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CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY SALAGEAN OPERATOR WITH VARYING ARGUMENTS

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Abstract. In the paper we derive results for certain new of analytic functions defined by using Salagean operator with varying arguments.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For a function $f \in \mathcal{A}$, where $f(z)$ is given by (1.1), we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z) \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))' \ (n \in \mathbb{N} = \{1, 2, ...\}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [9].
It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.5}$$

In this paper we define the class $G(n, \lambda, A, B)$ as follows:
Definition 1. Let $G(n, \lambda, A, B)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) such that

$$(D^n f(z))' + \lambda z (D^n f(z))'' \prec \frac{1 + Az}{1 + Bz} \tag{1.6}$$

$$(\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0; z \in U).$$

Specializing the parameters λ, A, B and n, we can obtain different classes studied by various authors:
(i) $G(0, \lambda, 2\alpha - 1, 1) = R(\lambda, \alpha)$ $(0 \leq \alpha < 1, \lambda \geq 0)$ (see Altintas [2]);
(ii) $G(0, 0, 2\alpha - 1, 1) = T^{**}(\alpha)$ $(0 \leq \alpha < 1)$ (see Sarangi and Uralegaddi [10] and Al-Amiri [1]);
(iii) $G(0, 0, (2\alpha - 1)\beta, \beta) = P^*(\alpha, \beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ (see Gupta and Jain [6]);

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(iv) $G(0,0,((1+\mu)\alpha-1)\beta,\mu\beta)=P^{*}(\alpha,\beta,\mu)$ $(0\leq\alpha<1,0<\beta\leq1,0<\mu\leq1)$ (see Owa and Aouf [7]).

Also we note that:

(i) $G(0,\lambda,A,B)=R(\lambda,A,B)=\Big\{f(z)\in\mathcal{A}:f^{'}(z)+\lambda zf^{''}(z)\prec\dfrac{1+Az}{1+Bz}(\lambda\geq0;$
$-1\leq A<B\leq1;0<B\leq1;z\in U)\Big\};$ (1.7)

(ii) $G(n,0,A,B)=G_{n}(A,B)=\Big\{f(z)\in\mathcal{A}:(D^{n}f(z))^{'}\prec\dfrac{1+Az}{1+Bz}(\lambda\geq0;$
$-1\leq A<B\leq1;0<B\leq1;n\in\mathbb{N}_{0};z\in U)\Big\};$ (1.8)

(iii) $G(n,0,2\alpha-1,1)=G_{n}(\alpha)=f(z)\in\mathcal{A}:\Re\{(D^{n}f(z))^{'}\}>\alpha;$
$0\leq\alpha<1;n\in\mathbb{N}_{0};z\in U)\Big\}$. Silverman [11] defined the class of univalent functions $f(z)$ are given by (1.1) for which $arg(a_{k})$ prescribed in such a way that $f(z)$ is univalent if and only if $f(z)$ is starlike as follows:

Definition 2 A function $f(z)$ of the form (1.1) is said to be in the class $V(\theta_{k})$ if $f\in\mathcal{A}$ and $arg(a_{k})=\theta_{k}$ for all $k\geq2$. If furthermore there exist a real number δ such that $\theta_{k}+(k-1)\delta\equiv\pi\ (mod\ 2\pi)$ $(k\geq2)$, then $f(z)$ is said to be in the class $V(\theta_{k},\delta)$. The union of $V(\theta_{k},\delta)$ taken over all possible sequences $\{\theta_{k}\}$ and all possible real numbers δ is denoted by V.

Let $VG(n,\lambda,A,B)$ denote the subclass of V consisting of functions $f(z)\in G(n,\lambda,A,B)$.

We note that:

(i)$VG(0,0,2\alpha-1,1)=C_{n}(\alpha)=\Big\{f\in V:\Re\{f^{'}(z)\}>\alpha;0\leq\alpha<1\Big\}$, studied by Srivastava and Owa [12].

