CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY
SALAGEAN OPERATOR WITH VARYING ARGUMENTS

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Abstact. In the paper we derive results for certain new of analytic functions
defined by using Salagean operator with varying arguments.

1. Introduction

Let \( A \) denote the class of functions of the form:
\[
  f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]
which are analytic and univalent in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). For a function \( f \in A \), where \( f(z) \) is given by (1.1), we define
\[
  D^0 f(z) = f(z),
\]
\[
  D^1 f(z) = Df(z) = zf'(z)
\]
and
\[
  D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))' \quad (n \in \mathbb{N} = \{1, 2, \ldots \}).
\]
The differential operator \( D^n \) was introduced by Salagean [9].
It is easy to see that
\[
  D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).
\]
In this paper we define the class \( G(n, \lambda, A, B) \) as follows:

**Definition 1.** Let \( G(n, \lambda, A, B) \) denote the subclass of \( A \) consisting of functions \( f(z) \) of the form (1.1) such that
\[
  (D^n f(z))' + \lambda z (D^n f(z))'' < \frac{1 + Az}{1 + Bz}
\]
\( (\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0; z \in U) \).
Specializing the parameters \( \lambda, A, B \) and \( n \), we can obtain different classes studied by various authors:
(i) \( G(0, \lambda, 2\alpha - 1, 1) = R(\lambda, \alpha) \) \((0 \leq \alpha < 1, \lambda \geq 0)\) (see Altintas [2]);
(ii) \( G(0, 0, 2\alpha - 1, 1) = T^* \) \((0 \leq \alpha < 1)\) (see Sarangi and Uralegaddi [10] and Al-Amiri [1]);
(iii) \( G(0, 0, (2\alpha - 1) \beta, \beta) = P^* \) \((0 \leq \alpha < 1, 0 < \beta \leq 1)\) (see Gupta and Jain [6]);
Also we note that:

(i) \( G(0, 0, (1 + \mu)(\alpha - 1)\beta, \mu) = P^*(\alpha, \beta, \mu) \) \((0 \leq \alpha < 1, 0 < \beta \leq 1, 0 < \mu \leq 1) \) (see Owa and Aouf [7]).

Also we note that:

(ii) \( G((0, 0, (1 + \mu)(\alpha - 1)\beta, \mu) = R(\lambda, A, B) = \left\{ f(z) \in A : f'(z) + \lambda f''(z) < \frac{1 + Az}{1 + Bz} (\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; z \in U) \right\} \). (1.7)

(iii) \( G(n, 0, 2(\alpha - 1), 1) = G_n(\alpha) = \{ f(z) \in A : \Re\{D^n f(z)\} > \alpha; 0 \leq \alpha < 1; n \in \mathbb{N}_0; z \in U \} \). Silverman [11] defined the class of univalent functions \( f(z) \) are given by (1.1) for which \( \arg(a_k) \) prescribed in such a way that \( f(z) \) is univalent if and only if \( f(z) \) is starlike as follows:

Definition 2 A function \( f(z) \) of the form (1.1) is said to be in the class \( V(\theta_k) \) if \( f \in A \) and \( \arg(a_k) = \theta_k \) for all \( k \geq 2 \). If furthermore there exist a real number \( \delta \) such that \( \theta_k + (k - 1)\delta \equiv \pi (\bmod 2\pi) \) \((k \geq 2) \), then \( f(z) \) is said to be in the class \( V(\theta_k, \delta) \). The union of \( V(\theta_k, \delta) \) taken over all possible sequences \( \{\theta_k\} \) and all possible real numbers \( \delta \) is denoted by \( V \).

Let \( VG(n, \lambda, A, B) \) denote the subclass of \( V \) consisting of functions \( f(z) \in G(n, \lambda, A, B) \).

We note that:

(i) \( VG(0, 0, 2\alpha - 1, 1) = C_n(\alpha) = \left\{ f(z) \in V : \Re\{f'(z)\} > \alpha; 0 \leq \alpha < 1 \right\} \), studied by Srivastava and Owa [12].

(ii) \( V \) and \( A, B \) and \( n \) we can obtain different classes with varying arguments:

(i) \( VG(0, 0, 2\alpha - 1, 1) = R(\lambda, \alpha) \) \((0 \leq \alpha < 1, \lambda \geq 0) \);

(ii) \( VG(n, 0, 2\alpha - 1, 1) = VG_n(\alpha) \) \((0 \leq \alpha < 1, n \in \mathbb{N}_0) \);

(iii) \( VG(0, 0, (2\alpha - 1)\beta, \beta) = PV^*(\alpha, \beta) \) \((0 \leq \alpha < 1, 0 < \beta \leq 1) \);

(iv) \( VG(0, 0, ((1 + \mu)(\alpha - 1)\beta, \mu) = PV^*(\alpha, \beta, \mu) \) \((0 \leq \alpha < 1, 0 < \beta \leq 1, 0 < \mu \leq 1) \);

(v) \( VG(0, 0, A, B) = VR(\lambda, A, B) \) \((\lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1) \).

