt\right] \quad \text{using Lemma 3} \\ &= O(n+1)\left[\int_0^{\frac{1}{n+1}}|\psi(t)|\ dt\right] \\ &= O(n+1)\left[o\left\{\frac{1}{(n+1)\ \ \alpha(n+1)\,C_{n+1}}\right\}\right] \text{ by (8)} \\ &= o\left\{\frac{1}{\alpha(n+1)\,C_{n+1}}\right\} \\ &= o\left\{\frac{1}{\log(n+1)}\right\} \text{ by (7)} \\ &= o(1), \text{ as } n\to\infty\end{aligned} \tag{19}$$

Now,

$$
\begin{aligned}
|I_{2.2}| &\leq \int_{\frac{1}{n+1}}^{\delta} |\psi(t)| \; \left|\bar{K}_n(t)\right| dt \\
&\leq \left[k \int_{\frac{1}{n+1}}^{\delta} \left[\frac{1}{t} + \left(\frac{1}{t\,(n+1)(n+2)} \right) 2^{\tau} \sum_{k=\tau}^{n} \left(\frac{n-k+1}{2^k} \right) \right] |\psi(t)| \; dt \right] \\
&= O\left[\int_{\frac{1}{n+1}}^{\delta} \left(\frac{1}{t} \right) |\psi(t)|\, dt \right] \text{ by (9)} \\
&= O\left[\left\{ \frac{1}{t} \; \Psi(t) \right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2} \; \Psi(t) \; dt \right] \\
&= O\left[o\left\{ \frac{1}{\alpha\left(\frac{1}{t}\right) C_{\tau}} \right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} o\left(\frac{1}{t\;\alpha\left(\frac{1}{t}\right) C_{\tau}} \right) dt \right] \qquad \text{by (8)}
\end{aligned}
$$

Putting $\frac{1}{t} = u$ in second term,

$$
\begin{aligned}
|I_{2.2}| &= O\left[o\left\{ \frac{1}{\alpha(n+1)\, C_{n+1}} \right\} + \int_{\frac{1}{\delta}}^{n+1} o\left(\frac{1}{u\;\alpha(u)\, C_u} \right) du \right] \\
&= o\left\{ \frac{1}{\alpha(n+1)\, C_{n+1}} \right\} + o\left\{ \frac{1}{(n+1\;)\,\alpha(n+1)\, C_{n+1}} \right\} \int_{\frac{1}{\delta}}^{n+1} 1.du \\
&= o\left\{ \frac{1}{\log(n+1)} \right\} + o\left\{ \frac{1}{\log(n+1)} \right\} \text{by (7)} \\
&= o(1) + o(1), \quad \text{as } n \to \infty \\
&= o(1), \text{as } n \to \infty
\end{aligned} \tag{20}
$$

By Riemann- Lebesgue lemma and by regularity condition of $(C,2)(E,1)$ method of summability,

$$
\begin{aligned}
|I_{2.3}| &\leq \int_{\delta}^{\pi} |\psi(t)| \; \left|\bar{K}_n(t)\right| \; dt \\
&= o(1), \text{ as } n \to \infty
\end{aligned} \tag{21}
$$

Combining (18) to (21),

$$
\bar{t}^{(C,2)(E,1)} - \bar{f}(x) = o(1), \text{ as } n \to \infty
$$

This completes the proof of theorem 2.

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