Also we note that by specializing the parameters λ, A, B and n we can obtain different classes with varying arguments:

(i) $VG(0,\lambda,2\alpha-1,1)=VR(\lambda,\alpha)$ $(0\leq\alpha<1,\lambda\geq0)$;

(ii) $VG(n,0,2\alpha-1,1)=VG_{n}(\alpha)$ $(0\leq\alpha<1,n\in\mathbb{N}_{0})$;

(iii) $VG(0,0,(2\alpha-1)\beta,\beta)=VP^{*}(\alpha,\beta)$ $(0\leq\alpha<1,0<\beta\leq1)$;

(iv) $VG(0,0,((1+\mu)\alpha-1)\beta,\mu\beta)=VP^{*}(\alpha,\beta,\mu)$ $(0\leq\alpha<1,0<\beta\leq1,0<\mu\leq1)$;

(v) $VG(0,\lambda,A,B)=VR(\lambda,A,B)$ $(\lambda\geq0,-1\leq A<B\leq1,0<B\leq1)$.

Some subclasses of analytic functions with varying arguments were itroduced and studied by various authors (see [3], [4] , [5] and [8]). In this paper we obtain coefficient bounds for functions in the class $VG(n,\lambda,A,B)$, further we obtain distortion bounds and the extreme points for functions in this class.

2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that, $\lambda\geq0$, $-1\leq A<B\leq1$, $0<B\leq1$, $n\in\mathbb{N}_{0}$ and $z\in U$.

Theorem 1. Let the function $f(z)$ defined by (1.1) be in V. Then $f(z)\in VG(n,\lambda,A,B)$, if and only if

$$\sum_{k=2}^{\infty}k^{n+1}C_{k}\left|a_{k}\right|\leq(B-A)\,, \tag{2.1}$$

where

$$C_{k}=(1+B)\left[1+\lambda(k-1)\right]. \tag{2.2}$$

Proof. Assume that $f(z)\in VG(n,\lambda,A,B)$. Then

$$h(z)=(D^{n}f(z))^{'}+\lambda z\,(D^{n}f(z))''=\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where

$$w\in H=\{w\text{ analytic},\ w(0)=0\text{ and }\ |w(z)|<1,\ z\in U\}.$$

Thus we get

$$w(z) = \frac{1-h(z)}{Bh(z)-A}.$$

Therefore

$$h(z) = 1 + \sum_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1},$$

and $|w(z)| < 1$ implies

$$\left| \frac{\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}}{(B-A) + B\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}} \right| < 1. \tag{2.4}$$

Since $f(z)$ lies in the class $V(\theta_k, \delta)$ for some sequence $\{\theta_k\}$ and a real number δ such that

$$\theta_k + (k-1)\delta \equiv \pi (mod\ 2\pi) \quad (k \geq 2).$$

Set $z = re^{i\delta}(\delta \in \mathbb{R})$ in (2.4), we get

$$\left| \frac{\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]\,|a_k|\, r^{k-1}}{(B-A) - B\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]\,|a_k|\, r^{k-1}} \right| < 1. \tag{2.5}$$

Since $\Re\{w(z)\} < |w(z)| < 1$, we have

$$\Re\left\{ \frac{\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]\,|a_k|\, r^{k-1}}{(B-A) - B\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]\,|a_k|\, r^{k-1}} \right\} < 1. \tag{2.6}$$

Hence

$$\sum_{k=2}^{\infty} k^{n+1} C_k\, |a_k|\, r^{k-1} \leq (B-A). \tag{2.7}$$

Letting $r \longrightarrow 1^-$ in (2.7), we get (2.1).
Conversely, $f(z) \in V$ and satisfies (2.1). Since $r^{k-1} < 1$. So we have

$$\left| \frac{\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}}{(B-A) + B\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}} \right|$$

$$\leq \quad \frac{\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}}{(B-A) - B\sum\limits_{k=2}^{\infty} k^{n+1}[1+\lambda(k-1)]a_k z^{k-1}} < 1,$$

that is $f(z) \in VG(n, \lambda, A, B)$. This completes the proof of Theorem 1.
Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $VG(n, \lambda, A, B)$. Then

$$|a_k| \leq \frac{(B-A)}{k^{n+1}C_k}\ (k \geq 2).$$

The result (2.1) is sharp for the function $f(z)$ defined by

$$f(z) = z + \frac{(B-A)}{k^{n+1}C_k} e^{i\theta_k} z^k\ (k \geq 2). \tag{2.8}$$

3. Distortion theorems

Theorem 2. Let the function $f(z)$ defined by (1.1) be in the class $VG(n,\lambda,A,B)$. Then

$$|z|-\frac{B-A}{2^{n+1}C_2}|z|^2\le|f(z)|\le|z|+\frac{B-A}{2^{n+1}C_2}|z|^2. \tag{3.1}$$

The result is sharp.

