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BLOW-UP ANALYSIS FOR A DEGENERATE PARABOLIC SYSTEM WITH POSITIVE DIRICHLET BOUNDARY VALUE*

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Abstract. This paper investigates a nonlocal degenerate parabolic system with positive Dirichlet boundary value conditions. By means of the super- and sub-solution techniques and piecewise functions, some results of interactions among the multi-nonlinearities in the system described by four exponents, global boundedness and blow-up criteria of positive solutions are determined. The results show the positive boundary value ε_0 plays an important role in the case of blow-up.

1. Introduction

In this paper, we investigate the following nonlocal degenerate parabolic system

$$u_t = \Delta u^m + a\|v\|_\alpha^p, \qquad v_t = \Delta v^n + b\|u\|_\beta^q, \qquad x \in \Omega,\ t > 0 \tag{1}$$

with positive Dirichlet boundary value conditions

$$u(x,t) = \varepsilon_0 > 0, \quad v(x,t) = \varepsilon_0 > 0, \qquad x \in \partial\Omega,\ t > 0 \tag{2}$$

and initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{3}$$

where Ω be a bounded domain in R^N($N \geq 1$) with smooth boundary $\partial\Omega$, and constants $m, n > 1$, $\alpha, \beta \geq 1$, $a, b, p, q > 0$, where $u_0(x), v_0(x) > \varepsilon_0$ are nonnegative bounded functions on Ω, and where $\|\cdot\|_\alpha^\alpha = \int_\Omega |\cdot|^\alpha \mathrm{d}x$.

The coupled parabolic system (1)-(3) can be interpreted as the porous medium or diffusion equations such as thermoelasticity ([1]). They are worth to study because of the applications to heat and mass transport processes. In addition, there exist interesting interactions among the multi-nonlinearities described by four exponents m, n, p and q in the model (1)-(3).

In the past two decades, many physical phenomena were formulated into nonlocal mathematical models (see [2]-[6] and references therein) and studied by many authors. For example, Bebernes and Bressan ([2]) studied an ignition model for a compressible reactive gas which is a nonlocal reaction-diffusion equation. Pao ([3])

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discussed a nonlocal model arising from combustion theory. On the other hand, some authors ([7]-[8]) studied a class of nonlocal degenerate parabolic equations which arise in a model of population that communicates through chemical means.

In recent years, many important results have appeared on blow-up problems for nonlinear parabolic systems. We will recall some of those results concerning the first initial-boundary value problems. For other related works on the global existence and blow-up of solutions of nonlinear parabolic systems, we refer the reader to [9]-[11] and references therein.

In [12], Galaktionov et al. considered the system

$$u_t = \Delta u^{\nu+1} + v^p, \qquad v_t = \Delta v^{\mu+1} + u^q \tag{4}$$

with homogeneous Dirichlet boundary conditions. They proved that $p_c = pq - (\nu + 1)(\mu + 1)$ is the critical exponent of (4). Later, Song in [13] and Deng in [14] studied the following problem

$$u_t = \Delta u^m + u^\alpha v^p, \quad v_t = \Delta v^n + u^q v^\beta \tag{5}$$

by different methods. Some results, which concern the global boundedness and blow-up criteria of solutions were determined.

In 2003, Deng [15] et al. investigated the system (1), (3) with homogeneous Dirichlet boundary condition

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{6}$$

Several interesting results are established. We only state some of them here.

Theorem 1 Under the above assumptions.
(1) If $pq < mn$, then the every nonnegative solution of (1), (3) and (6) is global.
(2) If $pq > mn$, then

(i) the nonnegative solution of (1),(3) and (6) is global if the initial data u_0, v_0 are sufficiently small;

(ii) the nonnegative solution of (1),(3) and (6) blows up in finite time if the initial data u_0, v_0 are sufficiently large.

(3) If $pq = mn$, then

(i) the nonnegative solution of (1),(3) and (6) is global if the domain Ω is sufficiently small;

(ii) the nonnegative solution of (1),(3) and (6) blows up in finite time if the domain Ω contains a sufficiently large ball, and u_0, v_0 are positive and continuous in Ω.

For the degenerate parabolic system (1)-(3), due to the positivity of the boundary value, we can deal with the comparison principle and classical solutions by similar arguments as [15] and [16]. We point out that blow-up behavior of positive solutions is similar for the system (1),(3) with either homogeneous Dirichlet boundary condition (6) or positive Dirichlet boundary value condition (2), but the globality of positive solutions is somewhat different, see Remark 1.

The rest of the paper is organized as follows. Section 2 deals with global boundedness of solutions, we will prove Theorems 2 and 3. Theorems 4 and 5 about blow-up criteria are proved in Section 3. Some remarks are given in Section 4.

2. Global boundedness

Clearly, any solution (u, v) of (1)-(3) satisfies $u(x,t) \geq \varepsilon_0, v(x,t) \geq \varepsilon_0$ by the comparison principle due to the boundary condition (2) and the initial data $u_0(x), v_0(x) \geq \varepsilon_0$. Therefore, the local classical solutions to (1)-(3) do exist. We say a solution (u, v) of (1)-(3) blows up in finite time T if

$$\lim_{t\to T} \max_{\overline{\Omega}}(|u(\cdot,t)| + |v(\cdot,t)|) = +\infty.$$

Now, we deal with the following theorems on global boundedness of solutions.

