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SOLUTION OF KDVB EQUATION VIA BLOCK PULSE FUNCTIONS METHOD

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Abstract. In this paper, based on two dimension Block Pulse functions (2D-BPFs), a numerical technique is introduced to solve the solitary wave equations. The operational matrices for partial derivatives of the first and second ordered are deduced. Using this technique, Korteweg de Vries Burger equation (KdVB) is transformed to its corresponding system of algebraic equations. Some numerical examples are presented to illustrate the effectiveness and accuracy of the technique.

1. Introduction

Many different bases functions have been used to estimate the solution to differential equations, such as orthogonal bases [1, 2] , wavelets [3, 4] and hybrid [2]. The various systems of orthogonal functions may be classified into two categories. The first is piecewise continuous function (PCBF) to which the orthogonal systems of Walsh functions [5], Block-pulse functions [4, 6] and Haar functions [7, 8] belong. The second group consists in continuous orthogonal functions such as orthogonal polynomials and sin-cos basis [9]. We notice that, by approximating a discontinuous function by a continuous basis we can't properly model the discontinuities and therefore we must approximate such a function by PCBFs. One of the main characteristics of the orthogonal function techniques for solving different problems is to reduce these problems to those of solving a system of algebraic equations. These techniques have been presented, among others by Hwang and Shih [9] and Lepik [7]. In this paper, an effective numerical method is introduced for treatment of nonlinear KdVB equation. Here, we use the so called two dimensional block pulse functions (2D-BPFs) [10, 12] which was presented by Harmuth [10]. First, the two dimensional block pulse operational matrix of integration and differentiation has been presented, then by using these matrices, the KdVB equation has been reduced to an algebraic system. The Block-Pulse Functions (BPFs) are a set of orthogonal functions with piecewise constant values which are defined in the time interval $[0, T]$

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as:

$$\phi_i = \begin{cases} 1, & (i-1)\frac{T}{m} \le t < i\frac{T}{m} \\ 0, & otherwise \end{cases} \tag{1}$$

where $i = 1, ..., m$ with m as a positive integer. We usually call the block pulse functions containing two variables as two dimensional block pulse functions (2D-BPFs). $A(m_1 m_2)$ set of 2D-BPFs are defined in region $x \in [0, T_1)$ and $t \in [0, T_2)$ as:

$$\phi_{i_1,i_2} = \begin{cases} 1, & (i_1-1)h_1 \le x < i_1 h_1, (i_2-1)h_2 \le x < i_2 h_2 \\ 0, & otherwise \end{cases} \tag{2}$$

where $i_1 = 1, ..., m_1$ and $i_2 = 1, ..., m_2$ with positive integer values for m_1, m_2 and $h_1 = \frac{T_1}{m_1}$ and $h_2 = \frac{T_2}{m_2}$. There are some properties for 2D-BPFs; e.g. disjointness, orthogonality, and completeness.

1.1. **Disjointness.** The two-dimensional block-pulse functions are disjoined with each other, i.e.

$$\phi_{i_1,i_2}(x,t)\phi_{j_1,j_2}(x,t) = \begin{cases} \phi_{i_1,i_2}(x,t), & i_1 = i_2, j_1 = j_2 \\ 0, & otherwise \end{cases} \tag{3}$$

1.2. **Orthogonality.** The two-dimensional block-pulse functions are orthogonal with each other, i.e.

$$\int_0^{T_2}\int_0^{T_1} \phi_{i_1,i_2}(x,t)\phi_{j_1,j_2}(x,t)dxdt = \begin{cases} h_1 h_2, & i_1 = i_2, j_1 = j_2 \\ 0, & otherwise \end{cases} \tag{4}$$

In the region of $t \in [0, T_1)$ and $x \in [0, T_2)$ where $i_1 = 1, ..., m_1$ and $i_2 = 1, ..., m_2$.

1.3. **Completeness.** For every $f \in L^2([0, T_1) \times [0, T_2))$, Parseval identity holds:

$$\int_0^{T_2}\int_0^{T_1} f^2(x,t)dxdt = \sum_{i_1}^{\infty}\sum_{i_2}^{\infty} f^2_{i_1,i_2}(x,t)\|\phi_{i_1,i_2}(x,t)\|^2 \tag{5}$$

where

$$f_{i_1,i_2} = \frac{1}{h_1 h_2}\int_0^{T_2}\int_0^{T_1} f(x,t)(x,t)\phi_{i_1,i_2}(x,t)dxdt \tag{6}$$

