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ON CERTAIN SUBCLASS OF $p-$VALENT FUNCTIONS DEFINED BY THE JUNG-KIM-SRIVASTAVA INTEGRAL OPERATOR

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Abstract. The purpose of the present paper is to introduce certain subclass of p-valent functions by using the Jung-Kim-Srivastava integral operator in the open unit disc and to obtain the sufficient conditions for this subclass.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and $p-$valent in the open unit disc $U = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $S_p^*(\eta)$ of $p-$valent starlike functions of order η, if it satisfies the following inequality:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \eta \qquad (0 \leq \eta < p; z \in U). \tag{2}$$

Motivated essentially by Jung et al. [3], Shams et al. [5] introduced the integral operator $I_p^\alpha : \mathcal{A}_p \to \mathcal{A}_p$ as follows (see also Aouf et al. [1]):

$$I_p^\alpha f(z) = \begin{cases} \dfrac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \dfrac{z}{t}\right)^{\alpha-1} f(t)\, dt & (\alpha > 0; p \in \mathbb{N}), \\ f(z) & (\alpha = 0; p \in \mathbb{N}). \end{cases} \tag{3}$$

For $f \in \mathcal{A}_p$ given by (1), then from (3), we deduce that

$$I_p^\alpha f(z) = z^p + \sum_{n=k}^{\infty} \left(\frac{p+1}{n+1}\right)^\alpha a_{p+n} z^{p+n} \quad (\alpha \geq 0; p \in \mathbb{N}). \tag{4}$$

Using the above relation, it is easy to verify the identity:

$$z\left(I_p^\alpha f(z)\right)' = (p+1)\, I_p^{\alpha-1} f(z) - I_p^\alpha f(z). \tag{5}$$

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We note that the one-parameter family of integral operator $I_1^\alpha = I^\alpha$ was defined by Jung et al. [3].

Definition 1. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\alpha,\eta)$ if it satisfies

$$Re\left\{\frac{I_p^\alpha f(z)}{I_p^{\alpha+1} f(z)}\right\} > \frac{\eta}{p}\ (0 \le \eta < p; p \in \mathbb{N}; \alpha \ge 0; z \in U). \tag{6}$$

Putting $p = 1$ in (6), then the class $\mathcal{S}_1^*(\alpha,\eta)$ reduces to the class $\mathcal{S}^*(\alpha,\eta)$, which is defined by:

$$Re\left\{\frac{I^\alpha f(z)}{I^{\alpha+1} f(z)}\right\} > \eta\ (0 \le \eta < 1; \alpha \ge 0; z \in U). \tag{1.7}$$

In the present paper, our aim is to determine sufficient conditions for a function $f \in \mathcal{A}_p$ to be a member of the class $\mathcal{S}_p^*(\alpha,\eta)$.

2. Main results involving the operator I_p^α

We begin by recalling the following result (Jack's Lemma), which we shall apply in proving our results below.

Lemma 1 ([2] and see also [4])**.** *Suppose $w(z)$ be a not constant analytic function in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in U$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = \zeta w(z_0)$, where $\zeta \ge 1$ is some real number.*

Theorem 1. *If $f \in \mathcal{A}_p$ satisfies the following condition:*

$$\left|\frac{I_p^\alpha f(z)}{I_p^{\alpha+1} f(z)} - 1\right|^\gamma \left|\frac{I_p^{\alpha-1} f(z)}{I_p^\alpha f(z)} - 1\right|^\delta < M_p(\alpha,\eta,\delta,\gamma)\ \ (z \in U), \tag{7}$$

for some real numbers η, δ and γ such that $0 \le \eta < p$, $\delta + \gamma \ge 0$, $\alpha \ge 1$ and $p \in \mathbb{N}$, then $f \in \mathcal{S}_p^*(\alpha,\eta)$, where

$$M_p(\alpha,\eta,\delta,\gamma) = \begin{cases} \left(1 - \dfrac{\eta}{p}\right)^\gamma \left(1 - \dfrac{\eta}{p} + \frac{1}{2(p+1)}\right)^\delta & \left(0 \le \eta \le \dfrac{p}{2}\right), \\ \left(1 - \dfrac{\eta}{p}\right)^{\delta+\gamma} \left(\dfrac{p+2}{p+1}\right)^\delta & \left(\dfrac{p}{2} \le \eta < p\right). \end{cases} \tag{8}$$