Theorem 2 Assume $pq < mn$. Then the nonnegative solutions of system (1)-(3) are globally bounded.

Proof. According to the comparison principle, we only need to construct bounded, positive super-solutions for any $T > 0$. Let $\psi(x)$ be the unique positive solution of the following linear elliptic problem

$$-\Delta\psi(x) = 1, \quad x \in \Omega; \quad \psi(x) = 1, \quad x \in \partial\Omega.$$

Denote $C = \max_{x\in\Omega} \psi(x)$, then $1 \leq \psi(x) \leq C$. Now, we define the functions $\bar{u}, \bar{v}$ as

$$\bar{u}(x,t) = k_1\psi^{\frac{1}{m}}(x), \quad \bar{v}(x,t) = k_2\psi^{\frac{1}{n}}(x), \quad x \in \Omega,\ t > 0 \tag{7}$$

with positive constants k_1, k_2 to be determined later. Clearly, for any $T > 0$, $(\bar{u}, \bar{v})$ is a bounded function and $\bar{u} \geq k_1 > 0, \bar{v} \geq k_2 > 0$. Then, a series of computations yields

$$\bar{u}_t - \Delta\bar{u}^m = k_1^m, \qquad \|\bar{v}\|_\alpha^p = k_2^p\|\psi(x)^{\frac{1}{n}}\|_\alpha^p \leq k_2^p C^{\frac{p}{n}}|\Omega|^{p/\alpha}, \tag{8}$$

$$\bar{v}_t - \Delta\bar{v}^n = k_2^n, \qquad \|\bar{u}\|_\beta^q = k_1^q\|\psi(x)^{\frac{1}{m}}\|_\beta^q \leq k_1^q C^{\frac{q}{m}}|\Omega|^{q/\beta}. \tag{9}$$

Denote

$$C_1 = \left(aC^{\frac{p}{n}}|\Omega|^{\frac{p}{\alpha}}\right)^{\frac{1}{m}}, \quad C_2 = \left(b^{\frac{1}{q}}C^{\frac{1}{m}}|\Omega|^{\frac{1}{\beta}}\right)^{-1}. \tag{10}$$

The assumption $pq < mn$ implies $\frac{p}{m} < \frac{n}{q}$. So, for the positive constants C_1 and C_2, there exist sufficiently large constants $k_1, k_2 > 0$ such that $C_1k_2^{\frac{p}{m}} \leq k_1 \leq C_2k_2^{\frac{n}{q}}$, which implies that

$$\bar{u}_t - \Delta\bar{u}^m \geq a\|\bar{v}\|_\alpha^p, \quad \bar{v}_t - \Delta\bar{v}^n \geq b\|\bar{u}\|_\beta^q, \quad x \in \Omega, t > 0. \tag{11}$$

In addition, we may assume k_1, k_2 to be so large that

$$\bar{u}(x,0) = k_1\psi^{\frac{1}{m}}(x) \geq u_0(x), \quad \bar{v}(x,0) = k_2\psi^{\frac{1}{n}}(x) \geq v_0(x), \quad x \in \Omega \tag{12}$$

and

$$\bar{u}(x,t) = k_1\psi^{\frac{1}{m}}(x) \geq \varepsilon_0, \quad \bar{v}(x,t) = k_2\psi^{\frac{1}{n}}(x) \geq \varepsilon_0, \quad x \in \partial\Omega, t > 0. \tag{13}$$

Thus we have shown that $(\bar{u}, \bar{v})$ is a positive super-solution of (1)-(3), which implies the global boundedness of solutions to the problem (1)-(3). The proof is complete. □

Now consider the critical case of $pq = mn$. Denote by $\varphi_1(x)$ the first eigenfunction of the problem

$$\Delta\varphi(x) + \lambda\varphi(x) = 0,\ x \in \Omega; \quad \varphi(x) = 0,\ x \in \partial\Omega \tag{14}$$

with the first eigenvalue λ_1. Then $\varphi_1(x) > 0$ in Ω with $\lambda_1 > 0$. It is well known that λ_1 can be used to describe the size of Ω([14]).

Theorem 3 Assume $pq = mn$. If the diameter of Ω is sufficiently small, then the solutions of (1)-(3) are globally bounded.

Proof. Because of the continuous dependence of λ_1 upon the domain Ω([17]), we know that for any constant $\widetilde{\lambda} \in (0, \lambda_1)$, there is a bounded domain $\widetilde{\Omega} \supset \Omega$ such that $\widetilde{\lambda}$ is the first eigenvalue of the following problem

$$\Delta\varphi(x) + \lambda\varphi(x) = 0, \ x \in \widetilde{\Omega}; \quad \varphi(x) = 0, \ x \in \partial\widetilde{\Omega}. \tag{15}$$

Let $\widetilde{\varphi}(x)$ be the first eigenfunction of (15) with the first eigenvalue $\widetilde{\lambda}$, normalized by $\|\widetilde{\varphi}\|_\infty = 1$, then $\widetilde{\varphi}(x) > 0$ in $\widetilde{\Omega}$, and hence $\widetilde{\varphi}(x) \geq \delta_0 > 0$ on Ω for some positive constant δ_0.