The set of 2D-BPFs may be written as a $(m_1 m_2)$ vector $\phi(t,x)$:

$$\phi(t,x) = [\phi_{1,1}(x,t), \cdots, \phi_{1,m_2}(x,t), \cdots, \phi_{m_1,m_2}(x,t)]^T, \tag{7}$$

where m_1 and m_2 goes to infinity. From the above representation and disjointness property, it follows that:

$$\phi(t,x)\phi^T(t,x) = \begin{pmatrix} \phi_{1,1} & 0 & \cdots & 0 \\ 0 & \phi_{2,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \phi_{m_1,m_2} \end{pmatrix} \tag{8}$$

$$\phi^T(t,x)\phi(t,x) = 1 \tag{9}$$

$$\phi(t,x)\phi^T(t,x)V = \tilde{V}\phi(t,x) \tag{10}$$

where V is an m_1m_2 vector and $\tilde{V} = diag(V)$. Moreover, it can be clearly concluded that for every $(m_1m_2) \times (m_1m_2)$ matrix A:

$$\phi^T(t,x)A\phi(t,x) = \tilde{A}^T\phi(t,x) \tag{11}$$

where $\tilde{A}$ is a vector with elements equal to the diagonal entries of matrix A.

2. 2D-BPFs expansion

A function $f(t,x) \in L^2([0,T_1) \times [0,T_2))$ may be expanded by the 2D-BPFs as:

$$\sum_{i_1}^{m_1}\sum_{i_2}^{m_2} f_{i_1,i_2}(x,t)\phi_{i_1,i_2}(x,t) = F^T\phi(x,t) = \phi^T(x,t)F \tag{12}$$

where F is a (m_1m_2) vector given by:

$$F = [f_{1,1}(t,x)\cdots f_{1,m_2}(t,x)\cdots f_{m_1,m_2}(t,x)]^T \tag{13}$$

The block-pulse coefficients f_{i_1,i_2} are obtained by:

$$f_{i_1,i_2} = \frac{1}{h_1h_2}\int_{(i_1-1)h_1}^{i_1h_1}\int_{(i_2-1)h_2}^{i_2h_2} f(x,t)dxdt \tag{14}$$

2.1. **Operational matrix of integration.** The integration of the vector $\phi(t,x)$ defined in (7) may be obtained as

$$\int_0^t\int_0^x \phi(\tau_1,\tau_2)d\tau_1d\tau_2 \cong \gamma\phi(t,x) = E_{(m_1\times m_1)} \otimes E_{(m_2\times m_2)}\phi(t,x) \tag{15}$$

where γ is a $(m_1m_1) \times (m_1m_1)$operational matrix of integration for 2D-BPFs and E is the operational matrix of one dimension Block Pulse functions (1D-BPFs) de?ned over $[0,1)$ with $= \frac{1}{m}$ as follows,

$$E = \frac{h}{2}\begin{pmatrix} 1 & 2 & \cdots & 2 \\ 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & 2 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{16}$$

In (15), $\otimes$ denotes the Kronecker product defined as:

$$A \otimes B = a_{ij}B \tag{17}$$

2.2. **Operational matrix of differentiation.** In this work we need to compute the operational matrix of differentiation. For this, let

$$\begin{aligned} u(x,t) &= U^T\phi(x,t) \\ u(x,0) &= U_{x0}^T\phi(x,t) \\ u(0,t) &= U_{0t}^T\phi(x,t) \\ u_x(x,t) &= U_x^T\phi(x,t) \\ u_t(x,t) &= U_x^T\phi(x,t) \\ u_x(0,t) &= U_{x0t}^T\phi(x,t) \\ u_t(x,0) &= U_{tx0}^T\phi(x,t) \\ u_{xx}(x,t) &= U_{xx}^T\phi(x,t) \\ u_{tt}(x,t) &= U_{tt}^T\phi(x,t) \end{aligned} \tag{18}$$

so, we can write:

$$u(x,t) - u(0,t) = \int_0^x u_x(x,\tau)dx \tag{19}$$

Then from (18), we obtain

$$U^T\phi(x,t) - U_{0t}^T\phi(x,t) = \int_0^x u_x^T\phi(x,\tau)d\tau \tag{20}$$

$$U^T\phi(x,t) - U_{0t}^T\phi(x,t) = U_x^T\int_0^x \phi(x,\tau)d\tau = U_x^T E\phi(x,t) \tag{21}$$

so we get

$$U^T - U_{0t}^T = U_x^T E \tag{22}$$

hence,

$$U_x^T = (U^T - U_{0t}^T)E^{-1} \tag{23}$$

Similarly, for the partial derivative of $u(t,x)$ with respect to t, it can be shown that

$$U_t^T = (U^T - U_{x0}^T)E^{-1} \tag{24}$$

Moreover, for the second order partial derivatives of $u(t,x)$, the following equations can be written:

$$u_x(x,t) - u_x(0,t) = \int_0^x u_{xx}(x,\tau)dx \tag{25}$$

Using (18), we have

$$U_{xx}^T\phi(x,t) - U_{x0t}^T\phi(x,t) = \int_0^x u_{xx}^T\phi(x,\tau)d\tau \tag{26}$$