Some subclasses of analytic functions with varying arguments were introduced and studied by various authors (see [3], [4], [5] and [8]). In this paper we obtain coefficient bounds for functions in the class \( VG(n, \lambda, A, B) \), further we obtain distortion bounds and the extreme points for functions in this class.

2. COEFFICIENT ESTIMATES

Unless otherwise mentioned, we assume in the reminder of this paper that \( \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1 \), \( n \in \mathbb{N}_0 \) and \( z \in U \).

Theorem 1 Let the function \( f(z) \) defined by (1.1) be in \( V \). Then \( f(z) \in VG(n, \lambda, A, B) \), if and only if

\[
\sum_{k=2}^{\infty} k^{n+1} C_k |a_k| \leq (B - A),
\]

where

\[
C_k = (1 + B) [1 + \lambda(k - 1)].
\]

Proof. Assume that \( f(z) \in VG(n, \lambda, A, B) \). Then

\[
h(z) = (D^n f(z)) + \lambda z (D^n f(z))' = \frac{1 + Aw(z)}{1 + Bw(z)}
\]

where \( w \in H = \{ w \text{ analytic, } w(0) = 0 \text{ and } |w(z)| < 1, z \in U \}. \)
Thus we get
\[ w(z) = \frac{1 - h(z)}{Bh(z) - A}. \]

Therefore
\[ h(z) = 1 + \sum_{k=2}^{\infty} k^{n+1} [1 + \lambda(k - 1)]a_k z^{k-1}, \]
and \(|w(z)| < 1\) implies
\[ \left| \frac{\sum_{k=2}^{\infty} k^{n+1} [1 + \lambda(k - 1)]a_k z^{k-1}}{(B - A) + B \sum_{k=2}^{\infty} k^{n+1} [1 + \lambda(k - 1)]a_k z^{k-1}} \right| < 1. \] (2.4)

Since \(f(z)\) lies in the class \(V(\theta_k, \delta)\) for some sequence \(\{\theta_k\}\) and a real number \(\delta\) such that \(\theta_k + (k - 1)\delta \equiv \pi \mod 2\pi\) (\(k \geq 2\)).

Set \(z = re^{i\delta} (\delta \in \mathbb{R})\) in (2.7), we get (2.1).

Letting \(r \longrightarrow 1^-\) in (2.7), we get (2.1).

Conversely, \(f(z) \in V\) and satisfies (2.1). Since \(r^{k-1} < 1\). So we have
\[ \left| \frac{\sum_{k=2}^{\infty} k^{n+1} [1 + \lambda(k - 1)]a_k z^{k-1}}{(B - A) + B \sum_{k=2}^{\infty} k^{n+1} [1 + \lambda(k - 1)]a_k z^{k-1}} \right| \leq \frac{\sum_{k=2}^{\infty} k^{n+1} [1 + \lambda(k - 1)]a_k z^{k-1}}{(B - A) - B \sum_{k=2}^{\infty} k^{n+1} [1 + \lambda(k - 1)]a_k z^{k-1}} < 1, \]
that is \(f(z) \in VG(n, \lambda, A, B)\). This completes the proof of Theorem 1.

**Corollary 1.** Let the function \(f(z)\) defined by (1.1) be in the class \(VG(n, \lambda, A, B)\). Then
\[ |a_k| \leq \frac{(B - A)}{k^{n+1}C_k} (k \geq 2). \]

The result (2.1) is sharp for the function \(f(z)\) defined by
\[ f(z) = z + \frac{(B - A)}{k^{n+1}C_k} e^{i\theta_k} z^k (k \geq 2). \] (2.8)
3. Distortion theorems

**Theorem 2.** Let the function \( f(z) \) defined by (1.1) be in the class \( VG(n, \lambda, A, B) \). Then

\[
|z - \frac{B - A}{2^{n+1}C_2}|^2 \leq |f(z)| \leq |z| + \frac{B - A}{2^{n+1}C_2}|z|^2.
\]

(3.1)

The result is sharp.

**Proof.** We employ the same technique as used by Silverman [11]. In view of Theorem 1, since

\[
\Phi(k) = kn^{k+1}C_k,
\]

(3.2)

is an increasing function of \( k \) (\( k \geq 2 \)), we have

\[
\Phi(2) \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} \Phi(k) |a_k| \leq (B - A),
\]

that is

\[
\sum_{k=2}^{\infty} |a_k| \leq \frac{(B - A)}{\Phi(2)} \leq \frac{(B - A)}{2^{n+1}C_2}.
\]

(3.3)

Thus we have

\[
|f(z)| \leq |z| + \sum_{k=2}^{\infty} |a_k| |z|^k \leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k|,
\]

Thus

\[
|f(z)| \leq |z| + \frac{(B - A)}{2^{n+1}C_2}|z|^2.
\]

(3.4)

Similarly, we get

\[
|f(z)| \geq |z| - \sum_{k=2}^{\infty} |a_k| |z|^k \geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k|.
\]

Thus

\[
|f(z)| \geq |z| - \frac{(B - A)}{2^{n+1}C_2}|z|^2.
\]

This completes the proof of Theorem 2. Finally the result is sharp for the function

\[
f(z) = z + \frac{(B - A)}{2^{n+1}C_2} e^{i \theta_2} z^2,
\]

(3.4)

at \( z = \pm |z| e^{-i \theta} \).