Define

$$\bar{u}(x,t) = M^{a_1}\widetilde{\varphi}^{\frac{1}{m}}(x), \quad \bar{v}(x,t) = M^{b_1}\widetilde{\varphi}^{\frac{1}{n}}(x), \quad x \in \Omega, \ t > 0 \tag{16}$$

with positive constants a_1 and b_1 satisfying

$$\begin{pmatrix} -m & p \\ q & -n \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

Since $pq = mn$, there exists positive solutions to linear system (17). With a pair of such positive constants a_1 and b_1, by calculating directly, we have $\bar{u}_t = 0, \ \bar{v}_t = 0$ and

$$\Delta\bar{u}^m + a\|\bar{v}\|_\alpha^p = M^{ma_1}\widetilde{\varphi}\big(-\widetilde{\lambda} + ac_1M^{pb_1 - ma_1}\widetilde{\varphi}^{-1}\big) \leq M^{ma_1}\widetilde{\varphi}\big(-\widetilde{\lambda} + \frac{ac_1}{\delta_0}\big), \tag{18}$$

$$\Delta\bar{v}^n + b\|\bar{u}\|_\beta^q = M^{nb_1}\widetilde{\varphi}\big(-\widetilde{\lambda} + bc_2M^{qa_1 - nb_1}\widetilde{\varphi}^{-1}\big) \leq M^{nb_1}\widetilde{\varphi}\big(-\widetilde{\lambda} + \frac{bc_2}{\delta_0}\big), \tag{19}$$

where

$$c_1 = \|\widetilde{\varphi}^{\frac{1}{n}}\|_\alpha^p > 0, \quad c_2 = \|\widetilde{\varphi}^{\frac{1}{m}}\|_\beta^q > 0.$$

As the domain Ω becomes smaller and smaller, the corresponding first eigenvalue λ_1 of (14) will become bigger and bigger. Therefore, assume the diameter of Ω is sufficiently small such that

$$\lambda_1 > \lambda_0 =: \max\Big\{\frac{ac_1}{\delta_0}, \ \frac{bc_2}{\delta_0}\Big\}. \tag{20}$$

Moreover, choose the $\widetilde{\lambda}$ such that $\lambda_0 < \widetilde{\lambda} < \lambda_1$. Thus, from (18)-(20) we obtain

$$\Delta\bar{u}^m + a\|\bar{v}\|_\alpha^p \leq \bar{u}_t, \quad \Delta\bar{v}^n + b\|\bar{u}\|_\beta^q \leq \bar{v}_t$$

for $x \in \Omega$ and $t > 0$. On the other hand, taking M large enough such that

$$\bar{u}(x,t) = M^{a_1}\widetilde{\varphi}^{\frac{1}{m}}(x) \geq \varepsilon_0, \quad \bar{v}(x,t) = M^{b_1}\widetilde{\varphi}^{\frac{1}{n}}(x) \geq \varepsilon_0, \quad x \in \partial\Omega, \ t > 0$$

and

$$\bar{u}(x,0) = M^{a_1}\widetilde{\varphi}^{\frac{1}{m}}(x) \geq u_0(x), \quad \bar{v}(x,0) = M^{b_1}\widetilde{\varphi}^{\frac{1}{n}}(x) \geq v_0(x), \quad x \in \Omega.$$

So we have proved that $(\bar{u}, \bar{v})$ is a super-solution of (1)-(3), which implies the global boundedness of solutions to the problem (1)-(3). The proof is complete. $\square$

3. Blow-up criteria

The following theorems is concerning blow-up criteria for the solutions of (1)-(3). Due to the requirement of the comparison principle, we will construct blow-up positive sub-solutions to complete the proofs of theorems.

Theorem 4 Assume $pq > mn$, and the initial data $u_0(x), v_0(x)$ are sufficiently large. Then the nonnegative solution of (1)-(3) blows up in a finite time.

Proof. Since $pq > mn$, and hence there exists two positive constants α_1, β_1 large enough that

$$\frac{p}{m} > \frac{\alpha_1}{\beta_1} > \frac{n}{q} \quad \text{and} \quad (m-1)\alpha_1 > 1,\ (n-1)\beta_1 > 1. \tag{21}$$

