SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper, by using convolution between analytic functions we define two subclasses of bi-univalent functions and obtain estimates on the initial coefficients of these functions.

1. Introduction

Let \( A \) be the class of all analytic functions in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k.
\]  

(1)

For analytic functions \( f, g \) in \( U \), we say that \( f \) is subordinate to \( g \), written \( f(z) \prec g(z) \) if there exists a Schwarz function \( w \), which is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(w(z)) \), \( z \in U \). Furthermore, if \( g \) is univalent in \( U \), then we have the following equivalence, (cf., e.g.,[7], [8] see also [2]):

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

For function \( f \) given by (1) and function \( g \in A \) given by

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k,
\]  

(2)

the Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z).
\]

Further, let \( S \) denotes the class of all functions in \( A \) which are univalent in \( U \). The Koebe one-quarter theorem (see [5]) ensures that the image of \( U \) under every
univalent function \( f \in A \) contains a disc of radius \( \frac{1}{4} \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) defined by

\[
f^{-1}(f(z)) = z \quad \text{and} \quad f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4),
\]

where

\[
f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \tag{3}
\]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). Let \( \sum \) denote the class of bi-univalent functions in \( U \) given by (1). For a brief history and interesting examples in the class \( \sum \) see ([6] and [9] and its references).

Let the function \( \varphi \) be analytic in \( U \) with positive real part satisfying \( \varphi(0) = 1, \varphi'(0) > 0 \) and \( \varphi(U) \) is symmetric with respect to the real axis. Such a function has a series expansion of the form:

\[
\varphi(z) = 1 + B_1z + B_2z^2 + \ldots \quad (B_1 > 0). \tag{4}
\]

Denote by \( \sum \{g, \alpha, \varphi\} \) \( (0 \leq \alpha \leq 1) \), \( g \) given by (2) be the class of functions \( f \in \sum \) given by (1) and satisfies

\[
(1 - \alpha)\frac{f * g(z)}{z} + \alpha(f * g(z))' \prec \varphi(z). \tag{5}
\]

Also denote by \( \sum^*(g, \mu, \varphi) \) \( (0 \leq \mu \leq 1) \), \( g \) given by (2) be the class of functions \( f \in \sum \) given by (1) and satisfies

\[
\left(\frac{(f * g(z))}{z}\right)^\mu (\frac{(f * g(z))'}{z})^{1-\mu} \prec \varphi(z). \tag{6}
\]

Specializing the parameters \( \alpha, \mu \) and the function \( g \), we have the following sub-classes.

i) \( \sum^*(\frac{1}{z^2}, 1, \varphi) = \sum^*(\frac{1}{z^2}, 0, \varphi) = H^*_\Sigma(\varphi) \) (see Ali et al. [1]);

ii) \( \sum\{1, \alpha, \varphi\} = \sum(\alpha, \varphi) = \left\{ f \in A : (1 - \alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \varphi(z) \right\} \);

iii) \( \sum(g, 0, \varphi) = \left\{ f \in A : \frac{(f * g(z))}{z} \prec \varphi(z) \right\} \);

iv) \( \sum(g, 1, \varphi) = \left\{ f \in A : (f * g(z))' \prec \varphi(z) \right\} \);

v) \( \sum(g, 1, \varphi) = \sum^*(g, 0, \varphi) \);

vi) \( \sum(z + \sum_{k=2}^\infty \sigma_k(\alpha_1)z^k, \alpha, \varphi) = \sum(\alpha_1, \alpha, \varphi) = \left\{ f \in A : (1 - \alpha)\frac{H_{l,s}(\alpha_1)f(z)}{z} + \alpha(H_{l,s}(\alpha_1)f(z))' \prec \varphi(z) \right\} \), where

\[
\sigma_k(\alpha_1) = \frac{(\alpha_1)_{k-l} \cdots (\alpha_1)_{k-1}}{(\beta_1)_{k-1} \cdots (\beta_s)_{k-1}(1)_{k-1}} \tag{7}
\]

and \( H_{l,s}(\alpha_1) \) is the Dziok-Srivastava operator (see Dziok and Srivastava [4]);

vii) \( \sum(z + \sum_{k=2}^\infty \left[\frac{1+l+l(k-1)}{l+1}\right]^k z^k (\lambda \geq 0; l, s \in N_0, \alpha, \varphi) = \sum(l, \lambda, \alpha, \varphi) = \left\{ f \in A : (1 - \alpha)\frac{I^*(l, \lambda)f(z)}{z} + \alpha(I^*(l, \lambda)f(z))' \prec \varphi(z) \right\} \), where \( I^*(l, \lambda) \) is the generalized multiplier transformation which was introduced and studied by Cătăș et al. [3] and \( N_0 = N \cup \{0\}, N = \{1, 2, \ldots\} \);

viii) \( \sum^*(z + \sum_{k=2}^\infty \sigma_k(\alpha_1)z^k, \mu, \varphi) = \sum^*(\alpha_1, \mu, \varphi) \)
for functions in the subclasses \( \sum(g, \alpha, \varphi) \) and \( \sum^*(g, \mu, \varphi) \).