Proof. Case (i) Let $0 \le \eta \le \dfrac{p}{2}$. Define a function $w(z)$ as

$$\frac{I_p^\alpha f(z)}{I_p^{\alpha+1} f(z)} = \frac{1 + \left(1 - \dfrac{2\eta}{p}\right) w(z)}{1 - w(z)}\quad (w(z) \ne 1; z \in U). \tag{9}$$

Then w is analytic in U, $w(0) = 0$ in U. Differentiating (9) logarithmically with respect to z and multiplying the resulting equation by z, we have

$$\frac{z(I_p^\alpha f(z))'}{I_p^\alpha f(z)} - \frac{z(I_p^{\alpha+1} f(z))'}{I_p^{\alpha+1} f(z)} = \frac{\left(1 - \dfrac{2\eta}{p}\right) z w'(z)}{1 + \left(1 - \dfrac{2\eta}{p}\right) w(z)} + \frac{z w'(z)}{1 - w(z)}. \tag{10}$$

By using (5), we obtain

$$(p+1)\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}-(p+1)\frac{I_p^{\alpha}f(z)}{I_p^{\alpha+1}f(z)}=\frac{2\left(1-\frac{\eta}{p}\right)zw^{'}(z)}{\left[1+\left(1-\frac{2\eta}{p}\right)w(z)\right]\left[(1-w\left(z\right)\right]},$$

$$\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}=1+\frac{2(1-\frac{\eta}{p})w(z)}{1-w(z)}+\frac{2\left(1-\frac{\eta}{p}\right)zw^{'}\left(z\right)}{(p+1)\left[1+\left(1-\frac{2\eta}{p}\right)w(z)\right]\left[(1-w\left(z\right)\right]},$$

$$\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}-1=\frac{2\left(1-\frac{\eta}{p}\right)w(z)}{1-w(z)}+\frac{2\left(1-\frac{\eta}{p}\right)zw^{'}\left(z\right)}{(p+1)\left[1+\left(1-\frac{2\eta}{p}\right)w(z)\right]\left[(1-w\left(z\right)\right]} \tag{11}$$

and

$$\frac{I_p^{\alpha}f(z)}{I_p^{\alpha+1}f(z)}-1=\frac{2\left(1-\frac{\eta}{p}\right)w(z)}{1-w(z)}.$$

Thus, we have

$$\begin{aligned}
&\left|\frac{I_p^{\alpha}f(z)}{I_p^{\alpha+1}f(z)}-1\right|^{\gamma}\left|\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}-1\right|^{\delta}\\
=&\left|\frac{2\left(1-\frac{\eta}{p}\right)w\left(z\right)}{1-w\left(z\right)}\right|^{\gamma}\left|\frac{2\left(1-\frac{\eta}{p}\right)w(z)}{1-w(z)}+\frac{2\left(1-\frac{\eta}{p}\right)zw^{'}\left(z\right)}{(p+1)\left[1+\left(1-\frac{2\eta}{p}\right)w(z)\right]\left[(1-w\left(z\right)\right]}\right|^{\delta}\\
=&\left|\frac{2\left(1-\frac{\eta}{p}\right)w\left(z\right)}{1-w\left(z\right)}\right|^{\gamma+\delta}\left|1+\frac{zw^{'}\left(z\right)}{(p+1)\left[1+\left(1-\frac{2\eta}{p}\right)w(z)\right]w\left(z\right)}\right|^{\delta}.
\end{aligned}$$

Suppose that there exists a point $z_0\in U$ such that $\max\limits_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$. Then by using Lemma 1, we have $w(z_0)=e^{i\theta}$ $(0<\theta\leq 2\pi)$ and $z_0w^{'}(z_0)=\zeta w(z_0)$,