Construct the piecewise functions $\tilde{u}, \tilde{v}$ as follows

$$\tilde{u}(x,t) = \begin{cases} \dfrac{M^{\alpha_1}\varepsilon_0}{[(1-ct)+M]^{\alpha_1}}\Big(1+\varphi_1(x)\Big)^{\alpha_1}, & (x,t) \in \Omega \times \Big(0, \dfrac{1}{c}\Big], \\ \dfrac{M^{\alpha_1}\varepsilon_0}{[(1-ct)+M]^{\alpha_1}}\varphi_1^{\alpha_1}(x), & (x,t) \in \Omega \times \Big(\dfrac{1}{c}, \dfrac{1+M}{c}\Big), \end{cases} \tag{22}$$

$$\tilde{v}(x,t) = \begin{cases} \dfrac{M^{\beta_1}\varepsilon_0}{[(1-ct)+M]^{\beta_1}}\Big(1+\varphi_1(x)\Big)^{\beta_1}, & (x,t) \in \Omega \times \Big(0, \dfrac{1}{c}\Big], \\ \dfrac{M^{\beta_1}\varepsilon_0}{[(1-ct)+M]^{\beta_1}}\varphi_1^{\beta_1}(x), & (x,t) \in \Omega \times \Big(\dfrac{1}{c}, \dfrac{1+M}{c}\Big), \end{cases} \tag{23}$$

where M and c are positive constants to be determined later, $\varphi_1(x)$ is the first eigenfunction (normalized by $\|\varphi_1\|_\infty = 1$) of the problem (14) with the corresponding first eigenvalue λ_1. Then $\lambda_1 > 0$ and $\varphi_1(x) > 0$ in Ω. It is easy to see that

$$\begin{cases} \tilde{u}(x,t) = \dfrac{M^{\alpha_1}\varepsilon_0}{[(1-ct)+M]^{\alpha_1}} \le \varepsilon_0,\ \tilde{v}(x,t) = \dfrac{M^{\beta_1}\varepsilon_0}{[(1-ct)+M]^{\beta_1}} \le \varepsilon_0, (x,t) \in \partial\Omega \times \big(0, \dfrac{1}{c}\big], \\ \tilde{u}(x,t) = 0 \le \varepsilon_0, \quad \tilde{v}(x,t) = 0 \le \varepsilon_0, \qquad (x,t) \in \partial\Omega \times \big(\dfrac{1}{c}, \dfrac{1+M}{c}\big). \end{cases} \tag{24}$$

In addition, by calculating direct, we have

$$\begin{aligned} \tilde{u}_t &= \frac{c\alpha_1 M^{\alpha_1}\varepsilon_0}{[(1-ct)+M]^{\alpha_1+1}}\Big(1+\varphi_1(x)\Big)^{\alpha_1}, \\ \Delta\tilde{u}^m &= \frac{M^{m\alpha_1}\varepsilon_0^m m\alpha_1(m\alpha_1-1)}{[(1-ct)+M]^{m\alpha_1}}\big(1+\varphi_1\big)^{m\alpha_1-2}|\nabla\varphi_1|^2 \\ &\qquad - \frac{\lambda_1 M^{m\alpha_1}\varepsilon_0^m m\alpha_1}{[(1-ct)+M]^{m\alpha_1}}\big(1+\varphi_1\big)^{m\alpha_1-1}\varphi_1, \\ \|\tilde{v}\|_\alpha^p &= \frac{\tilde{c}_{11} M^{\beta_1 p}\varepsilon_0^p}{[(1-ct)+M]^{\beta_1 p}}, \qquad \tilde{c}_{11} = \|(1+\varphi_1)^{\beta_1}\|_\alpha^p > 0, \end{aligned}$$

and hence

$$
\begin{aligned}
\tilde{u}_t - \Delta\tilde{u}^m - a\|\tilde{v}\|_\alpha^p \le\ & \frac{c\alpha_1 M^{\alpha_1}\varepsilon_0}{[(1-ct)+M]^{\alpha_1+1}}\big(1+\varphi_1\big)^{\alpha_1} \\
& + \frac{\lambda_1 M^{m\alpha_1}\varepsilon_0^m m\alpha_1}{[(1-ct)+M]^{m\alpha_1}}\big(1+\varphi_1\big)^{m\alpha_1-1}\varphi_1 - \frac{a\tilde{c}_{11}M^{\beta_1 p}\varepsilon_0^p}{[(1-ct)+M]^{\beta_1 p}} \\
\le\ & \frac{c\alpha_1 M^{\alpha_1}\varepsilon_0\big(1+\varphi_1\big)^{\alpha_1}}{[(1-ct)+M]^{\alpha_1+1}} + \frac{\lambda_1 M^{m\alpha_1}\varepsilon_0^m m\alpha_1}{[(1-ct)+M]^{m\alpha_1}}\big(1+\varphi_1\big)^{m\alpha_1} - \frac{a\tilde{c}_{11}M^{\beta_1 p}\varepsilon_0^p}{[(1-ct)+M]^{\beta_1 p}} \\
=\ & \frac{M^{\alpha_1}\varepsilon_0(1+\varphi_1)^{\alpha_1}}{[(1-ct)+M]^{\alpha_1+1}}\Big[c\alpha_1 + \frac{M^{(m-1)\alpha_1}\varepsilon_0^{m-1}}{[(1-ct)+M]^{m\alpha_1-\alpha_1-1}}\Big(\lambda_1 m\alpha_1(1+\varphi_1)^{(m-1)\alpha_1} \\
& - \frac{a\tilde{c}_{11}M^{\beta_1 p-m\alpha_1}\varepsilon_0^{p-m}}{[(1-ct)+M]^{\beta_1 p-m\alpha_1}}\big(1+\varphi_1\big)^{-\alpha_1}\Big)\Big] \\
\le\ & \frac{M^{\alpha_1}\varepsilon_0(1+\varphi_1)^{\alpha_1}}{[(1-ct)+M]^{\alpha_1+1}}\Big[c\alpha_1 - \frac{M^{(m-1)\alpha_1}\varepsilon_0^{m-1}}{[(1-ct)+M]^{m\alpha_1-\alpha_1-1}}\Big(-\lambda_1 m\alpha_1 2^{(m-1)\alpha_1} \\
& + \frac{a\tilde{c}_{11}M^{\beta_1 p-m\alpha_1}\varepsilon_0^{p-m}}{(1+M)^{\beta_1 p-m\alpha_1}}2^{-\alpha_1}\Big)\Big] \qquad (25)
\end{aligned}
$$