$$U_x^T\phi(x,t) - U_{x0t}^T\phi(x,t) = U_{xx}^T\int_0^x \phi(x,\tau)d\tau = U_{xx}^T E\phi(x,t) \tag{27}$$

i.e

$$U_{xx}^T = (U_x^T - U_{x0t}^T)E^{-1} \tag{28}$$

Similarly, to approximate the second order partial derivatives of $u(t,x)$ with respect to t, the following equation has been obtained:

$$U_{tt}^T = (U_t^T - U_{tx0}^T)E^{-1} \tag{29}$$

Finally, the following procedure can used to approximate $u_{xxx}(t,x)$:

$$U_{xxx}^T = (U_{xx}^T - U_{xx0t}^T)E^{-1} \tag{30}$$

3. BPFs applied to solitary waves equations.

Consider the KdVB equation $u_t + \epsilon uu_x + \nu u_{xx} + \mu u_{xxx} = 0$. By using the proposed scheme to approximate the partial derivatives, we have,

$$U_t^T\phi(x,t) + \epsilon U^T\phi(x,t)U_x^T\phi(x,t) + \nu U_{xx}^T\phi(x,t) + \mu U_{xxx}^T\phi(x,t) = 0 \tag{31}$$

$$U_t^T + \epsilon U^T\phi(x,t)U_x^T + \nu U_{xx}^T + \mu U_{xxx}^T = 0 \tag{32}$$

now, by using the equations (18), (22), (23), (28) and (30) we can obtain:

$$\mathrm{AU} = \mathrm{F} \tag{33}$$

where A and F are the combination of block-pulse coefficient matrix and U can be obtained from solving nonlinear systems.

4. Error analysis.

In this section, we investigate the representation error of a differentiable function $f(t,x)$ when it is represented in a series of 2D-BPFs over the region $D = [0,1) \times [0,1)$. For this we use some results from [13, 15]. We put $m_1 = m_2 = m$, so$h_1 = h_2 = \frac{1}{m}$. We define the representation error between $f(t,x)$and its 2D-BPFs expansion, over every subregion D_{i_1,i_2} as follows:

$$e_{i_1,i_2} = f_{i_1,i_2}\phi_{i_1,i_2}(x,t) - f(x,t) = f_{i_1,i_2} - f(x,t), \qquad x,t \in D_{i_1,i_2} \tag{34}$$

$$D_{i_1,i_2} = \{(x,t) : \frac{i_1 - 1}{m} \le x \le \frac{i_1}{m}, \frac{i_2 - 1}{m} \le t \le \frac{i_2}{m}\} \tag{35}$$

Using the mean value theorem, it can be shown that:

$$\|e_{i_1,i_2}\|^2 \le \frac{2}{m^4}M^2 \tag{36}$$

where $|(f(x,t))| \le M$, error between $f(t,x)$ and its 2D-BPFs expansion $f_m(x,t)$, over the region D, can be obtained as follows:

$$e(x,t) = f_m(x,t) - f(x,t) \tag{37}$$

By using equations (36) and (37), it can be shown that

$$\|e(x,t)\|^2 \le \frac{2}{m^2}M^2 \tag{38}$$

Hence, $\| \ e(t,x) \ \| = O(\frac{1}{m})$ which is similar to the proposed scheme in [14, 15]. Suppose that $f(t,x)$ is approximated by,

$$f_m(x,t) = \sum_{i_1}^{m}\sum_{i2}^{m} f_{i_1,i_2}\phi_{i_1,i_2}(x,t) \tag{39}$$

we get, $\bar{f}_{i_1,i_2}$ the approximation of f_{i_1,i_2} and

$$\bar{f}_{i_1,i_2} = \sum_{i_1}^{m}\sum_{i2}^{m} \bar{f}_{i_1,i_2}\phi_{i_1,i_2}(x,t) \tag{40}$$

then from equation (39) for $(t,x) \in D_{i_1,i_2}$ we have

$$\begin{aligned} \| \ \bar{f}_{i_1,i_2} - f(x,t) \ \| &\le \| \ \bar{f}_{i_1,i_2} - f_{i_1,i_2} \ \| + \| \ f_{i_1,i_2} - f(x,t) \ \| \\ &\le \frac{\sqrt{2}M}{m} + \frac{\| \ \bar{f}_m - f \ \|_\infty}{m} \end{aligned} \tag{41}$$

Therefore, from equation (41), it can be shown that:

$$\lim_{m\to\infty} f_m(x,t) = f(x,t) \tag{42}$$

For error estimation, reconsider the following KdVB equation,

$$u_t + \epsilon u u_x + \nu u_{xx} + \mu u_{xxx} = 0, \qquad (t,x) \in [0,1] \times [0,1] \tag{43}$$