$\zeta \geq 1$. Therefore

$$\begin{aligned}
&\left|\frac{I_p^{\alpha} f(z_0)}{I_p^{\alpha+1} f(z_0)} - 1\right|^{\gamma} \left|\frac{I_p^{\alpha-1} f(z_0)}{I_p^{\alpha} f(z_0)} - 1\right|^{\delta} \\
&= \left|\frac{2\left(1-\frac{\eta}{p}\right) w\left(z_0\right)}{1-w\left(z_0\right)}\right|^{\gamma+\delta} \left|1 + \frac{\zeta w(z_0)}{(p+1)\left[1+\left(1-\frac{2\eta}{p}\right) w(z_0)\right] w\left(z_0\right)}\right|^{\delta} \\
&= \frac{2^{\delta+\gamma}\left(1-\frac{\eta}{p}\right)^{\delta+\gamma}}{\left|1-e^{i\theta}\right|^{\delta+\gamma}} \left|1 + \frac{\zeta}{(p+1)\left[1+\left(1-\frac{2\eta}{p}\right) e^{i\theta}\right]}\right|^{\delta} \\
&\geq \left(1-\frac{\eta}{p}\right)^{\delta+\gamma} \left(1 + \frac{\zeta}{2\left(p+1\right)\left(1-\frac{\eta}{p}\right)}\right)^{\delta} \\
&\geq \left(1-\frac{\eta}{p}\right)^{\delta+\gamma} \left(1 + \frac{1}{2\left(p+1\right)\left(1-\frac{\eta}{p}\right)}\right)^{\delta} \\
&= \left(1-\frac{\eta}{p}\right)^{\gamma} \left(1-\frac{\eta}{p}+\frac{1}{2\left(p+1\right)}\right)^{\delta}
\end{aligned}$$

which contradicts (7) for $0 \leq \eta \leq \frac{p}{2}$. Therefore, we must have $|w\left(z\right)| < 1$ for all $z \in U$ and hence $f \in \mathcal{S}_p^*(\alpha, \eta)$.

Case (ii) When $\frac{p}{2} \leq \eta < p$. Let $w(z)$ be defined by

$$\frac{I_p^{\alpha} f(z)}{I_p^{\alpha+1} f(z)} = \frac{\frac{\eta}{p}}{\frac{\eta}{p} - \left(1-\frac{\eta}{p}\right) w(z)} \quad (z \in U),$$

where $w\left(z\right) \neq \frac{\eta}{p-\eta}$ in U. Then $w\left(z\right)$ is analytic in U and $w(0) = 0$. Proceeding as in Case (i) and using (5), we have

$$\frac{z(I_p^{\alpha} f(z))^{'}}{I_p^{\alpha} f(z)} - \frac{z(I_p^{\alpha+1} f(z))^{'}}{I_p^{\alpha+1} f(z)} = \frac{\left(1-\frac{\eta}{p}\right) z w^{'}(z)}{\frac{\eta}{p} - \left(1-\frac{\eta}{p}\right) w(z)},$$

$$(p+1)\,\frac{I_p^{\alpha-1} f(z)}{I_p^{\alpha} f(z)} - (p+1)\frac{I_p^{\alpha} f(z)}{I_p^{\alpha+1} f(z)} = \frac{(1-\frac{\eta}{p}) z w^{'}(z)}{\frac{\eta}{p} - (1-\frac{\eta}{p}) w(z)},$$

$$\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}=\frac{\frac{\eta}{p}}{\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)w(z)}+\frac{(1-\frac{\eta}{p})zw^{'}(z)}{(p+1)\left[\frac{\eta}{p}-(1-\frac{\eta}{p})w(z)\right]},$$

$$\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}-1=\frac{\left(1-\frac{\eta}{p}\right)w(z)}{\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)w(z)}+\frac{(1-\frac{\eta}{p})zw^{'}(z)}{(p+1)\left[\frac{\eta}{p}-(1-\frac{\eta}{p})w(z)\right]},$$

and

$$\frac{I_p^{\alpha}f(z)}{I_p^{\alpha+1}f(z)}-1=\frac{\left(1-\frac{\eta}{p}\right)w(z)}{\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)w(z)}.$$