for $(x,t)\in\Omega\times(0,1/c]$. Similarly,

$$
\begin{aligned}
\tilde{v}_t - \Delta\tilde{v}^n - b\|\tilde{u}\|_\beta^q \le\ & \frac{M^{\beta_1}\varepsilon_0(1+\varphi_1)^{\beta_1}}{[(1-ct)+M]^{\beta_1+1}} \\
& \Big[c\beta_1 - \frac{M^{(n-1)\beta_1}\varepsilon_0^{n-1}}{[(1-ct)+M]^{n\beta_1-\beta_1-1}}\Big(-\lambda_1 n\beta_1 2^{(n-1)\beta_1} + \frac{b\tilde{c}_{21}M^{\alpha_1 q-n\beta_1}\varepsilon_0^{q-n}}{(1+M)^{\alpha_1 q-n\beta_1}}2^{-\beta_1}\Big)\Big] \qquad (26)
\end{aligned}
$$

for $(x,t)\in\Omega\times(0,1/c]$ and

$$
\begin{aligned}
\tilde{u}_t - \Delta\tilde{u}^m - a\|\tilde{v}\|_\alpha^p \le\ & \frac{M^{\alpha_1}\varepsilon_0\varphi_1^{\alpha_1}}{[(1-ct)+M]^{\alpha_1+1}} \\
& \Big[c\alpha_1 - \frac{M^{(m-1)\alpha_1}\varepsilon_0^{m-1}}{[(1-ct)+M]^{m\alpha_1-\alpha_1-1}}\Big(-\lambda_1 m\alpha_1 + \frac{a\tilde{c}_{12}M^{\beta_1 p-m\alpha_1}\varepsilon_0^{p-m}}{(1+M)^{\beta_1 p-m\alpha_1}}\Big)\Big], \qquad (27) \\
\tilde{v}_t - \Delta\tilde{v}^n - b\|\tilde{u}\|_\beta^q \le\ & \frac{M^{\beta_1}\varepsilon_0\varphi_1^{\beta_1}}{[(1-ct)+M]^{\beta_1+1}} \\
& \Big[c\beta_1 - \frac{M^{(n-1)\beta_1}\varepsilon_0^{n-1}}{[(1-ct)+M]^{n\beta_1-\beta_1-1}}\Big(-\lambda_1 n\beta_1 + \frac{b\tilde{c}_{22}M^{\alpha_1 q-n\beta_1}\varepsilon_0^{q-n}}{(1+M)^{\alpha_1 q-n\beta_1}}\Big)\Big] \qquad (28)
\end{aligned}
$$

for $(x,t)\in\Omega\times(1/c,(1+M)/c)$, where

$$\tilde{c}_{21} = \|(1+\varphi_1)^{\alpha_1}\|_\beta^q > 0,\ \tilde{c}_{12} = \|\varphi_1^{\beta_1}\|_\alpha^p > 0,\ \tilde{c}_{22} = \|\varphi_1^{\alpha_1}\|_\beta^q > 0.$$

Denote

$$
\begin{aligned}
M_{11} &= \Big(\frac{2^{m\alpha_1}\lambda_1 m\alpha_1}{a\tilde{c}_{11}\varepsilon_0^{p-m}}\Big)^{\frac{1}{\beta_1 p-m\alpha_1}}, \quad & M_{21} &= \Big(\frac{2^{n\beta_1}\lambda_1 n\beta_1}{b\tilde{c}_{21}\varepsilon_0^{q-n}}\Big)^{\frac{1}{\alpha_1 q-n\beta_1}}, \\
M_{12} &= \Big(\frac{\lambda_1 m\alpha_1}{a\tilde{c}_{12}\varepsilon_0^{p-m}}\Big)^{\frac{1}{\beta_1 p-m\alpha_1}}, \quad & M_{22} &= \Big(\frac{\lambda_1 n\beta_1}{b\tilde{c}_{22}\varepsilon_0^{q-n}}\Big)^{\frac{1}{\alpha_1 q-n\beta_1}}.
\end{aligned}
$$