This equation may take the form,

$$u_t + \frac{\epsilon}{2}u_x^2 + \nu u_{xx} + \mu u_{xxx} = 0 \tag{44}$$

Let $e_m^2(x,t) = u^2(x,t) - u_m^2(x,t)$ be the error function of the approximate solution $u_m(t,x)$ to

$$R_m(x,t) + (u_t + \frac{\epsilon}{2}u_x^2 + \nu u_{xx} + \mu u_{xxx})_m = 0 \tag{45}$$

where $R_m(x,t)$ is the perturbation function that depends on, $(u_{xx}(t,x))_m$ and $(u_{xxx}(t,x))_m$. Substituting into the equation (43) as:

$$R_m(x,t) = -(u_t + \frac{\epsilon}{2}u_x^2 + \nu u_{xx} + \mu u_{xxx})_m \tag{46}$$

Subtracting (46) from (44) yields,

$$R_m(x,t) = (e_t + \frac{\epsilon}{2}e_x^2 + \nu e_{xx} + \mu e_{xxx})_m \tag{47}$$

Finally, in the proposed scheme we, can approximate $e_m(x,t)$ from equation (46).

5. Numerical application of BPFs.

In this section we are going to apply the BPFs method for solving KdVB, KdV and Burger equations. The grid points are selected as (x,t) where $x = t = \frac{2\gamma-1}{64}$, $\gamma = 1, 2, 3, 4, 5$.

Example(1)

Consider the KdVB equation of the form $u_t - \nu u_{xx} + \epsilon u u_x = 0, 0 \leq x \leq 1, 0 \leq t \leq 1$which is Burger's equation with $\epsilon = 1, \nu = 2, \mu = 0$ initial approximation $u(x,0) = 2x, u(0,t) = 0$ and $u_x(0,t) = \frac{2}{(1+2t)}$. The equation have the exact solution $u(x,t) = \frac{2x}{(1+2t)}$. Numerical results of Burger equation for different values of m and γ are given in Table .1

Table .1 solution of Burger equation.

γ	$m = 8$	$m = 16$	$m = 32$
$\gamma = 1$	$4.3 * 10^{-3}$	$2.8 * 10^{-4}$	$1.2 * 10^{-6}$
$\gamma = 2$	$7.1 * 10^{-3}$	$3.9 * 10^{-4}$	$3.4 * 10^{-6}$
$\gamma = 3$	$9.4 * 10^{-3}$	$7.3 * 10^{-4}$	$2.7 * 10^{-5}$
$\gamma = 4$	$8.4 * 10^{-2}$	$2.5 * 10^{-3}$	$5.2 * 10^{-4}$
$\gamma = 5$	$5.2 * 10^{-2}$	$8.7 * 10^{-3}$	$7.1 * 10^{-4}$

Example(2)

Consider the KdVB equation of the form $u_t + \epsilon u u_x + \mu u_{xxx} = 0, 0 \leq x \leq 1, 0 \leq t \leq 1$which is Burger's equation with $\epsilon = -6, \nu = 0, \mu = 1$ (KdV equation) we start with an initial approximation $u(x,0) = -2\text{sech}^2(X), u(0,t) = -2\text{sech}^2(4t)$ and $u_x(0,t) = -4\text{sech}^2(4t)\tanh(4t), u_{xx}(0,t) = 4\text{sech}^2(4t)\left[2\tanh^2(4t)\text{sech}^2(4t)-\right]$. The equation have the exact solution $u(x,t) = -2\text{sech}^2(x-4t)$. Numerical results of KDV equation for different values of m and γ are given in Table .2

Table .2 solution of KDV equation.

γ	$m = 8$	$m = 16$	$m = 32$
$\gamma = 1$	$7.7 * 10^{-3}$	$3.1 * 10^{-4}$	$8.9 * 10^{-7}$
$\gamma = 2$	$8.4 * 10^{-3}$	$4.3 * 10^{-4}$	$7.2 * 10^{-6}$
$\gamma = 3$	$2.6 * 10^{-2}$	$6.7 * 10^{-3}$	$4.6 * 10^{-5}$
$\gamma = 4$	$7.3 * 10^{-2}$	$7.1 * 10^{-3}$	$1.9 * 10^{-4}$
$\gamma = 5$	$9.1 * 10^{-2}$	$7.8 * 10^{-3}$	$2.8 * 10^{-4}$

6. Conclusion.

Based on the 2D-BPFs and there operational matrices for partial derivatives, a new numerical scheme is introduced. This method can be used for solving linear and nonlinear partial differential equations. Numerical results, show the effectiveness and accuracy of the proposed scheme.

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