Proof. We employ the same technique as used by Silverman [11]. In view of Theorem 1, since

$$\Phi(k)=k^{n+1}C_k\ , \tag{3.2}$$

is an increasing function of $k(k\ge2)$, we have

$$\Phi(2)\sum_{k=2}^{\infty}|a_k|\le\sum_{k=2}^{\infty}\Phi(k)|a_k|\le(B-A),$$

that is

$$\sum_{k=2}^{\infty}|a_k|\le\frac{(B-A)}{\Phi(2)}\le\frac{(B-A)}{2^{n+1}C_2}. \tag{3.3}$$

Thus we have

$$|f(z)|\le|z|+\sum_{k=2}^{\infty}|a_k||z|^k\le|z|+|z|^2\sum_{k=2}^{\infty}|a_k|,$$

Thus

$$|f(z)|\le|z|+\frac{(B-A)}{2^{n+1}C_2}|z|^2.$$

Similarly, we get

$$|f(z)|\ge|z|-\sum_{k=2}^{\infty}|a_k||z|^k\ge|z|-|z|^2\sum_{k=2}^{\infty}|a_k|.$$

Thus

$$|f(z)|\ge|z|-\frac{(B-A)}{2^{n+1}C_2}|z|^2.$$

This completes the proof of Theorem 2. Finally the result is sharp for the function

$$f(z)=z+\frac{(B-A)}{2^{n+1}C_2}e^{i\theta_2}z^2, \tag{3.4}$$

at $z=\pm|z|e^{-i\theta_2}$.

Corollary 2. Under the hypotheses of Theorem 2, $f(z)$ is included in a disc with center at the origin and radius r_1 given by

$$r_1=1+\frac{(B-A)}{2^{n+1}C_2}. \tag{3.5}$$

Theorem 3. Let the function $f(z)$ defined by (1.1) be in the class $VG(n,\lambda,A,B)$. Then

$$1-\frac{(B-A)}{2^nC_2}|z|\le\left|f^{'}(z)\right|\le1+\frac{(B-A)}{2^nC_2}|z|. \tag{3.6}$$

The result is sharp.

Proof. Similarly $\frac{\Phi(k)}{k}$ is an increasing function of $k(k\ge2)$, where $\Phi(k)$ is defined by (3.2). In view of Theorem 1, we have

$$\frac{\Phi(2)}{2}\sum_{k=2}^{\infty}k|a_k|\le\sum_{k=2}^{\infty}\Phi(k)|a_k|\le(B-A),$$

that is

$$\sum_{k=2}^{\infty}k|a_k|\le\frac{(B-A)}{\Phi(2)}\le\frac{(B-A)}{2^nC_2}.$$

Thus we have

$$\left|f^{'}(z)\right| \leq 1+|z|\sum_{k=2}^{\infty}k\,|a_k| \leq 1+\frac{(B-A)}{2^n C_2}\,|z|\,. \tag{3.7}$$

Similarly

$$\left|f^{'}(z)\right| \geq 1-|z|\sum_{k=2}^{\infty}k\,|a_k| \geq 1-\frac{(B-A)}{2^n C_2}\,|z|\,. \tag{3.8}$$

Finally, we can see that the assertions of Theorem 3 are sharp for the function $f(z)$ defined by (3.4). This completes the proof of Theorem 3.

Corollary 3. Under the hypotheses of Theorem 3, $f^{'}(z)$ is included in a disc with center at the origin and radius r_2 given by

$$r_2 = 1+\frac{(B-A)}{2^n C_2}. \tag{3.9}$$

4. Extreme points

Theorem 4. Let the function $f(z)$ defined by (1.1) be in the class $VG(n,\lambda,A,B)$, with arg $a_k=\theta_k$, where $\theta_k+(k-1)\delta \equiv \pi\ (mod\ 2\pi)\ (k\geq 2)$. Define

$$f_1(z)=z$$

and

$$f_k(z)=z+\frac{(B-A)}{k^{n+1}C_k}e^{i\theta_k}z^k \quad (k\geq 2; z\in U)\,.$$

Then $f(z)\in VG(n,\lambda,A,B)$ if and only if $f(z)$ can expressed in the form $f(z)=\sum\limits_{k=1}^{\infty}\mu_k f_k(z)$, where $\mu_k\geq 0$ and $\sum\limits_{k=1}^{\infty}\mu_k=1$.