Letting

$$M > \max\Big(\frac{M_{11}}{1-M_{11}}, \frac{M_{21}}{1-M_{21}}, \frac{M_{12}}{1-M_{12}}, \frac{M_{22}}{1-M_{22}}\Big) \qquad (29)$$

and then

$$\begin{cases} \delta_{11} = \dfrac{a\tilde{c}_{11} M^{\beta_1 p - m\alpha_1} \varepsilon_0^{p-m}}{(1+M)^{\beta_1 p - m\alpha_1}} 2^{-\alpha_1} - \lambda_1 m\alpha_1 2^{(m-1)\alpha_1} > 0, \\ \delta_{12} = \dfrac{a\tilde{c}_{12} M^{\beta_1 p - m\alpha_1} \varepsilon_0^{p-m}}{(1+M)^{\beta_1 p - m\alpha_1}} - \lambda_1 m\alpha_1 > 0, \\ \delta_{21} = \dfrac{b\tilde{c}_{21} M^{\alpha_1 q - n\beta_1} \varepsilon_0^{q-n}}{(1+M)^{\alpha_1 q - n\beta_1}} 2^{-\beta_1} - \lambda_1 n\beta_1 2^{(n-1)\beta_1} > 0, \\ \delta_{22} = \dfrac{b\tilde{c}_{22} M^{\alpha_1 q - n\beta_1} \varepsilon_0^{q-n}}{(1+M)^{\alpha_1 q - n\beta_1}} - \lambda_1 n\beta_1 > 0. \end{cases}$$

Define

$$A_{11} = \frac{M^{(m-1)\alpha_1} \varepsilon_0^{m-1} \delta_{11}}{\alpha_1 (1+M)^{m\alpha_1 - \alpha_1 - 1}}, \quad A_{21} = \frac{M^{(n-1)\beta_1} \varepsilon_0^{n-1} \delta_{21}}{\beta_1 (1+M)^{n\beta_1 - \beta_1 - 1}},$$
$$A_{12} = \frac{M^{(m-1)\alpha_1} \varepsilon_0^{m-1} \delta_{12}}{\alpha_1 (1+M)^{m\alpha_1 - \alpha_1 - 1}}, \quad A_{22} = \frac{M^{(n-1)\beta_1} \varepsilon_0^{n-1} \delta_{22}}{\beta_1 (1+M)^{n\beta_1 - \beta_1 - 1}}.$$

Taking

$$c \le \min\big(A_{11}, A_{12}, A_{21}, A_{22}\big). \tag{30}$$

Thus, from (25)-(30) we get

$$\tilde{u}_t - \Delta\tilde{u}^m - a\|\tilde{v}\|_\alpha^p \le 0, \quad \tilde{v}_t - \Delta\tilde{v}^n - b\|\tilde{u}\|_\beta^q \le 0, \ \text{ for } \ (x,t) \in \Omega\times\big(0, (1+M)/c\big). \tag{31}$$

Choose $u_0(x), v_0(x)$ properly large such that

$$\tilde{u}(x,0) \le u_0(x)\ ,\ \tilde{v}(x,0) \le v_0(x), \qquad \text{on } \ x \in \Omega. \tag{32}$$

We know from (24), (31) and (32) that $(\tilde{u}, \tilde{v})$ is a sub-solution of (1)-(3), which means that the solutions of (1)-(3) will blow up in time $T \le (1+M)/c$. The proof is complete. □

Theorem 5 Assume $pq = mn$, then the nonnegative solution of (1)-(3) blows up in finite time if the domain contains a sufficiently large ball, and $u_0(x), v_0(x)$ are positive and continuous in Ω.

Proof. Since $pq = mn$, clearly, there exists two positive constants l_1, l_2 large enough that

$$\frac{p}{m} = \frac{l_1}{l_2} = \frac{n}{q} \quad \text{ and } \quad (m-1)l_1 > 1,\ (n-1)l_2 > 1. \tag{33}$$

Without loss of generality, we may assume that $0 \in \Omega$. Let $B_R = B(0,R)$ be a ball such that $B_R \subset\subset \Omega$. In the following, we will prove that (u,v) blows up in finite time in the ball B_R. Because if so, (u,v) does blow up in the large domain Ω.

Denote by $\lambda_{B_R} > 0$ and $\phi_R(r)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\phi''(r) - \frac{N-1}{r}\phi'(r) = \lambda\phi(r), \quad r \in (0,R); \quad \phi'(0) = 0, \quad \phi(R) = 0.$$

It is well known that $\phi_R(r)$ can be normalized as $\phi_R(r) > 0$ in B_R and $\phi_R(0) = \max_{B_R}\phi_R(r) = 1$. By the property (let $\tau = r/R$) of eigenvalues and eigenfunctions we see that $\lambda_{B_R} = R^{-2}\lambda_{B_1}$ and $\phi_R(r) = \phi_1(r/R) = \phi_1(\tau)$, where λ_{B_1} and