**Corollary 2.** Under the hypotheses of Theorem 2, \( f(z) \) is included in a disc with center at the origin and radius \( r_1 \) given by

\[
r_1 = 1 + \frac{(B - A)}{2^{n+1}C_2}.
\]

(3.5)

**Theorem 3.** Let the function \( f(z) \) defined by (1.1) be in the class \( VG(n, \lambda, A, B) \). Then

\[
1 - \frac{(B - A)}{2^n C_2} |z| \leq \left| f'(z) \right| \leq 1 + \frac{(B - A)}{2^n C_2} |z|.
\]

(3.6)

The result is sharp.

**Proof.** Similarly \( \Phi(k) \) is an increasing function of \( k \) (\( k \geq 2 \)), where \( \Phi(k) \) is defined by (3.2). In view of Theorem 1, we have

\[
\frac{\Phi(2)}{2} \sum_{k=2}^{\infty} k |a_k| \leq \sum_{k=2}^{\infty} \Phi(k) |a_k| \leq (B - A),
\]

that is

\[
\sum_{k=2}^{\infty} k |a_k| \leq \frac{(B - A)}{\Phi(2)} \leq \frac{(B - A)}{2^n C_2}.
\]
Thus we have
\[ |f'(z)| \leq 1 + |z| \sum_{k=2}^{\infty} k |a_k| \leq 1 + \frac{(B - A)}{2^n C_2} |z|. \]  
(3.7)

Similarly
\[ |f'(z)| \geq 1 - |z| \sum_{k=2}^{\infty} k |a_k| \geq 1 - \frac{(B - A)}{2^n C_2} |z|. \]  
(3.8)

Finally, we can see that the assertions of Theorem 3 are sharp for the function \( f(z) \) defined by (3.4). This completes the proof of Theorem 3.

**Corollary 3.** Under the hypotheses of Theorem 3, \( f'(z) \) is included in a disc with center at the origin and radius \( r_2 \) given by
\[ r_2 = 1 + \frac{(B - A)}{2^n C_2}. \]  
(3.9)

4. Extreme Points

**Theorem 4.** Let the function \( f(z) \) defined by (1.1) be in the class \( VG(n, \lambda, A, B) \), with arg \( a_k = \theta_k \), where \( \theta_k + (k - 1)\delta \equiv \pi \ (mod \ 2\pi) \ (k \geq 2) \). Define
\[ f_1(z) = z \]
and
\[ f_k(z) = z + \frac{(B - A)}{k^{n+1} C_k} e^{i\theta_k} z^k \quad (k \geq 2; z \in U). \]

Then \( f(z) \in VG(n, \lambda, A, B) \) if and only if \( f(z) \) can expressed in the form \( f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \), where \( \mu_k \geq 0 \) and \( \sum_{k=1}^{\infty} \mu_k = 1 \).

**Proof.** If \( f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \) with \( \mu_k \geq 0 \) and \( \sum_{k=1}^{\infty} \mu_k = 1 \), then
\[ f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) \]
\[ = z + \sum_{k=2}^{\infty} \mu_k \left( \frac{(B - A)}{k^{n+1} C_k} e^{i\theta_k} z^k \right). \]

Therefore,
\[ \sum_{k=2}^{\infty} \left( \frac{(B - A)}{k^{n+1} C_k} e^{i\theta_k} \right) \mu_k = \sum_{k=2}^{\infty} (B - A) \mu_k \]
\[ = (1 - \mu_1) (B - A) \leq (B - A). \]
Hence \( f(z) \in VG(n, \lambda, A, B) \).

Conversely, let the function \( f(z) \) defined by (1.1) be in the class \( VG(n, \lambda, A, B) \), define
\[ \mu_k = \frac{k^{n+1} C_k}{(B - A) e^{i\theta_k} a_k} \quad (k \geq 2) \]
and
\[ \mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k. \]

From Theorem 1, \( \sum_{k=2}^{\infty} \mu_k \leq 1 \) and so \( \mu_1 \geq 0 \). Since \( \mu_k f_k(z) = \mu_k z + a_k z^k \),
then
\[ \sum_{k=1}^{\infty} \mu_k f_k(z) = z + \sum_{k=2}^{\infty} a_k z^k = f(z). \]
This completes the proof of Theorem 4.

Remarks. (i) Putting $\lambda = n = 0$, $A = 2\alpha - 1$ ($0 \leq \alpha < 1$) and $B = 1$ in all the above results, we obtain the corresponding results obtained by Srivastava and Owa [12];
(ii) Specializing the parameters $\lambda, A, B$ and $n$, we obtain results corresponding to the classes $VR(\lambda, \alpha)$, $VG_{\alpha}(\alpha)$, $VP^*(\alpha, \beta)$, $VP^*(\alpha, \beta, \mu)$ and $VR(\lambda, A, B)$, mentioned in the introduction.

References


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