Proof. If $f(z)=\sum\limits_{k=1}^{\infty}\mu_k f_k(z)$ with $\mu_k\geq 0$ and $\sum\limits_{k=1}^{\infty}\mu_k=1$, then

$$\begin{aligned} f(z) &= \mu_1 f_1(z)+\sum_{k=2}^{\infty}\mu_k f_k(z) \\ &= z+\sum_{k=2}^{\infty}\mu_k\frac{(B-A)}{k^{n+1}C_k}e^{i\theta_k}z^k\ . \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{k=2}^{\infty}(k^{n+1}C_k)e^{i\theta_k}\frac{(B-A)}{(k^{n+1}C_k)e^{i\theta_k}}\mu_k=\sum_{k=2}^{\infty}(B-A)\,\mu_k \\ = \ &(1-\mu_1)\,(B-A)\leq (B-A)\,. \end{aligned}$$

Hence $f(z)\in VG(n,\lambda,A,B)$.

Conversely, let the function $f(z)$ defined by (1.1) be in the class $VG(n,\lambda,A,B)$, define

$$\mu_k=\frac{k^{n+1}C_k}{(B-A)\,e^{i\theta_k}}a_k \qquad (k\geq 2)$$

and

$$\mu_1=1-\sum_{k=2}^{\infty}\mu_k.$$

From Theorem 1, $\sum\limits_{k=2}^{\infty}\mu_k\leq 1$ and so $\mu_1\geq 0$. Since $\mu_k f_k(z)=\mu_k z+a_k z^k$, then

$$\sum_{k=1}^{\infty}\mu_k f_k(z)=z+\sum_{k=2}^{\infty}a_k z^k=f(z).$$

This completes the proof of Theorem 4.

Remarks. (i) Putting $\lambda = n = 0$, $A = 2\alpha - 1\,(0 \leq \alpha < 1)$ and $B = 1$ in all the above results, we obtain the corresponding results obtained by Srivastava and Owa [12];

(ii) Specializing the parameters λ, A, B and n, we obtain results corresponding to the classes $VR(\lambda, \alpha)$, $VG_n(\alpha)$, $VP^*(\alpha, \beta)$, $VP^*(\alpha, \beta, \mu)$ and $VR(\lambda, A, B)$, mentioned in the introduction.

References

[1] H. S. Al-Amiri, On a subclass of close-to-convex functions with negative coefficients, Math. (Cluj), 31 (1989), no. 54, 1-7.

[2] O. Altintas, A subclass of analytic functions with negative coefficients, Hacettepe Bull. Natur. Sci. Engrg., 19 (1990), 15-24.

[3] J. Dziok, Certain inequalities for classes of analytic functions with varying argument of coefficients, Math. Inequal. Appl., 14 (2011), no. 2, 389-398.

[4] R. M. El-Ashwah, M. K. Aouf, A. A. M. Hassan and A. H. Hassan, Certain new classes of analytic functions with varying arguments, J. Complex Anal., (2013), Art. ID 95820, 1-5.

[5] R. M. El-Ashwah, M. K. Aouf, A. A. M. Hassan and A. H. Hassan, Multivalent functions with varying arguments, J. Classical Anal., (2013), (To appear).

[6] V. P. Gupta and P. K. Jain, Certain classes of univalent functions with negative coefficients II, Bull. Austral. Math. Soc., 15 (1976), 467-473.

[7] S. Owa and M. K. Aouf, On subclasses of univalent functions with negative coefficients II, Pure Appl. Math. Sci., 29 (1989), no. 1:2, 131-139.

[8] N. Ravikumar and S. Latha, Riemman-Liouvlle fractinal derviative with varying arguments, Mat. Vesnik, 1 (2012), no. 64, 17-23.

[9] G. S. Salagean, Subclasses of univalent function, Lecture Notes in Math. (Springer-Verlag) 1013 (1983), 368-372.

[10] S. M. Sarangi and B. A. Uralegaddi, The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients I, Rend. Acad. Naz. Lincei, 65 (1978), 38-42.

[11] H. Silverman, Univalent functions with varying arguments, Houston J. Math., 17(1981), 283-287.

[12] H. M. Srivastava and S. Owa, Certain classes of analytic functions with varying arguments, J. Math. Anal. Appl., 136 (1988), no. 1, 217-228.

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