$\phi_1(\tau)$ are the first eigenvalue and the corresponding normalized eigenfunction of the eigenvalue problem in the unit ball $B_1(0)$. Moreover,

$$\max_{B_1} \phi_1(\tau) = \phi_1(0) = \phi_R(0) = \max_{B_R} \phi_R(r) = 1.$$

Similar to (22) and (23), we define the following piecewise functions

$$\tilde{u}(x,t) = \begin{cases} \frac{M^{l_1}\varepsilon_0}{[(1-ct)+M]^{l_1}}\Big(1+\phi_R(|x|)\Big)^{l_1}, & (x,t) \in \Omega \times \Big(0, \frac{1}{c}\Big], \\ \frac{M^{l_1}\varepsilon_0}{[(1-ct)+M]^{l_1}}\phi_R^{l_1}(|x|), & (x,t) \in \Omega \times \Big(\frac{1}{c}, \frac{1+M}{c}\Big), \end{cases} \tag{34}$$

$$\tilde{v}(x,t) = \begin{cases} \frac{M^{l_2}\varepsilon_0}{[(1-ct)+M]^{l_2}}\Big(1+\phi_R(|x|)\Big)^{l_2}, & (x,t) \in \Omega \times \Big(0, \frac{1}{c}\Big], \\ \frac{M^{l_2}\varepsilon_0}{[(1-ct)+M]^{l_2}}\phi_R^{l_2}(|x|), & (x,t) \in \Omega \times \Big(\frac{1}{c}, \frac{1+M}{c}\Big), \end{cases} \tag{35}$$

where M and c are positive constants to be determined. Similar to (25)-(28), by calculating directly, we have

$$\begin{aligned} \tilde{u}_t - \Delta\tilde{u}^m - a\|\tilde{v}\|_\alpha^p \le & \frac{M^{l_1}\varepsilon_0(1+\phi_R)^{l_1}}{[(1-ct)+M]^{l_1+1}} \\ & \Big[cl_1 - \frac{M^{(m-1)l_1}\varepsilon_0^{m-1}}{[(1-ct)+M]^{(m-1)l_1-1}}\Big(a\tilde{c}_{31}\varepsilon_0^{p-m}2^{-l_1} - \lambda_{B_R} ml_1 2^{(m-1)l_1}\Big)\Big], \end{aligned} \tag{36}$$

$$\begin{aligned} \tilde{v}_t - \Delta\tilde{v}^n - b\|\tilde{u}\|_\beta^q \le & \frac{M^{l_2}\varepsilon_0(1+\phi_R)^{l_2}}{[(1-ct)+M]^{l_2+1}} \\ & \Big[cl_2 - \frac{M^{(n-1)l_2}\varepsilon_0^{n-1}}{[(1-ct)+M]^{(n-1)l_2-1}}\Big(b\tilde{c}_{41}\varepsilon_0^{q-n}2^{-l_2} - \lambda_{B_R} nl_2 2^{(n-1)l_2}\Big)\Big] \end{aligned} \tag{37}$$

for $(x,t) \in \Omega \times (0, 1/c]$ and

$$\begin{aligned} \tilde{u}_t - \Delta\tilde{u}^m - a\|\tilde{v}\|_\alpha^p \le & \frac{M^{l_1}\varepsilon_0\phi_R^{l_1}}{[(1-ct)+M]^{l_1+1}} \\ & \Big[cl_1 - \frac{M^{(m-1)l_1}\varepsilon_0^{m-1}}{[(1-ct)+M]^{(m-1)l_1-1}}\Big(a\tilde{c}_{32}\varepsilon_0^{p-m} - \lambda_{B_R} ml_1\Big)\Big], \end{aligned} \tag{38}$$

$$\begin{aligned} \tilde{v}_t - \Delta\tilde{v}^n - b\|\tilde{u}\|_\beta^q \le & \frac{M^{l_2}\varepsilon_0\phi_R^{l_2}}{[(1-ct)+M]^{l_2+1}} \\ & \Big[cl_2 - \frac{M^{(n-1)l_2}\varepsilon_0^{n-1}}{[(1-ct)+M]^{(n-1)l_2-1}}\Big(b\tilde{c}_{42}\varepsilon_0^{q-n} - \lambda_{B_R} nl_2\Big)\Big] \end{aligned} \tag{39}$$

for $(x,t) \in \Omega \times (1/c, (1+M)/c)$, where

$$\tilde{c}_{31} = \|(1+\phi_R)^{l_2}\|_\alpha^p, \quad \tilde{c}_{41} = \|(1+\phi_R)^{l_1}\|_\beta^q, \quad \tilde{c}_{32} = \|\phi_R^{l_2}\|_\alpha^p, \quad \tilde{c}_{42} = \|\phi_R^{l_1}\|_\beta^q > 0.$$