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that: \( 0 \leq \alpha \leq 1, 0 \leq \mu \leq 1, g \) given by (2), \( \varphi \) given by (4), \( z \in U \) and the powers are considered principal ones.

**Theorem 1.** Let the function \( f(z) \) given by (1) belongs to \( \sum(g, \alpha, \varphi) \). Then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{(1 + 2\alpha)B_1^2 + (1 + \alpha)^2(B_1 - B_2)b_2}
\]

and

\[
|a_3| \leq \left[ \left( \frac{B_1}{1 + \alpha} \right)^2 + \frac{B_1}{1 + 2\alpha} \right] \frac{1}{b_3}.
\]

**Proof.** Let \( f \in \sum(g, \alpha, \varphi) \) and \( F = f \ast g \). Then there are analytic functions \( u, v : U \to U \), with \( u(0) = v(0) = 0 \) and satisfying

\[
F'(z) = \varphi(u(z)) \quad \text{and} \quad (F^{-1}(w))' = \varphi(v(w)).
\]  

Let the functions \( p_1 \) and \( p_2 \) be defined by

\[
p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1z + c_2z^2 + ...
\]

and

\[
p_2(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + d_1z + d_2z^2 + ....
\]

We see that \( p_1 \) and \( p_2 \) are analytic in \( U \) with \( p_1(0) = p_2(0) = 1 \), \( p_1 \) and \( p_2 \) have positive real part in \( U \) and \( |c_i| \leq 2 \) and \( |d_i| \leq 2 \).

From (11) and (12), we have:

\[
u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left[ c_1z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + ... \right]
\]

and

\[
v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2} \left[ d_1z + \left( d_2 - \frac{d_1^2}{2} \right) z^2 + ... \right].
\]

From (4), (10), (12) and (13) we see that

\[
F'(z) = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1z + \left( \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + ...
\]

and

\[
(F^{-1}(w))' = \varphi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right) = 1 + \frac{1}{2} B_1 d_1w + \left( \frac{1}{2} B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} B_2 d_1^2 \right) w^2 + ...
\]

For \( f \) of the form (1), \( g \) of the form (2) and \( F = f \ast g \), we have

\[
F(z) = z + a_2b_2z^2 + a_3b_3z^3 + ...,
\]

\[
F'(z) = 1 + 2a_2b_2z + 3a_3b_3z^2 + ..., \quad (17)
\]
\[ F^{-1}(w) = w - a_2 b_2 w^2 + (2(a_2 b_2)^2 - a_3 b_3) w^3 - (5(a_2 b_2)^3 - 5a_2 a_3 b_2 b_3 + a_4 b_4) w^4 + \ldots \]

and
\[ (F^{-1}(w))' = 1 - 2a_2 b_2 w + 3(2(a_2 b_2)^2 - a_3 b_3) w^2. \] (18)

From (15) – (18), we have
\[
\begin{align*}
2a_2 b_2 &= \frac{1}{2} B_1 c_1, \\
3a_3 b_3 &= \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2, \\
-2a_2 b_2 &= \frac{1}{2} B_1 d_1 d_2
\end{align*}
\] (19)

and
\[
3(2(a_2 b_2)^2 - a_3 b_3) = \frac{1}{2} B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} B_2 d_1^2.
\] (20)

Simple computations show that:
\[ c_1 = -d_1 \] (21)

and
\[
(a_2 b_2)^2 = \frac{B_3^2 (c_2 + d_2)}{4 \left[ (1 + 2\alpha) B_1^2 + (1 + \alpha)^2 (B_1 - B_2) \right]}.
\] (22)

Since, for functions with positive real part \(|c_2| \leq 2 \text{ and } |d_2| \leq 2\). Then (24) leads to the estimate (8).