Then we have

$$\begin{aligned}
&\left|\frac{I_p^{\alpha}f(z)}{I_p^{\alpha+1}f(z)}-1\right|^{\gamma}\left|\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}-1\right|^{\delta}\\
=&\left|\frac{\left(1-\frac{\eta}{p}\right)w(z)}{\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)w(z)}\right|^{\gamma}\left|\frac{\left(1-\frac{\eta}{p}\right)w(z)}{\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)w(z)}+\frac{(1-\frac{\eta}{p})zw^{'}(z)}{(p+1)\left[\frac{\eta}{p}-(1-\frac{\eta}{p})w(z)\right]}\right|^{\delta}\\
=&\left|\frac{\left(1-\frac{\eta}{p}\right)w(z)}{\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)w(z)}\right|^{\gamma+\delta}\left|1+\frac{zw^{'}(z)}{(p+1)\,w(z)}\right|^{\delta}.
\end{aligned}$$

Suppose that there exists a point $z_0 \in U$ such that $\max\limits_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$, then by using Lemma 1, we obtain $w(z_0)=e^{i\theta}$ and $z_0w^{'}(z_0)=\zeta w(z_0)$, $\zeta\geq 1$. Therefore

$$\begin{aligned}
\left|\frac{I_p^{\alpha}f(z_0)}{I_p^{\alpha+1}f(z_0)}-1\right|^{\gamma}\left|\frac{I_p^{\alpha-1}f(z_0)}{I_p^{\alpha}f(z_0)}-1\right|^{\delta} &= \left|\frac{\left(1-\frac{\eta}{p}\right)w(z_0)}{\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)w(z_0)}\right|^{\gamma+\delta}\left|1+\frac{\zeta w(z_0)}{(p+1)\,w(z_0)}\right|^{\delta}\\
&\geq \frac{\left(1-\frac{\eta}{p}\right)^{\gamma+\delta}}{\left|\frac{\eta}{p}-\left(1-\frac{\eta}{p}\right)e^{i\theta}\right|}\left|1+\frac{\zeta}{p+1}\right|^{\delta}\\
&\geq \left(1-\frac{\eta}{p}\right)^{\gamma+\delta}\left(1+\frac{1}{p+1}\right)^{\delta}\\
&= \left(1-\frac{\eta}{p}\right)^{\gamma+\delta}\left(\frac{p+2}{p+1}\right)^{\delta},
\end{aligned}$$

which contradicts (7) for $\frac{p}{2} \leq \eta < p$. Therefore, we must have $|w(z)| < 1$ for all $z \in U$ and hence $f \in \mathcal{S}_p^*(\alpha, \eta)$. This completes the proof of Theorem 1.

□

Putting $p = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *If the function $f \in \mathcal{A}$ satisfies*

$$\left|\frac{I^{\alpha} f(z)}{I^{\alpha+1} f(z)} - 1\right|^{\gamma} \left|\frac{I^{\alpha-1} f(z)}{I^{\alpha} f(z)} - 1\right|^{\delta} < \mathcal{K}(\alpha, \eta, \delta, \gamma) \quad (z \in U), \tag{12}$$

for some real numbers η, δ and γ such that $0 \leq \eta < 1$, $\delta + \gamma \geq 0$ and $\alpha \geq 1$, then $f \in \mathcal{S}^*(\alpha, \eta)$, where $\mathcal{K}(\alpha, \eta, \delta, \gamma)$ is given by

$$\mathcal{K}(\alpha, \eta, \delta, \gamma) = \begin{cases} (1-\eta)^{\gamma} \left(1 - \eta + \frac{1}{4}\right)^{\delta} & \left(0 \leq \eta \leq \dfrac{1}{2}\right), \\ \\ (1-\eta)^{\delta+\gamma} \left(\dfrac{3}{2}\right)^{\delta} & \left(\dfrac{1}{2} \leq \eta < 1\right). \end{cases} \tag{13}$$

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