Then, in view of $\lambda_{B_R} = R^{-2}\lambda_{B_1}$, we may assume that R, that is , the ball B_R , is sufficiently large that

$$\lambda_{B_R} < \min\Big(\frac{a\tilde{c}_{31}\varepsilon_0^{p-m}}{2^{ml_1}ml_1}, \frac{b\tilde{c}_{41}\varepsilon_0^{q-n}}{2^{nl_2}nl_2}, \frac{a\tilde{c}_{32}\varepsilon_0^{p-m}}{ml_1}, \frac{b\tilde{c}_{42}\varepsilon_0^{q-n}}{nl_2}\Big) \tag{40}$$

and hence

$$\begin{cases} \delta_{31} = a\tilde{c}_{31}\varepsilon_0^{p-m}2^{-l_1} - \lambda_{B_R} ml_1 2^{(m-1)l_1} > 0, & \delta_{32} = a\tilde{c}_{32}\varepsilon_0^{p-m} - \lambda_{B_R} ml_1 > 0, \\ \delta_{41} = b\tilde{c}_{41}\varepsilon_0^{q-n}2^{-l_2} - \lambda_{B_R} nl_2 2^{(n-1)l_2} > 0, & \delta_{42} = b\tilde{c}_{42}\varepsilon_0^{q-n} - \lambda_{B_R} nl_2 > 0. \end{cases}$$

Define

$$A_{31} = \frac{M^{(m-1)l_1}\varepsilon_0^{m-1}\delta_{31}}{l_1(1+M)^{(m-1)l_1-1}}, \quad A_{32} = \frac{M^{(m-1)l_1}\varepsilon_0^{m-1}\delta_{32}}{l_1(1+M)^{(m-1)l_1-1}},$$

$$A_{41} = \frac{M^{(n-1)l_2}\varepsilon_0^{n-1}\delta_{41}}{l_2(1+M)^{(n-1)l_2-1}}, \quad A_{42} = \frac{M^{(n-1)l_2}\varepsilon_0^{n-1}\delta_{42}}{l_2(1+M)^{(n-1)l_2-1}}.$$

Taking

$$c \leq \min\big(A_{31},\ A_{32},\ A_{41},\ A_{42}\big). \tag{41}$$

Therefore, (36)-(41) imply that (31) holds for $(x,t) \in B_R \times (0,(1+M)/c$. In addition, it is easy to see that

$$\tilde{u}(x,t) \leq \varepsilon_0, \quad \tilde{v}(x,t) \leq \varepsilon_0, \quad (x,t) \in \partial\Omega \times (0,(1+M)/c).$$

Thus, $(\tilde{u},\tilde{v})$ is a positive sub-solution of (1)-(3) in the ball B_R, which blows up in finite time provided we choose M small enough to satisfy (32) in the ball B_R. The proof is complete. □

4. Some remarks

Remark 1 Assume $pq > mn$ and the initial data $u_0(x), v_0(x)$ are sufficiently small. Then if the boundary value ε_0 is small enough, by the proof of Theorem 2 we know the nonnegative solution of $(1)-(3)$ exists globally.

Remark 2 The results in this paper show the interactions among the multi-nonlinearities in the parabolic system (1)-(3). Roughly speaking, either small diffusion exponents m,n or large coupling exponents p,q benefit the occurrence of the finite blow-up. The key condition is $pq > mn$ or $pq < mn$, the critical case of $pq = mn$ belongs to the situations of global existence (or blow-up), where one needs some other assumptions that the size of Ω should be smaller (or larger). The boundary value is taken as positive constant ε_0 in (2), which guarantees the local existence of classical solutions to the problem (1)-(3).

We know from Theorems 2 and 3 that the global boundedness conditions for (1)-(3) are independent of the value of ε_0. While Theorems 4 and 5 show that the case of blow-up is quite different. In particular, the value of ε_0 plays an important role in Theorem 5. Indeed, in addition to the key condition $pq = mn$, the global non-existence depends essentially on the relation between λ_{B_R} (the description of the size of ball B_R) and ε_0 (the boundary value). To make a finite blow-up to problem (1)-(3), for fixed ε_0, the size of B_R should be properly large (i.e. λ_{B_R} is properly small) such that the inequality (40) holds, i.e.

$$\lambda_{B_R} < \min\Big(\frac{a\tilde{c}_{31}\varepsilon_0^{p-m}}{2^{ml_1}ml_1},\ \frac{b\tilde{c}_{41}\varepsilon_0^{q-n}}{2^{nl_2}nl_2},\ \frac{a\tilde{c}_{32}\varepsilon_0^{p-m}}{ml_1},\ \frac{b\tilde{c}_{42}\varepsilon_0^{q-n}}{nl_2}\Big).$$

Equivalently, we can understand the same inequality (40) as that for fixed B_R, ε_0 should be properly large with $p-m > 0, q-n > 0$ such that

$$\varepsilon_0 > \max\Big\{\Big(\frac{2^{ml_1}ml_1\lambda_{B_R}}{a\tilde{c}_{31}}\Big)^{\frac{1}{p-m}},\ \Big(\frac{2^{nl_2}nl_2\lambda_{B_R}}{b\tilde{c}_{41}}\Big)^{\frac{1}{q-n}},\ \Big(\frac{ml_1\lambda_{B_R}}{a\tilde{c}_{32}}\Big)^{\frac{1}{p-m}},\ \Big(\frac{nl_2\lambda_{B_R}}{b\tilde{c}_{41}}\Big)^{\frac{1}{q-n}}\Big\}.$$

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