Also, using (19), (20), (22) and (24), we have
\[ 2(1 + 2\alpha) a_3 b_3 = \frac{(1 + 2\alpha) B_1^2 c_1^2}{2(1 + \alpha)^2} + \frac{1}{2} B_1 (c_2 - d_2), \]

which in turn leads to the estimate (9). This completes the proof of Theorem 1.

Remark 1. Taking \(\alpha = 1\) and \(g(z) = \frac{z}{1 - z}\) or \(b_k = 1, k \geq 2\) in Theorem 1, we have the result obtained by Ali et al. [1, Theorem 2.1].

Theorem 2. Let the function \(f(z)\) given by (1) belongs to \(\sum^*(g, \mu, \varphi)\). Then
\[
|a_2| \leq \frac{B_1 \sqrt{B_3}}{\sqrt{2\left[ (\mu^2 - 5\mu + 6) B_1^2 + 2(2 - \mu)^2 (B_1 - B_2) \right]} b_2}.
\] (25)

and
\[
|a_3| \leq \left[ \frac{B_1}{3 - 2\mu} + \frac{B_1^2}{(2 - \mu)^2} \right] \frac{1}{b_3}.
\] (26)

Proof. Let \(f \in \sum^*(g, \mu, \varphi)\) and \(F = f * g\). Then there are analytic functions \(u, v : U \rightarrow U\), with \(u(0) = v(0) = 0\) and satisfying:
\[
\left( \frac{F(z)}{z} \right)^\mu (F'(z))^{1 - \mu} = \varphi(u(z))
\] (27)

and
\[
\left( \frac{F^{-1}(w)}{w} \right)^\mu (F^{-1}(w))^{1 - \mu} = \varphi(v(w)),
\] (28)

where \(\varphi, u\) and \(v\) are given by (4), (13) and (14), respectively. Since
\[
\left( \frac{F(z)}{z} \right)^\mu (F'(z))^{1 - \mu} = 1 + (2 - \mu) a_2 b_2 z + \frac{1}{2} \left[ \mu (\mu - 1) (a_2 b_2)^2 + 2(3 - 2\mu) a_3 b_3 \right] z^2 + \ldots
\] (29)
and
\[
\left( \frac{F^{-1}(w)}{w} \right)^{\mu} \left( \left( F^{-1}(w) \right) \right)^{1-\mu} = 1 - (2 - \mu)a_2b_2w + \frac{1}{2} \left[ (\mu^2 - 9\mu + 12)(a_2b_2)^2 - 2(3 - 2\mu)a_3b_3 \right] w^2 + \ldots 30
\] (4)

Using (4), (13), (14), (29) and (30), we have
\[
(2 - \mu)a_2b_2 = \frac{1}{2} B_1 c_1,
\] (31)
\[
\mu(\mu - 1)(a_2b_2)^2 + 2(3 - 2\mu)a_3b_3 = B_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{2} B_2 c_1^2,
\] (32)
\[
- (2 - \mu)a_2b_2 = \frac{1}{2} B_1 d_1
\] (33)

and
\[
(\mu^2 - 9\mu + 12)(a_2b_2)^2 - 2(3 - 2\mu)a_3b_3 = B_1 (d_2 - \frac{d_1^2}{2}) + \frac{1}{2} B_2 d_1^2.
\] (34)

From (31) and (33), we have
\[
c_1 = -d_1
\] (35)

and from (32)-(34), we have
\[
(a_2b_2)^2 = \frac{B_1^3 (c_2 + d_2)}{2 \left[ (\mu^2 - 5\mu + 6)B_1^2 + 2(2 - \mu)^2(B_1 - B_2) \right]}.
\] (36)

Since $|c_2| \leq 2$ and $|d_2| \leq 2$ for functions of positive real part and $B_1 > 0$, inequality (25) holds.

From (32) – (35), we have
\[
a_3b_3 = \frac{B_1 (c_2 - d_2)}{4(3 - 2\mu)} + \frac{B_1^2 c_1^2}{4(2 - \mu)^2}.
\] (37)

which in turn leads to the estimate (26). This completes the proof of Theorem 2.

**Remark 2.** Taking $\mu = 0$ and $g(z) = \frac{z}{1-z}$ or $b_k = 1, k \geq 2$ in Theorem 2, we have the result obtained by Ali et al. [1, Theorem 2.1].

**Remark 3.** Specilizing the parameters $\alpha, \mu$ and the function $g$ in Theorems 1 and 2, we obtain results corresponding to the classes $i$ – $vii$ defined in the introduction.

**References**


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